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Ai-ni HSIEH

SOME LOCALLY TABULAR LOGICS WITH CONTRACTION AND MINGLE

A b s t r a c t. Anderson and Belnap's implicational system $\mathbf{RMO}_{\rightarrow}$ can be extended conservatively by the usual axioms for fusion and for the Ackermann truth constant t. The resulting system $\mathbf{RMO}^*$ is algebraized by the quasivariety IP of all idempotent commutative residuated po-monoids. Thus, the axiomatic extensions of $\mathbf{RMO}^*$ are in one-to-one correspondence with the relative subvarieties of IP. An algebra in IP is called *semiconic* if it decomposes subdirectly (in IP) into algebras where the identity element t is order-comparable with all other elements. The semiconic algebras in IP are locally finite. It is proved here that a relative subvariety of IP consists of semiconic algebras if and only if it satisfies $x \approx (x \rightarrow \mathsf{t}) \rightarrow x$. It follows that if an axiomatic extension of $\mathbf{RMO}^*$ has $((p \rightarrow \mathsf{t}) \rightarrow p) \rightarrow p$ among its theorems then it is locally tabular. In particular, such an extension is strongly decidable, provided that it is finitely axiomatized.

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1. Introduction

There are now several different motivations for the study of logics that lack the *weakening* axiom $p \to (q \to p)$. The first systems of this kind were developed by relevance logicians, who also debated the merits of the weaker *mingle* postulate $p \to (p \to p)$. In the principal relevance logic **R**, and more generally in extensions of the intensional fragments of **R**, this postulate amounts to idempotence of the fusion connective $(\cdot)$, so its adoption as an axiom leads to a reduction in the number of independent formulas, improving the chances of decidability.

In [1, p. 98], Anderson and Belnap introduced the purely implicational formal system $\mathbf{RMO}_{\to}$ axiomatized by

(B)	$(p \to q) \to ((r \to p) \to (r \to q))$	(*prefixing*)
(C)	$(p \to (q \to r)) \to (q \to (p \to r))$	(*exchange*)
(I)	$p \to p$	(*identity*)
(W)	$(p \to (p \to q)) \to (p \to q)$	(*contraction*)
(M)	$p \to (p \to p)$	(*mingle*),

where the sole inference rule is *modus ponens*, viz.

(MP) $\langle \{p,\, p \to q\},\, q \rangle.$

The postulates other than (M) axiomatize the implication fragment of **R**, and they are intuitionistically valid. In $\mathbf{RMO}_{\to}$, the identity axiom is redundant, since it can be derived from (W), (M) and (MP).

Information about $\mathbf{RMO}_{\to}$ can be found in [1, 13, 16]. It follows from a result of Church [8, 9] that $\mathbf{RMO}_{\to}$ enjoys a variant of the classical deduction theorem:

$$\Gamma \cup \{\varphi\} \vdash_{\mathbf{RMO}_{\to}} \psi \quad \text{iff} \quad (\Gamma \vdash_{\mathbf{RMO}_{\to}} \varphi \to \psi \ \text{ or } \ \Gamma \vdash_{\mathbf{RMO}_{\to}} \psi).$$

As Church observed (in greater generality), this meta-theorem persists even when we extend $\mathbf{RMO}_{\to}$ by arbitrary new axioms, possibly involving new connectives or sentential constants, *provided* that we do not add any new inference rules.

If we add a negation to $\mathbf{RMO}_{\to}$, as well as the usual axioms of double negation and contraposition, we obtain a definable fusion of the form

$p \cdot q := \neg (p \to \neg q)$, but we also obtain new theorems in the purely implicational vocabulary [2]; systems of this kind have been analyzed in detail in [3, 4, 5, 11].

On the other hand, we might choose to omit negation and to add to $\mathbf{RMO}_{\to}$ a *primitive* fusion and the Ackermann truth constant, accompanied by the usual postulates, as follows:

Definition 1.1. $\mathbf{RMO}^*$ *shall denote the formal system with language* $\cdot, \to, \mathsf{t}$ *that is axiomatized by the postulates of* $\mathbf{RMO}_{\to}$*, together with*

$$q \to (p \to (p \cdot q))$$
$$(q \to (p \to r)) \to ((p \cdot q) \to r)$$
$$\mathsf{t}$$
$$\mathsf{t} \to (p \to p).$$

It turns out that the purely implicational theorems of $\mathbf{RMO}^*$ are just those of $\mathbf{RMO}_{\to}$. The same applies to derivable rules, in view of Church's deduction theorem. This conservation result is explained, for instance, in [17, Remark, p. 267].

The finitely axiomatized extensions of $\mathbf{RMO}^*$ include two well understood systems, viz. the $\wedge, \to$ fragment of intuitionistic propositional logic and the intensional fragment of $\mathbf{RM}^{\mathbf{t}}$. These two mutually incomparable systems and all of their finitely axiomatized extensions are *decidable*, because both systems are *locally tabular*—this means that for each finite number n, there are only finitely many logically inequivalent formulas in n variables.

In this paper, we shall prove a simultaneous generalization of these facts by considering the equivalent algebraic semantics for $\mathbf{RMO}^*$, which is the quasivariety of *idempotent commutative residuated po-monoids*. It follows from a result in [12] that an axiomatic extension of $\mathbf{RMO}^*$ will be locally tabular whenever its algebraic counterpart consists of *semiconic* algebras (defined in Section 5). We prove here that this happens exactly when the extension includes the formula $((p \to \mathsf{t}) \to p) \to p$ among its theorems. The result encompasses the intuitionistic case and the case of $\mathbf{RM}^{\mathbf{t}}$.

2. Preliminaries

Given a fixed algebraic language (or type) and an infinite set of variables, let $\boldsymbol{Fm}$ denote the absolutely free algebra, freely generated by the variables. *Formulas* are just elements of the universe of $\boldsymbol{Fm}$, and *substitutions* are endomorphisms of $\boldsymbol{Fm}$.

