REPORTS ON MATHEMATICAL LOGIC
46 (2011), 3–15

Martín FIGALLO

SOME RESULTS ON DIAGONAL-FREE TWO-DIMENSIONAL CYLINDRIC ALGEBRAS

A b s t r a c t. Formulas for computing the number of $\mathbf{Df}_2$-algebra structures that can be defined over $\mathbb{B}_n$, where $\mathbb{B}_n$ is the Boolean algebra with n atoms, as well as the fine spectrum of $\mathbf{Df}_2$ are obtained. Properties of the lattice of all subvarieties of $\mathbf{Df}_2$, $\Lambda(\mathbf{Df}_2)$, are exhibited. In particular, the poset $\mathrm{Si}_{fin}(\mathbf{Df}_2)$ is described.

1. Introduction and Preliminaries

Cylindric algebras were first introduced by A. Tarski in the 1940's. As a general reference we mention the fundamental work by Henkin, Monk and

Received 7 October 2008

Dedicated to the memory of my mother Aedda H. Orellano

Key words. Diagonal–free two–dimensional cylindric algebra; finite algebras; lattice of subvarieties.

2000 Mathematics Subject Classification: 03G15, 08A99, 08B15

Tarski [5]. Following the notation introduced in [5], we shall denote by $\mathbf{Df}_2$ the variety diagonal–free two–dimensional cylindric algebras, that is to say, the variety of the Boolean algebras with two quantifiers which commute. This variety has been widely studied, but little research has pursued to investigate those problems inherent to finite algebras.

In [5, Part II Lemma 5.1.24 (p.188) and Theorem 5.1.7 (ii) (p.185)] it is proved that $\mathbf{Df}_2$ is generated by its finite members and $\mathbf{Df}_2$ is not locally finite. In [1], Bezhanishvili studied in depth the lattice $\Lambda(\mathrm{Df}_2)$ of all subvarieties of $\mathbf{Df}_2$ and, among other things, he proved that despite the fact that $\mathbf{Df}_2$ is not locally finite, every proper subvariety is.

One well–known fact about $\mathbf{Df}_2$ is that it is a discriminator variety. In consequence,

(I) the concepts of an algebra being simple, subdirectly irreducible or directly indecomposable are equivalent in $\mathbf{Df}_2$,

(II) the finite algebras all have unique direct factorization into simple algebras.

On the other hand, the fine spectrum $f_{\mathcal{V}}$ of a variety $\mathcal{V}$ is the function where $f_{\mathcal{V}}(n)$ is the number of isomorphism types of algebras of power n in $\mathcal{V}$ (see [8]). It is stated in [5, Part II p. 186] that $f_{\mathcal{V}}(1) = 1$ for $\mathcal{V} = \mathbf{Df}_2$

In what follows, we shall recall some known results about $\mathbf{Df}_2$ which are useful for the understanding of the present work (see [1], [3] and, [5]).

We shall denote by $\mathbb{B}_n$ the Boolean algebra with n atoms and with $\mathbb{A}_n$ the set of its atoms. Recall that $(B, \exists_1, \exists_2)$ is a $\mathbf{Df}_2$-algebra if and only if for all $x \in B$ it is verified

$$\begin{aligned} \exists_i 0 &= 0, \\ x &\leq \exists_i x, \\ \exists_i(x \wedge \exists_i y) &= \exists_i x \wedge \exists_i y, \text{ for } i = 1,2; \text{ and} \\ \exists_1\exists_2 x &= \exists_2\exists_1 x. \end{aligned}$$

Every quantifier $\exists$ defined on the Boolean algebra $\mathbb{B}_n$ induces a partition $\mathcal{P}_\exists$ of the set $\mathbb{A}_n$ of its atoms. It will be called partition associated to $\exists$ and it can be obtained in the following way: $C \in \mathcal{P}_\exists$ if and only if $\exists x = \exists y$ for $x, y \in C$. The following results will be used

Proposition 1.1. *(See [1], [3] or [5]) Let* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *be two partitions of* $\mathbb{A}_k$ *and* $\exists_1$, $\exists_2$ *their associated quantifiers. Then, the following conditions are equivalent:*

(i) $(\mathbb{B}_k, \exists_1, \exists_2)$ *is a simple* $\mathbf{Df}_2$*-algebra,*

(ii) $C \cap D \neq \emptyset$ *for every* $C \in \mathcal{P}_1$ *and* $D \in \mathcal{P}_2$.