A (finitary) *formal system* $\mathbf{F}$ over this language is meant here to consist of a set of formulas, called *axioms*, and a set of pairs $\langle \Phi, \varphi \rangle$, called *inference rules*, where $\Phi \cup \{\varphi\}$ is a finite set of formulas. The elements of Φ are called the *premisses* of $\langle \Phi, \varphi \rangle$, and φ is called the *conclusion*.

Given a formal system $\mathbf{F}$, the *deducibility relation* $\vdash_{\mathbf{F}}$ is the relation from sets of formulas to single formulas that contains a pair $\langle \Gamma, \alpha \rangle$ just when there is a *proof of* α *from* Γ *in* $\mathbf{F}$. A proof of this kind is any finite sequence of formulas terminating with α, such that every item in the sequence belongs to Γ or is a substitution instance of a formula that is either an axiom of $\mathbf{F}$ or the conclusion of an inference rule of $\mathbf{F}$, where in the last case, the same substitution turns the premisses of the rule into previous items in the sequence. To signify that such a proof exists, we write $\Gamma \vdash_{\mathbf{F}} \alpha$; then $\langle \Gamma, \alpha \rangle$ is called a *derivable rule* of $\mathbf{F}$. In this case, we omit Γ when it is empty. The *theorems* of $\mathbf{F}$ are the formulas α such that $\vdash_{\mathbf{F}} \alpha$.

Let K be a class of algebras in the language under discussion. The *equational consequence relation* $\models_{\mathsf{K}}$ from sets Σ of equations to single equations $\varphi \approx \psi$ is defined as follows: $\Sigma \models_{\mathsf{K}} \varphi \approx \psi$ iff for every homomorphism h from $\boldsymbol{Fm}$ into an algebra in K, if $h(\alpha) = h(\beta)$ for all $\alpha \approx \beta \in \Sigma$ then $h(\varphi) = h(\psi)$.

For sets of equations Σ and Ψ, the notation $\Sigma \models_{\mathsf{K}} \Psi$ means $\Sigma \models_{\mathsf{K}} \varphi \approx \psi$ for all $\varphi \approx \psi \in \Psi$, and similarly for $\vdash_{\mathbf{F}}$. We shall use $\Sigma \mathrel{=\!\!|\!\!\models_{\mathsf{K}}} \Psi$ as an abbreviation for the conjunction of $\Sigma \models_{\mathsf{K}} \Psi$ and $\Psi \models_{\mathsf{K}} \Sigma$, and similarly for $\dashv\vdash_{\mathbf{F}}$.

Blok and Pigozzi proposed a general notion of an algebraizable logic in [7]. In current terminology, a formal system $\mathbf{F}$ is said to be (elementarily) *algebraizable* if there exists a quasivariety K in the language of $\mathbf{F}$, as well as a finite family of unary equations $\delta_i(x) \approx \varepsilon_i(x)$, $i \in I$, and a finite family of binary formulas $\Delta_j(x, y)$, $j \in J$, such that for any set of formulas $\Gamma \cup \{\alpha\}$,

$$\Gamma \vdash_{\mathbf{F}} \alpha \quad \text{iff} \quad \{\delta_i(\gamma) \approx \varepsilon_i(\gamma) : \gamma \in \Gamma,\, i \in I\} \models_{\mathsf{K}} \{\delta_i(\alpha) \approx \varepsilon_i(\alpha) : i \in I\};$$

$$\{\delta_i(\Delta_j(x, y)) \approx \varepsilon_i(\Delta_j(x, y)) : i \in I,\, j \in J\} \mathrel{=\!\!|\!\!\models_{\mathsf{K}}} x \approx y.$$

In this case, for any set of equations $\Sigma \cup \{\varphi \approx \psi\}$, we also have

$$\Sigma \models_{\mathsf{K}} \varphi \approx \psi \quad \text{iff} \quad \{\Delta_j(\alpha,\beta) : \alpha \approx \beta \in \Sigma,\, j \in J\} \vdash_{\mathbf{F}} \{\Delta_j(\varphi,\psi) : j \in J\};$$
$$\{\Delta_j(\delta_i(p), \varepsilon_i(p)) : i \in I,\, j \in J\} \dashv\vdash_{\mathbf{F}} p.$$

Furthermore, the so-called *defining equations* $\delta_i(x) \approx \varepsilon_i(x)$, $i \in I$, and the *equivalence formulas* $\Delta_j(x,y)$, $j \in J$, are unique up to interderivability in $\models_{\mathsf{K}}$ and in $\vdash_{\mathbf{F}}$, respectively, and the quasivariety K is unique [7]. We call K the *equivalent quasivariety* of **F**.

3. Residuated Po-Monoids

In this section and the next, we discuss the algebraization of $\mathbf{RMO}^*$.

Definition 3.1. *A structure* $\langle A; \cdot, \rightarrow, \mathsf{t}, \leq \rangle$ *is called a* commutative residuated po-monoid *(briefly, a* CRP*) if* $\langle A; \leq \rangle$ *is a po-set,* $\langle A; \cdot, \mathsf{t} \rangle$ *is a commutative monoid, and* $\rightarrow$ *is a binary* residuation *operator—which means that for all* $a, b, c \in A$*, we have*

$$c \leq a \rightarrow b \quad \textit{iff} \quad a \cdot c \leq b.$$

This residuation law can be stated equivalently as follows: $\leq$ *is* compatible with $\cdot$ *(in the sense of (2) below) and for every* $a, b \in A$*, there is a* largest $c \in A$ *with* $a \cdot c \leq b$*. (The largest such* c *becomes* $a \rightarrow b$*.)*

Notation 3.2. *From now on,* $|x|$ *shall abbreviate* $x \rightarrow x$*.*

The following properties of CRPs are well known.

Proposition 3.3. *Every CRP satisfies*:

$$x \cdot (x \to y) \;\le\; y \tag{1}$$

$$x \le y \;\;\Longrightarrow\;\; z \cdot x \;\le\; z \cdot y \tag{2}$$

$$x \le y \;\;\Longrightarrow\;\; z \to x \;\le\; z \to y \quad \textit{and} \quad y \to z \;\le\; x \to z \tag{3}$$

$$(x \cdot y) \to z \;\approx\; y \to (x \to z) \;\approx\; x \to (y \to z) \tag{4}$$

$$x \le (x \to y) \to y, \quad \textit{hence} \tag{5}$$

$$((x \to y) \to y) \to y \;\approx\; x \to y \tag{6}$$

$$\mathsf{t} \le |x| \tag{7}$$

$$x \le y \;\;\Longleftrightarrow\;\; \mathsf{t} \le x \to y \;\;\Longleftrightarrow\;\; |x \to y| \;\le\; x \to y \tag{8}$$

$$x \;\approx\; \mathsf{t} \to x \;\approx\; |x| \to x \tag{9}$$

$$||\, x \,|| \;\approx\; |x|\,. \tag{10}$$

A CRP is said to be *idempotent* if it satisfies $x \cdot x \approx x$.

Proposition 3.4. *For any elements a, b of an idempotent CRP, we have*

$$a \le b \quad \textit{iff} \quad a \to b \;=\; |a \to b|\;; \quad \textit{in particular,} \tag{11}$$

$$\mathsf{t} \le a \quad \textit{iff} \quad a \;=\; |a|\;; \tag{12}$$

$$a \to |a| \;=\; |a|\;; \tag{13}$$

$$a \le (a \to \mathsf{t}) \to a. \tag{14}$$

$$\textit{if } \mathsf{t} \le a \le b \textit{ then } a \cdot b = b. \tag{15}$$

Proof. By idempotence, we have $a \cdot a \le a$ and thus $a \le a \to a = |a|$. So (11) follows immediately from (8). Then (12) follows from (11), because $\mathsf{t} \to a = a$ (by (9)). Also, (13) follows from (4) and idempotence.

By (1), we have $a \cdot (a \to \mathsf{t}) \le \mathsf{t}$, so $a \cdot a \cdot (a \to \mathsf{t}) \le a \cdot \mathsf{t}$, by (2). Thus, $a \cdot (a \to \mathsf{t}) \le a$, by idempotence, i.e., $a \le (a \to \mathsf{t}) \to a$.

If $\mathsf{t} \le a \le b$ then, by (2) and idempotence, $b = \mathsf{t} \cdot b \le a \cdot b \le b \cdot b = b$, so $a \cdot b = b$. □

It follows from (11) that an idempotent CRP $\langle A; \cdot, \to, \mathsf{t}, \le \rangle$ is definitionally equivalent to its pure algebra reduct $\boldsymbol{A} = \langle A; \cdot, \to, \mathsf{t} \rangle$. So, from now on, we treat these idempotent structures as pure algebras with an equationally definable partial order, always denoted by $\le$.

Notation 3.5. *For the remainder of the paper,* IP *shall denote the class of all idempotent CRPs.*

Obviously, IP is a quasivariety. It is not a variety, as it contains the idempotent CRP on the 3-element chain $-1 < 0 < 1$, where 0 is the identity for $\cdot$ and $1 \cdot -1 = -1$. It is well known that this 3-element algebra has a homomorphic image that is not an idempotent CRP (see for instance [12]).

In [12, Thm. 3.4], it is shown that every algebra in IP can be embedded into a lattice-ordered algebra in IP. This, together with [17, Cor. 9.4; Remark, p. 267] establishes that for any set of formulas $\Gamma \cup \{\alpha\}$ over the language of $\mathbf{RMO}^*$,

$$\Gamma \vdash_{\mathbf{RMO}^*} \alpha \quad \text{iff} \quad \{\gamma \approx |\gamma| : \gamma \in \Gamma\} \models_{\mathsf{IP}} \alpha \approx |\alpha| \, .$$

Since IP satisfies (10) and $x \leq y \iff x \to y \approx |x \to y|$, it is easy to see that

$$\{x \to y \approx |x \to y| \, , \; y \to x \approx |y \to x|\} \;=\!\!|\!\models_{\mathsf{IP}}\; x \approx y.$$

Thus, we have

Theorem 3.6. $\mathbf{RMO}^*$ *is algebraizable with equivalence formulas* $x \to y$ *and* $y \to x$ *and defining equation* $x \approx x \to x$, *and* IP *is the equivalent quasivariety of* $\mathbf{RMO}^*$.

4. Filters, Relative Congruences and Relative Subvarieties

Definition 4.1. *Let* $\mathbf{F}$ *be a formal system and* $\boldsymbol{A}$ *an algebra of the same type. A subset* X *of* A *is called an* $\mathbf{F}$–filter *of* $\boldsymbol{A}$ *if for every homomorphism* h *from* $\boldsymbol{Fm}$ *into* $\boldsymbol{A}$, *we have*

$h(\varphi) \in X$, *for every axiom* φ *of* $\mathbf{F}$;

if $h[\Phi] \subseteq X$ *then* $h(\varphi) \in X$, *for every inference rule* $\langle \Phi, \varphi \rangle$ *of* $\mathbf{F}$.

In this case, for any set of formulas $\Gamma \cup \{\alpha\}$ over the language of $\mathbf{F}$, if $\Gamma \vdash_{\mathbf{F}} \alpha$ and h is a homomorphism from $\boldsymbol{Fm}$ into $\boldsymbol{A}$ with $h[\Gamma] \subseteq X$, then $h(\alpha) \in X$. This follows by induction on the length of a proof of α from Γ in $\mathbf{F}$. Note that arbitrary intersections of $\mathbf{F}$–filters are still $\mathbf{F}$–filters.