This corresponds to the fact that a $\mathbf{Df}_2$-algebra is simple of and only if $\exists_1\exists_2 x = 1$ for all nonzero x.

Proposition 1.2. *(See [3]) If a finite* $\mathbf{Df}_2$*-algebra* $(A, \exists_1, \exists_2)$ *is simple and* $K_i = \exists_i \mathbb{B}_k$ *for* $i = 1, 2$*, then* $|\Pi(K_1)| \cdot |\Pi(K_2)| \leq |\Pi(A)|$*. Where* $\exists_i \mathbb{B}_k = \{\exists_i x : x \in \mathbb{B}_k\}$ *and* $\Pi(A)$ *is the set of atoms of the Boolean algebra* A.

This paper is organised in two main sections. Section 2 is devoted to some kind of problems related to finite algebras similar to the ones studied in [3] and [7]. More precisely, we exhibit a formula to calculate the fine spectrum of $\mathbf{Df}_2$. In Section 3, we describe de poset $\mathrm{Si}_{fin}(\mathbf{Df}_2)$ which generates the lattice $\Lambda(\mathbf{Df}_2)$.

2. $\mathbf{Df}_2$-algebra structures over a finite Boolean algebra; fine spectrum

Given the Boolean algebra $\mathbb{B}_n$, we are going to determine the number of $\mathbf{Df}_2$-algebra structures that can be defined over $\mathbb{B}_n$. We shall assume for simplicity that if $\mathcal{P} = \{C_1, \ldots, C_m\}$ is a partition of $\mathbb{A}_k$ then $|C_1| \leq \cdots \leq |C_m|$. We shall denote by $\mathbf{s}(k)$ the number of all simple $\mathbf{Df}_2$-algebras with k atoms. Then, from equation (3.3) of [3] we have that

$$\mathbf{s}(k) = \sum_{m=1}^{k} \sum_{\substack{\mathcal{P}_1 \in Part(\mathbb{A}_k, m) \\ \mathcal{P}_1 = \{C_1, \ldots, C_m\}}} \Gamma(|C_1|, \ldots, |C_m|). \tag{1}$$

where

$$\Gamma(s_1,\ldots,s_m) = \sum_{\substack{n\,\in\,\mathbb{N},\; n\cdot m\,\leq\,k\\ n\,\leq\, s_1}} \frac{\displaystyle\prod_{i=1}^{m}\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}(n-j)^{s_i}}{n!}. \qquad (2)$$

Now, we shall denote by $\mathcal{DF}_2(\mathbb{B}_n)$ the set of all pairs of quantifiers $(\exists_1, \exists_2)$ defined on $\mathbb{B}_n$ which verifies that $\exists_1\exists_2 = \exists_2\exists_1$.

Suppose that $(\exists_1, \exists_2) \in \mathcal{DF}_2(\mathbb{B}_n)$ and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the partitions of $\mathbb{A}_n$ associated to $\exists_1$ and $\exists_2$ respectively. We know that these partitions determine, in turn, two new partitions which are $\{U_C\}_{C\in\mathcal{P}_1}$ and $\{m_2(C)\}_{C\in\mathcal{P}_1}$ respectively (see Lemma 3.18 of [3]).

Let's make some remarks.

Remark 2.1. The sets

(i) $\{W : W = \bigcup\limits_{F\in U_C} F,$ for some $C \in \mathcal{P}_1\}$ and
$\{W' : W' = \bigcup\limits_{G\in m_2(C)} G,$ for some $C \in \mathcal{P}_1\}$ are partitions of $\mathbb{A}_n$.