A subset X of an idempotent CRP $\boldsymbol{A} = \langle A; \cdot, \to, \mathsf{t}\rangle$ is said to be *upward closed* provided that whenever $a \in X$ and $a \leq b \in A$, then $b \in X$. We call X a *submonoid* of $\boldsymbol{A}$ if $\mathsf{t} \in X$ and whenever $a, b \in X$, then $a \cdot b \in X$.

Lemma 4.2. *Let $\boldsymbol{A}$ be an idempotent CRP. Then the* **RMO****–filters of $\boldsymbol{A}$ are exactly the upward closed submonoids of $\boldsymbol{A}$.*

Proof. Suppose X is an **RMO***–filter of $\boldsymbol{A}$. If $a \in X$ and $a \leq b \in A$, then $a \to b = |a \to b|$, by (11), so $a \to b \in X$ (by the identity axiom of **RMO***). In this case, it follows that $b \in X$, by modus ponens. Thus, X is upward closed. Certainly, $\mathsf{t} \in X$ because t is an axiom of **RMO***. If $a, b \in X$ then, since $q \to (p \to (p \cdot q))$ is an axiom of **RMO***, we have $a \cdot b \in X$, by two applications of modus ponens. So X is a submonoid of $\boldsymbol{A}$.

Conversely, suppose X is an upward closed submonoid of $\boldsymbol{A}$. Let h be a homomorphism from $\boldsymbol{Fm}$ into $\boldsymbol{A}$, and let φ be an axiom of **RMO***. Then $\mathsf{t} \leq |h(\varphi)| = h(\varphi)$, by (7) and Theorem 3.6. So $h(\varphi) \in X$, because $\mathsf{t} \in X$ and X is upward closed. If $a, a \to b \in X$, where $b \in A$, then $a \cdot (a \to b) \leq b$, by (1), whence $b \in X$, because X is an upward closed submonoid of $\boldsymbol{A}$. This shows that X is an **RMO***–filter of $\boldsymbol{A}$. $\square$

Notation 4.3. *From now on, given any po-set $\langle A; \leq\rangle$ and $a \in A$, we use $[a)$ to abbreviate $\{b \in A : a \leq b\}$, and $(a]$ to abbreviate $\{b \in A : b \leq a\}$.*

If $\boldsymbol{A}$ is an idempotent CRP and $X \subseteq A$, then $\mathrm{Fg}\, X$ shall denote the smallest **RMO***–filter of $\boldsymbol{A}$ containing X. Lemma 4.2 yields:

Corollary 4.4. *For any element a of an idempotent CRP, we have*

$$\mathrm{Fg}\{a\} = [\mathsf{t}) \cup [a).$$

Clearly, the smallest **RMO***–filter of any idempotent CRP is $[\mathsf{t})$. Thus, every **RMO***–filter distinct from $[\mathsf{t})$ contains an element not above t.

Definition 4.5. *Let K be a quasivariety and $\boldsymbol{A}$ an algebra of the same type. A congruence θ of $\boldsymbol{A}$ is called a* K–congruence *if the factor algebra $\boldsymbol{A}/\theta$ belongs to K. We refer to K–congruences as* relative congruences *when K is understood.*

The K–congruences of $\boldsymbol{A}$ form an algebraic lattice under set inclusion, which coincides with the ordinary congruence lattice when K is a variety and $\boldsymbol{A} \in \mathsf{K}$.

Definition 4.6. *Given a subset X of an algebra $\boldsymbol{A}$, we use $\Omega(X)$ to denote the largest congruence of $\boldsymbol{A}$ such that X is a union of congruence classes.*

The congruence $\Omega(X)$ always exists. When X is a filter of an algebraizable formal system then $\Omega(X)$ has the internal characterization given in the next theorem. This result is one of several characterizations of algebraizable logics proved by Blok and Pigozzi in [7].

Theorem 4.7. *A formal system $\mathbf{F}$ is algebraizable with equivalent quasivariety K iff for every algebra $\boldsymbol{A}$ of the same type, the mapping $X \mapsto \Omega(X)$, restricted to the $\mathbf{F}$–filters X of $\boldsymbol{A}$, is an isomorphism between the lattices of $\mathbf{F}$–filters and K–congruences of $\boldsymbol{A}$.*

In this case, for every $\mathbf{F}$–filter X of an algebra $\boldsymbol{A}$, we have

$$\Omega(X) := \{\langle a, b\rangle \in A \times A : \Delta_j^{\boldsymbol{A}}(a, b) \in X \textit{ for all } j \in J\},$$

where $\Delta_j(x, y)$, $j \in J$, are the equivalence formulas.

Corollary 4.8. *Let X be an $\mathbf{RMO}^*$–filter of an algebra $\boldsymbol{A} \in \mathsf{IP}$. Then*

(i) $\Omega(X) = \{\langle a, b\rangle \in A \times A : a \to b,\ b \to a \in X\}$, *and this relation is an IP–congruence of $\boldsymbol{A}$.*

(ii) for any $a \in A$, we have $a \in X$ iff $\langle a, |a|\rangle \in \Omega(X)$.

(iii) for any elements a, b in A, we have $a \to b \in X$ iff $a/\Omega(X) \leq b/\Omega(X)$ in the factor algebra $\boldsymbol{A}/\Omega(X)$.

Proof. Item (i) follows from Theorems 3.6 and 4.7. Then (ii) follows from (i). Indeed, $|a| \to a = a$, by (9), while $a \to |a| = |a| \in X$, by (13) and the identity axiom of $\mathbf{RMO}^*$. Finally, (iii) follows from (ii), using (11). □

Definition 4.9. *An algebra $\boldsymbol{A}$ in a quasivariety K is said to be K–*subdirectly irreducible *(or* relatively subdirectly irreducible*) if the identity relation on A is completely meet irreducible in the K–congruence lattice of $\boldsymbol{A}$, i.e., $\boldsymbol{A}$ has a least non-identity K–congruence.*

Clearly, if a K–subdirectly irreducible algebra $\boldsymbol{A} \in \mathsf{K}$ is a subdirect product of a family of algebras $\boldsymbol{A}_i \in \mathsf{K}$ $(i \in I)$, then $\boldsymbol{A} \cong \boldsymbol{A}_i$ for some $i \in I$. The following adaptation of Birkhoff's subdirect decomposition theorem to quasivarieties is well known (see [15, Thm. 1.1]).

Theorem 4.10. *Every algebra in a quasivariety* K *is isomorphic to a subdirect product of relatively subdirectly irreducible algebras in* K*.*

Given a po-set $\langle A; \leq \rangle$, and an element $x \in A$, we say that x *splits* $\langle A; \leq \rangle$ if

$$A = [x) \overset{\bullet}{\cup} (a]$$

for some $a \in A$, where $\overset{\bullet}{\cup}$ indicates *disjoint* union (i.e., $x \nleq a$). That is to say, x splits $\langle A; \leq \rangle$ iff $a := \max_{\leq} \{b \in A : x \nleq b\}$ exists, i.e., iff A has a largest element not above x.

Theorem 4.11. *An idempotent CRP* $\boldsymbol{A}$ *is* IP*–subdirectly irreducible iff* t *splits the po-set* $\langle A; \leq \rangle$*.*

Proof. By Theorems 3.6 and 4.7, $\boldsymbol{A}$ is IP–subdirectly irreducible iff $\boldsymbol{A}$ has a least **RMO***–filter distinct from $[\mathsf{t})$. Since $[\mathsf{t})$ is contained in every **RMO***–filter, the latter demand means that there exists $a \in A$ such that $\mathsf{t} \nleq a$ and $\mathrm{Fg}\{a\} \subseteq \mathrm{Fg}\{b\}$ whenever $\mathsf{t} \nleq b \in A$. But for $a, b \in A$ with $\mathsf{t} \nleq a$, we have

$$\mathrm{Fg}\{a\} \subseteq \mathrm{Fg}\{b\} \text{ iff } a \in \mathrm{Fg}\{b\} = [\mathsf{t}) \cup [b) \text{ (Corollary 4.4) iff } a \in [b) \text{ iff } b \leq a.$$

So $\boldsymbol{A}$ is IP–subdirectly irreducible iff $a := \max_{\leq} \{b \in A : \mathsf{t} \nleq b\}$ exists. $\square$

Definition 4.12. *A* relative subvariety *of a quasivariety* K *is a subquasivariety* M *of* K *such that* $\mathsf{M} = \mathsf{K} \cap \mathsf{V}$ *for some variety* V*. Equivalently, it is a subclass of* K *that is axiomatized, relative to* K*, by some set of equations.*

If M is a relative subvariety of a quasivariety K then for every $\boldsymbol{A} \in \mathsf{M}$, the M–congruences of $\boldsymbol{A}$ are exactly the K–congruences of $\boldsymbol{A}$. So in this case, an algebra in M is M–subdirectly irreducible iff it is K–subdirectly irreducible. This need not be true if M is merely a subquasivariety of K.

Definition 4.13. *By a (*co-lingual*)* extension *of a formal system* $\mathbf{F}$, *we mean any formal system* $\mathbf{F}'$ *over the same language such that for any set of formulas* $\Gamma \cup \{\alpha\}$*, if* $\Gamma \vdash_{\mathbf{F}} \alpha$ *then* $\Gamma \vdash_{\mathbf{F}'} \alpha$*.*

In this case, we call $\mathbf{F}'$ *an* axiomatic extension *of* $\mathbf{F}$ *if there is a set* Π *of formulas, closed under substitution, such that for every set of formulas* $\Gamma \cup \{\alpha\}$*, we have* $\Gamma \vdash_{\mathbf{F}'} \alpha$ *iff* $\Gamma \cup \Pi \vdash_{\mathbf{F}} \alpha$*.*

In practice, axiomatic extensions of **F** are normally produced by adjoining new axioms to **F** but leaving the inference rules fixed.

In general, the extensions of an algebraizable system are themselves algebraizable, with the same defining equations and equivalence formulas. The next result is a consequence of this. It follows directly from [7, Cor. 4.9, Thm. 2.17].

Theorem 4.14. *If we identify formal systems that have the same deducibility relation, then the extensions of* $\mathbf{RMO}^*$ *are in one-to-one correspondence with the subquasivarieties of* IP, *and the axiomatic ones with the relative subvarieties of* IP. *In the case of the axiomatic extensions, the mutually inverse correspondences are*

$$\mathbf{F} \mapsto \{\boldsymbol{A} \in \mathsf{IP} : \boldsymbol{A} \text{ satisfies } \alpha \approx |\alpha| \text{ for every theorem } \alpha \text{ of } \mathbf{F}\};$$
$$\mathsf{Q} \mapsto \mathbf{RMO}^* \cup \{\alpha : \mathsf{Q} \text{ satisfies } \alpha \approx |\alpha|\}.$$

The former function takes an axiomatic extension to its equivalent quasivariety.

The one-to-one correspondences in this theorem are in fact lattice anti-isomorphisms.

5. Semiconic Algebras

Definition 5.1. *A CRP is said to be* conic *if each of its elements* a *is comparable with* t, *i.e.,* $a \leq \mathsf{t}$ *or* $\mathsf{t} \leq a$.

An idempotent CRP is said to be semiconic *if it is isomorphic to a subdirect product of conic idempotent CRPs.*

Proposition 5.2.

(i) For any element a *of a conic CRP, if* $a \to \mathsf{t} < \mathsf{t}$ *then* $\mathsf{t} < a$;

(ii) Every conic CRP satisfies the quasi-equation $x \to \mathsf{t} \leq x \implies \mathsf{t} \leq x$.

Proof. (i) If $a \leq \mathsf{t}$ then $\mathsf{t} \leq a \to \mathsf{t}$, by (8). So the result follows from conicity.