(ii) The above two partitions are equal. Furthermore, that is the partition of $\mathbb{A}_n$ associated to the quantifier $\exists = \exists_1\exists_2 = \exists_2\exists_1$.

From Remarks 2.1 we can assert that each pair $(\exists_1, \exists_2) \in \mathcal{DF}_2(\mathbb{B}_n)$ has associated one, and only one, partition of $\mathbb{A}_n$ which we shall denote by$P(\exists_1, \exists_2)$. Conversely, it is easy to verify that every partition of $\mathbb{A}_n$ has associated at least one pair $(\exists_1, \exists_2) \in \mathcal{DF}_2(\mathbb{B}_n)$ and generally, more than one.

We shall define an equivalence relation $\equiv$ on the set $\mathcal{DF}_2(\mathbb{B}_n)$ as follows:

$$(\exists_1, \exists_2) \equiv (\exists_1', \exists_2') \quad \text{if and only if } P(\exists_1, \exists_2) = P(\exists_1', \exists_2'),$$

and let's consider the quotient set of $\mathcal{DF}_2$ by $\equiv$ written $\mathcal{DF}_2(\mathbb{B}_n)/\equiv$. Then

Lemma 2.2. *The sets $\mathcal{DF}_2(\mathbb{B}_n)/\equiv$ and $Part(\mathbb{A}_n)$ have the same cardinality.*

Proof. It is a direct consequence of the above discussion. □

Now let's compute the cardinal of each equivalence class in $\mathcal{DF}_2(\mathbb{B}_n)/\equiv$. Let $(\exists_1', \exists_2') \in \mathcal{DF}_2(\mathbb{B}_n)$ and suppose that $P(\exists_1', \exists_2') = \{U_1, U_2, \ldots, U_r\}$ with $|U_i| = n_i$, for every $1 \leq i \leq r$. We shall denote by $\mathcal{S}(n)$ the set of all finite simple $\mathbf{Df}_2$-algebra with n atoms. Then the following assertion holds.

Lemma 2.3. *The sets* $(\exists'_1, \exists'_2)_\equiv$ *and* $\prod_{i=1}^{r} \mathcal{S}(n_i)$ *have the same cardinality.*

Proof. Let $(\exists_1, \exists_2) \in (\exists'_1, \exists'_2)_\equiv$. Then, the function $\phi : (\exists'_1, \exists'_2)_\equiv \to \prod_{i=1}^{r} \mathcal{S}(n_i)$ which matches every pair $(\exists_1, \exists_2)$ in $(\exists'_1, \exists'_2)_\equiv$ with the $\mathbf{Df}_2$-algebra $(\mathbb{B}_n, \exists_1, \exists_2)$ is well defined. Furthermore, it is easy to verify that ϕ is one–to–one and onto. □

Now, we can specify a formula which allows us to compute the cardinal we are looking for. Taking into account Lemmas 2.2 and 2.3 we have that:

$$\begin{aligned} |\mathcal{DF}_2(\mathbb{B}_n)| &= \sum_{(\exists_1,\exists_2)_\equiv \in \mathcal{DF}_2(\mathbb{B}_n)/\equiv} |(\exists_1, \exists_2)_\equiv| \\ &= \sum_{\mathcal{P} \in Part(\mathbb{A}_n)} |\prod_{C \in \mathcal{P}} \mathcal{S}(|C|)| \\ &= \sum_{\mathcal{P} \in Part(\mathbb{A}_n)} \prod_{C \in \mathcal{P}} |\mathcal{S}(|C|)| \end{aligned}$$

And from equation (1), $|\mathcal{S}(|C|)| = s(|C|)$ holds. Then,

$$|\mathcal{DF}_2(\mathbb{B}_n)| = \sum_{\mathcal{P} \in Part(\mathbb{A}_n)} \prod_{C \in \mathcal{P}} s(|C|). \qquad (3)$$