(ii) Let $\boldsymbol{A}$ be a conic CRP and $a \in A$. Suppose that $a \to \mathsf{t} \leq a$. By conicity, $a < \mathsf{t}$ or $\mathsf{t} \leq a$. If $a < \mathsf{t}$ then $\mathsf{t} \leq a \to \mathsf{t}$, by (8), and thus $a < a \to \mathsf{t}$, which contradicts $a \to \mathsf{t} \leq a$. So we must have $\mathsf{t} \leq a$, as required. □

In the idempotent case, the following additional properties are known. Proofs can be found in [12, 14].

Lemma 5.3. *Let $\boldsymbol{A}$ be a conic idempotent CRP. Then, for all $a, b \in A$,*

$$\text{if } a \leq b \text{ then } a \cdot b = a \text{ or } a \cdot b = b\,; \tag{16}$$

$$\text{if } a \leq \mathsf{t} \text{ then } a \to a = a \to \mathsf{t}\,; \tag{17}$$

$$\text{if } \mathsf{t} \leq a \leq b \text{ then } a \to b = b\,; \tag{18}$$

$$\text{if } \mathsf{t} \leq a < b \text{ then } b \to a = b \to \mathsf{t}\,; \tag{19}$$

$$\text{if } b \leq \mathsf{t} \leq a \text{ then } a \to b = (a \to \mathsf{t}) \cdot b \text{ and } b \to a = (b \to \mathsf{t}) \cdot a. \tag{20}$$

Notation 5.4. *We denote the class of all semiconic idempotent CRPs by* SCIP.

It is shown in [12] that SCIP is a quasivariety, but not a variety. The next theorem is also proved in [12].

Theorem 5.5. SCIP *is locally finite, i.e., every finitely generated semiconic idempotent CRP is finite.*

In the equivalent quasivariety of an algebraizable logic, finiteness results of this kind have implications for the decidability of the system and its extensions (see Section 6). So Theorem 5.5 prompts the question: which axiomatic extensions of $\mathbf{RMO}^*$ are algebraized by semiconic algebras? In view of Theorem 4.14, this problem amounts to finding a syntactic characterization of the relative subvarieties of IP that consist of semiconic algebras. The solution is given below, and this is the main algebraic result of the present paper.

Theorem 5.6. *A relative subvariety* W *of* IP *consists of semiconic algebras iff* W *satisfies* $x \approx (x \to \mathsf{t}) \to x$.

Proof. ($\Leftarrow$) Suppose W satisfies $x \approx (x \to \mathsf{t}) \to x$, and let $\boldsymbol{A}$ be a relatively subdirectly irreducible algebra in W. In view of Theorem 4.10, it suffices to show that $\boldsymbol{A}$ is conic. Since W is a relative subvariety of

IP, $\boldsymbol{A}$ is IP–subdirectly irreducible. So, by Theorem 4.11, $A = (a] \cup [\mathsf{t})$ for some $a \in A$ with $\mathsf{t} \nleq a$. In particular, $a \to \mathsf{t}$ belongs to $(a]$ or to $[\mathsf{t})$. If $a \to \mathsf{t} \in (a]$, then $\mathsf{t} \leq (a \to \mathsf{t}) \to a = a$, by (8) and the assumption. This contradicts $\mathsf{t} \nleq a$, so we must have $a \to \mathsf{t} \in [\mathsf{t})$, i.e., $\mathsf{t} \leq a \to \mathsf{t}$. Then $a < \mathsf{t}$ and, since $A = (a] \cup [\mathsf{t})$, this shows that $\boldsymbol{A}$ is conic.

($\Rightarrow$) Conversely, let W consist of semiconic algebras, and suppose that W does not satisfy $x \approx (x \to \mathsf{t}) \to x$. Since subdirect products preserve equations, Theorem 4.10 shows that there is a relatively subdirectly irreducible algebra $\boldsymbol{B}$ in W and an element $b \in B$ such that $b \neq (b \to \mathsf{t}) \to b$. Then, by (14), we must have $b < (b \to \mathsf{t}) \to b$.

Since $\boldsymbol{B} \in \mathsf{W}$ and W is a relative subvariety of IP, $\boldsymbol{B}$ is IP–subdirectly irreducible. But, by assumption, $\boldsymbol{B}$ is a subdirect product of conic algebras from IP, so one of these algebras is isomorphic to $\boldsymbol{B}$. Thus, $\boldsymbol{B}$ is conic.

Now if $\mathsf{t} \leq b \to \mathsf{t}$, then by (3) and (9), $(b \to \mathsf{t}) \to b \leq \mathsf{t} \to b = b$, contradicting $b < (b \to \mathsf{t}) \to b$. So $b \to \mathsf{t} < \mathsf{t}$, by conicity of $\boldsymbol{B}$. It then follows from Proposition 5.2(i) that $\mathsf{t} < b$. So $b \to \mathsf{t} < \mathsf{t} < b < (b \to \mathsf{t}) \to b$. Let

$$B' \;=\; \{b \to \mathsf{t},\ \mathsf{t},\ b,\ (b \to \mathsf{t}) \to b\}.$$

We shall show that B' is a subuniverse of $\boldsymbol{B}$. Since B' is linearly ordered, it follows from (16) that B' is closed under $\cdot$. Using (20), (5) and (15), we obtain $(b \to \mathsf{t}) \to b = ((b \to \mathsf{t}) \to \mathsf{t}) \cdot b = (b \to \mathsf{t}) \to \mathsf{t}$. So B' is closed under the term function of $x \to \mathsf{t}$, by (6). Using (17)–(20), we see that for any elements $c, d \in B'$,

$$c \to d \;=\; \begin{cases} d & \text{if } \ \mathsf{t} \leq c \leq d\,; \\ c \to \mathsf{t} & \text{if } \ c = d \leq \mathsf{t} \ \text{ or } \ \mathsf{t} \leq d < c\,; \\ (c \to \mathsf{t}) \cdot d & \text{if } \ c \leq \mathsf{t} \leq d \ \text{ or } \ d \leq \mathsf{t} \leq c. \end{cases}$$

Therefore, B' is closed under $\to$ (since it is closed under $\cdot$ and under the term function of $x \to \mathsf{t}$). This confirms that B' is the universe of a subalgebra $\boldsymbol{B}'$ of $\boldsymbol{B}$. Let $\boldsymbol{A} = \boldsymbol{B}' \times \boldsymbol{B}'$. Then $\boldsymbol{A} \in \mathsf{W}$, because quasivarieties are closed under subalgebras and products. Let

$$a' \;=\; \langle b, b \to \mathsf{t}\rangle, \quad b' \;=\; \langle (b \to \mathsf{t}) \to b,\, b \to \mathsf{t}\rangle \quad \text{and} \quad \mathsf{t}' \;=\; \langle \mathsf{t}, \mathsf{t}\rangle,$$

so $a', b', \mathsf{t}' \in A$. Now

$$(a' \to \mathsf{t}') \to a' \;\;=\;\; \langle b \to \mathsf{t},\, (b \to \mathsf{t}) \to \mathsf{t}\rangle \;\to\; \langle b, b \to \mathsf{t}\rangle$$

$$= \langle (b \to \mathsf{t}) \to b,\, b \to \mathsf{t} \rangle \quad \text{(by (20) and (6))}$$
$$= b'.$$

So $(a' \to \mathsf{t}') \to a' \in \mathrm{Fg}\{b'\}$. But, $a' \notin \mathrm{Fg}\{b'\}$, by Corollary 4.4, because neither $[\mathsf{t}')$ nor $[b')$ contains a'. This, together with Corollary 4.8(iii), shows that the factor algebra $\boldsymbol{A}/\Omega(\mathrm{Fg}\{b'\})$ does not satisfy the quasi-equation

$$x \to \mathsf{t} \leq x \implies \mathsf{t} \leq x \tag{21}$$

(as $a'/\Omega(\mathrm{Fg}\{b'\})$ violates this law). Since W is a relative subvariety of IP and $\boldsymbol{A} \in \mathsf{W}$, any IP–congruence of $\boldsymbol{A}$ is a W-congruence. So $\boldsymbol{A}/\Omega(\mathrm{Fg}\{b'\}) \in \mathsf{W}$, by Corollary 4.8(i). Thus, W does not satisfy (21).

On the other hand, because $\mathsf{W} \subseteq \mathsf{SCIP}$, every quasi-equation that holds in all conic idempotent CRPs must hold in W, and one of these is (21), by Proposition 5.2(ii). This contradiction completes the proof. □

The next example shows that SCIP itself does not satisfy the equation in Theorem 5.6.

Example 5.7. *The chain $-2 < 0 < 1 < 2$ is the order reduct of an idempotent CRP $\boldsymbol{A}$ with identity 0, in which*

$$a \cdot b \;=\; \begin{cases} \textit{the element of } \{a,b\} \textit{ with the larger absolute value, if } |a| \neq |b|\,; \\ \textit{min}_{\leq}\{a,b\}, \quad \textit{otherwise.} \end{cases}$$

To see quickly that $\cdot$ is associative, note that it is also the minimum operation of a different chain on A, viz. $-2 \prec 2 \prec 1 \prec 0$. (We shall make no further use of $\preceq$.) Now $\leq$ is compatible with $\cdot$, and for all $a, b \in A = \{-2, 0, 1, 2\}$, the set $\{c \in A : a \cdot c \leq b\}$ is non-empty, as $a \cdot -2 = -2$. So this set has a $\leq$–greatest element, which becomes $a \to b$.

Clearly, $\boldsymbol{A} \in \mathsf{SCIP}$. But in $\boldsymbol{A}$, we have $(1 \to 0) \to 1 = (-2) \to 1 = 2 > 1$. This shows that SCIP does not satisfy $x \approx (x \to \mathsf{t}) \to x$.

Corollary 5.8. SCIP *is not a relative subvariety of* IP.

Proof. This follows from Theorem 5.6 and Example 5.7. □

In other words, although SCIP is axiomatizable by quasi-equations, it cannot be axiomatized relative to IP by any set of equations. In fact, because of Corollary 5.8, the following problem is open:

Problem 1. *Axiomatize* SCIP *transparently. Is* SCIP *finitely axiomatizable?*

The analogous problem for the algebras in IP that are subdirect products of *chains* does not seem to be any easier.

6. Logical Consequences

Definition 6.1. *If a formal system* $\mathbf{F}$ *is algebraizable with equivalence formulas* $\Delta_j(x, y)$, $j \in J$, *then two formulas* φ *and* ψ *of* $\mathbf{F}$ *are said to be* logically equivalent *provided that* $\vdash_{\mathbf{F}} \Delta_j(\varphi, \psi)$ *for all* $j \in J$.

In this case, $\mathbf{F}$ *is said to be* locally tabular *if for each integer* $n \geq 0$, *there are only finitely many logically inequivalent formulas in* n *fixed variables.*

So in $\mathbf{RMO}^*$, logical equivalence of φ and ψ has the expected meaning: $\vdash_{\mathbf{RMO}^*} \varphi \to \psi$ and $\vdash_{\mathbf{RMO}^*} \psi \to \varphi$.