Example 2.4. (i) $|\mathcal{DF}_2(\mathbb{B}_1)| = 1$; $|\mathcal{DF}_2(\mathbb{B}_2)| = 4$

(ii) Let $\mathbb{A}_3 = \{a_1, a_2, a_3\}$ and $\mathcal{P}_0 = \mathbb{A}_3$, $\mathcal{P}_1 = \{\{a_1\}, \{a_2, a_3\}\}$, $\mathcal{P}_2 = \{\{a_2\}, \{a_1, a_3\}\}$, $\mathcal{P}_3 = \{\{a_3\}, \{a_1, a_2\}\}$ and $\mathcal{P}_4 = \{\{a_1\}, \{a_2\}, \{a_3\}\}$. Then, $|\mathcal{DF}_2(\mathbb{B}_3)| = 19$.

Therefore, there are only nineteen $\mathbf{Df}_2$-algebra's structures which can be defined over the Boolean algebra $\mathbb{B}_3$ and it is possible to verify that these algebras are: $(\mathbb{B}_3, \exists_{\mathcal{P}_0}, \exists_{\mathcal{P}_i})$, for $0 \le i \le 4$; $(\mathbb{B}_3, \exists_{\mathcal{P}_j}, \exists_{\mathcal{P}_0})$, for $1 \le j \le 4$; $(\mathbb{B}_3, \exists_{\mathcal{P}_i}, \exists_{\mathcal{P}_i})$, $(\mathbb{B}_3, \exists_{\mathcal{P}_i}, \exists_{\mathcal{P}_4})$ and $(\mathbb{B}_3, \exists_{\mathcal{P}_4}, \exists_{\mathcal{P}_i})$, for $1 \le i \le 3$ and $(\mathbb{B}_3, \exists_{\mathcal{P}_4}, \exists_{\mathcal{P}_4})$.

Now, we shall exhibit a formula for computing the fine spectrum of $\mathbf{Df}_2(n)$ for $n \in \mathbb{N}$. Let $A = (\mathbb{B}_k, \exists_1, \exists_2)$ be a finite $\mathbf{Df}_2$-algebra and let $\mathcal{P}_1 = \{C_1, \ldots, C_n\}$ and $\mathcal{P}_2 = \{D_1, \ldots, D_m\}$ be the partitions associated to $\exists_1$ and $\exists_2$ respectively. With S_k we shall denote the symmetric group of order k. Besides, we shall denote with $F(A)$ the family of all matrixs associated to A in the following way.

Definition 2.5. $F(A) = \{(a_{ij})_{n\times m}:$ there are permutations $\sigma \in S_n$ and $\tau \in S_m$ such that $a_{ij} = |C_{\sigma(i)} \cap D_{\tau(j)}|$ for $1 \leq i \leq n$, $1 \leq i \leq m\}$

Besides, we shall denote by $F((a_{ij})_{n\times m})$ the set

$$F((a_{ij})_{n\times m}) = \{(a_{\sigma(i)\tau(j)})_{n\times m} : \text{ for all } \sigma \in S_n \text{ and all } \tau \in S_m\}$$

that is to say that $F((a_{ij})_{n\times m})$ is the set of all matrixs that are obtained from $(a_{ij})_{n\times m}$ by interchanging rows and columns of it.

Remark 2.6. From Propositions 1.1 and 1.2 we know that, if A is simple and $(a_{ij})_{n\times m} \in F(A)$ then, $a_{ij} > 0$ for all i, j and $n{\cdot}m \leq \sum\limits_{i=1}^{n}\sum\limits_{j}^{m} a_{ij} = k$.

Proposition 2.7. *There exist $\sigma_0 \in S_n$, $\tau_0 \in S_m$ and an unique matrix $(\delta_{ij})_{n\times m} \in F(A)$ such that:*

(i) $\delta_{ij} = |C_{\sigma_0(i)} \cap D_{\tau_0(j)}|$,

(ii) *If $j < l$ then $\sum_{i=1}^{n} \delta_{ij} \leq \sum_{i=1}^{n} \delta_{ij}$,*

(iii) *If $\sum_{i=1}^{n} \delta_{ij} = \sum_{i=1}^{n} \delta_{ij}$, then there is $r \in \{1, \ldots, n\}$ such that $\delta_{rj} < \delta_{rl}$ and for allo s, $1 \leq s < r$, $\delta_{sj} = \delta_{sl}$.*

Proof. It is a routine. □

Lemma 2.8. *If $A, B \in \mathbf{Df}_2$ are finite algebras then the following conditions are equivalent:*

(i) *A and B are isomorphic algebras,*

(ii) *$F(A) = F(B)$.*

Proof. (i) $\Rightarrow$ (ii): Since $A \simeq B$ we can assume that $A = (\mathbb{B}_k, \exists_1, \exists_2)$ and $B = (\mathbb{B}_k, \exists_1, \exists_2)$ and if P_i and $\mathcal{P}'_i$ the partitions of $\mathbb{A}_k$ associated to $\exists_i$ and $\exists'_i$, for $i = 1, 2$, respectively, then we can assume that $\mathcal{P}_1 = \{C_1, \ldots, C_n\}$, $\mathcal{P}_2 = \{D_1, \ldots, D_m\}$, $\mathcal{P}'_1 = \{C'_1, \ldots, C'_n\}$ and $\mathcal{P}_2 = \{D'_1, \ldots, D'_m\}$. Let $h : A \to B$ an isomorphism, then for all C'_i and D'_j there are $C_{i'}$

and $D_{j'}$ such that $f_h : \mathbb{A}_k \to \mathbb{A}_k$ verifies that $f_h(C'_i) = C_{i'}$, $f_h(D'_i) = D_{i'}$ and $f_h(C'_i \cap D'_j) = C_{i'} \cap D_{j'}$. Therefore, $|C'_i \cap D'_j| = |C_{i'} \cap D_{j'}|$.

(ii) $\Rightarrow$ (i): If A and B are two finite $\mathbf{Df}_2$-algebras such that $F(A) = F(B)$ then we can assume that $\mathcal{P}_1 = \{C_1, \ldots, C_n\}$, $\mathcal{P}_2 = \{D_1, \ldots, D_m\}$, $\mathcal{P}'_1 = \{C'_1, \ldots, C'_n\}$ and $\mathcal{P}_2 = \{D'_1, \ldots, D'_m\}$ are the partitions of $\mathbb{A}_k$ associated to the quantifiers. Then we, can find $\sigma \in S_n$ and $\tau \in S_m$ such that $|C_i \cap D_j| = |C_{\sigma(i)} \cap D_{\tau(j)}|$. Therefore, we can define an one-to-one function f from $\mathbb{A}_k$ onto $\mathbb{A}_k$ such that f can be extended to an isomorphism from A to B. □

Let

$$\begin{aligned}\mathcal{M}_k &= \{(a_{ij})_{n\times m} : n, m \in \mathbb{N}; n \cdot m \leq k; a_{ij} > 0 \text{ for all } i, j;\\ &\qquad \text{and } \sum_{i=1}^{n}\sum_{j}^{m} a_{ij} = k\}\end{aligned}$$

and consider the equivalence relation $\sim$ defined over $\mathcal{M}_k$ as follows,

$$(a_{ij})_{n\times m} \sim (b_{ij})_{r\times s} \text{ if and only if } F((a_{ij})_{n\times m}) = F((b_{ij})_{r\times s}) \tag{4}$$

That is to say that $(a_{ij})_{n\times m} \sim (b_{ij})_{r\times s}$ if and only if $(b_{ij})_{r\times s}$ is obtained from $(a_{ij})_{n\times m}$ by interchaging a certain number of rows and columns.