When a formal system $\mathbf{F}$ is algebraizable with equivalent quasivariety K, then $\mathbf{F}$ is locally tabular if and only if K is locally finite. (This follows easily from a consideration of free algebras in K.) In this case, it is clear that $\mathbf{F}$ has the *strong finite model property*, i.e., whenever $\Gamma \not\vdash_{\mathbf{F}} \alpha$ (Γ finite) then some *finite* algebra in K witnesses the failure of

$$\big(\&_{i \in I\,;\, \gamma \in \Gamma} \;\; \delta_i(\gamma) \approx \varepsilon_i(\gamma) \big) \implies \delta_k(\alpha) \approx \varepsilon_k(\alpha)$$

for some $k \in I$, where $\delta_i(x) \approx \varepsilon_i(x)$, $i \in I$, are the defining equations. Indeed, some algebra $\boldsymbol{A} \in \mathsf{K}$ must witness such a failure (by algebraizability), and then the witnessing elements generate a finite witnessing subalgebra of $\boldsymbol{A}$ (by local finiteness). Theorem 5.6 has the following consequence:

Corollary 6.2. *An axiomatic extension* $\mathbf{F}$ *of* $\mathbf{RMO}^*$ *is locally tabular (and therefore has the strong finite model property) if its theorems include the formula* $((p \to \mathsf{t}) \to p) \to p$.

Proof. Let K be the equivalent quasivariety of $\mathbf{F}$. For any formulas α and β, Theorem 4.14 and (11) show that $\vdash_{\mathbf{F}} \alpha \to \beta$ iff K satisfies $\alpha \to \beta \approx |\alpha \to \beta|$ iff K satisfies $\alpha \leq \beta$.

In particular, if $\vdash_{\mathbf{F}} ((p \to \mathsf{t}) \to p) \to p$, then K satisfies $(x \to \mathsf{t}) \to x \leq x$, and therefore $x \approx (x \to \mathsf{t}) \to x$, by (14). Then, since K is a relative subvariety of IP, it follows from Theorem 5.6 that K consists of semiconic algebras. So K is locally finite, by Theorem 5.5, hence the result. □

Using a variant of Harrop's theorem [10] (cf. [6, Lemma 3.13]), we infer:

Corollary 6.3. *If an axiomatic extension* $\mathbf{F}$ *of* $\mathbf{RMO}^*$ *is finitely axiomatized and if* $\vdash_{\mathbf{F}} ((p \to \mathsf{t}) \to p) \to p$, *then* $\mathbf{F}$ *has a solvable deducibility problem, i.e., its set of finite derivable rules is recursive. In particular,* $\mathbf{F}$ *is decidable.*

Recall that the formal system **RM** ('**R**–*mingle*') is the extension of **R** by (M), and that $\mathbf{RM}^{\mathsf{t}}$ is the extension of **RM** by the constant t and the axioms t and $\mathsf{t} \to (p \to p)$. These systems are discussed for instance in [1]. Corollaries 6.2 and 6.3 both apply to the $\wedge, \to$ fragment of intuitionistic logic and to the $\cdot, \to, \mathsf{t}$ fragment of $\mathbf{RM}^{\mathsf{t}}$. For these two incomparable systems, the conclusions of the corollaries are of course well known, but their common explanation, via the shared theorem $((p \to \mathsf{t}) \to p) \to p$, is new.

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References

[1] A.R. Anderson and N.D. Belnap, Jnr., Entailment: The Logic of Relevance and Necessity, Volume 1, Princeton University Press, 1975.

[2] A. Avron, *Relevant entailment—semantics and the formal systems*, J. Symbolic Logic **49** (1984), pp. 334–432.

[3] A. Avron, *Relevance and paraconsistency—a new approach*, J. Symbolic Logic **55** (1990), pp. 707–732.

[4] A. Avron, *Relevance and paraconsistency—a new approach. Part II: The formal systems*; and *Part III: Cut-free Gentzen-type systems*, Notre Dame J. Formal Logic **31** (1990), pp. 169–202; and **32** (1991), pp. 147–160.

[5] A. Avron, *Whither relevance logic?*, J. Philosophical Logic **21** (1992), pp. 243–281.

[6] W.J. Blok and I.M.A. Ferreirim, *On the structure of hoops*, Algebra Universalis **43** (2000), pp. 233–257.

[7] W.J. Blok and D. Pigozzi, Algebraizable Logics, Memoirs of the American Mathematical Society, Number 396, Amer. Math. Soc., Providence, 1989.

[8] A. Church, *The weak theory of implication*, in A. Menne, A. Wilhelmy, H. Angsil (eds.), Kontrolliertes Denken, Untersuchungen zum Logikkalkül und zur Logik der Einzelwissenschaften, Kommissions-Verlag Karl Alber, 1951, pp. 22–37.

[9] A. Church, *The weak positive implicational propositional calculus* (abstract), J. Symbolic Logic **16** (1951), 238.

[10] R. Harrop, *On the existence of finite models and decision procedures for propositional calculi*, Proceedings of the Cambridge Philosophical Society **54** (1958), pp. 1–13.

[11] A. Hsieh and J.G. Raftery, *A finite model property for RMI_{min}*, Math. Logic Quarterly **52** (6) (2006), pp. 602–612.

[12] A. Hsieh and J.G. Raftery, *Semiconic idempotent residuated structures*, Algebra Universalis **61** (2009), pp. 413–430.

[13] J.M. Méndez, *The compatibility of relevance and mingle*, J. Philosophical Logic **17** (1988), pp. 279–297.

[14] J.S. Olson and J.G. Raftery, *Residuated structures, concentric sums and finiteness conditions*, Communications in Algebra **36** (10) (2008), pp. 3632–3670.

[15] D. Pigozzi, *Finite basis theorems for relatively congruence-distributive quasivarieties*, Trans. Amer. Math. Soc. **310** (1988), pp. 499–533.

[16] S. Tamura, *The implicational fragment of R-mingle*, Proceedings of the Japan Academy **47** (1971), pp. 71–75.

[17] C.J. van Alten and J.G. Raftery, *Rule separation and embedding theorems for logics without weakening*, Studia Logica **76** (2004), pp. 241–274.

School of Mathematical Sciences, University of KwaZulu-Natal,
Westville Campus, Private Bag X54001, Durban 4000, South Africa
akongrung@gmail.com