Now, let us consider the quotient $\mathcal{M}_k/\sim$. It is clear, from what we have seen so far, that the number of finite simple $\mathbf{Df}_2$-algebras with k atoms up to isomorphism is precisely $|\mathcal{M}_k/\sim|$. We shall denote such number with $s^\star(k)$.

Then, taking into account (II) we have that the fine spectrum of $\mathbf{Df}_2$ for k finite is given by

$$f_{\mathbf{Df}_2}(k) = \sum_{\substack{k_1 \leq \cdots \leq k_m \\ k_1 + \cdots + k_m = k}} \prod_{j=1}^{m} s^\star(k_j). \tag{5}$$

Example 2.9. $s^\star(1) = 1$, $s^\star(2) = 3$, $s^\star(3) = 5$, $s^\star(4) = 10$

(i) $f_{\mathbf{Df}_2}(1) = 1$; $f_{\mathbf{Df}_2}(2) = s^\star(1) \cdot s^\star(1) + s^\star(2) = 4$

(ii) $f_{\mathbf{Df}_2}(3) = s^\star(1) \cdot s^\star(1) \cdot s^\star(1) + s^\star(1) \cdot s^\star(2) + s^\star(3) = 8$.

Note that the algebras $(\mathbb{B}_3, \exists_{\mathcal{P}_0}, \exists_{\mathcal{P}_i})$, $(\mathbb{B}_3, \exists_{\mathcal{P}_i}, \exists_{\mathcal{P}_i})$, $(\mathbb{B}_3, \exists_{\mathcal{P}_i}, \exists_{\mathcal{P}_4})$, $(\mathbb{B}_3, \exists_{\mathcal{P}_4}, \exists_{\mathcal{P}_i})$ for $1 \leq i \leq 3$ are isomorphic respectively.

3. The lattice $\Lambda(\mathbf{Df_2})$

In this section, we give a description of the poset $\mathrm{Si}_{fin}(\mathrm{Df}_2)$. The tools that we shall use here are both the characterization of the finite subdirectly irreducible Df_2-algebras stated in [3] and the well-known results due to B. Jónsson ([6]) and B. Davey ([2]).

Given a class K of algebras, let $\mathrm{Si}(K)$ and $\mathrm{Si}_{fin}(K)$ consist of precisely one algebra from each of the isomorphism classes of the subdirectly irreducible algebras and finite subdirectly irreducible algebras respectively. Besides, we shall denote by $\mathcal{O}(P)$ the lattice of down-sets (order-ideals) of the poset P.

Let $A = (\mathbb{B}_k, \exists_1, \exists_2)$ be a (finite) $\mathbf{Df}_2$-algebra and let $\mathcal{P}_1 = \{C_1, \dots, C_n\}$ and $\mathcal{P}_2 = \{D_1, \dots, D_m\}$ be the partitions associated to $\exists_1$ and $\exists_2$ respectively.

Lemma 3.1. *The following conditions are equivalent:*

(i) $(\mathbb{B}_q, \exists)$ *is isomorphic to some subalgebra of* $(\mathbb{B}_k, \exists')$,

(ii) *there exists a partition* $\{I_1, \dots, I_m\}$ *of* $\{1, \dots, s\}$ *such that* $\alpha_l \leq \delta_t$ *whenever* $1 \leq l \leq m$ *and* $t \in I_l$ *where* $m, s, \alpha_l's, \delta_t's \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii): Let $(\mathbb{B}_q, \exists)$, $(\mathbb{B}_k, \exists')$ be two finite monadic algebras and $P = \{C_1, \dots, C_m\}$ with $|C_i| = \alpha_i$ for $1 \leq i \leq m$ and $\mathcal{P}' = \{D_1, \dots, D_s\}$ with $|D_j| = \gamma_j$ for $1 \leq j \leq s$ be the associated partitions of $\mathbb{A}_q$ and $\mathbb{A}_k$ to $\exists$ and $\exists'$, respectively.

By standard duality facts $(\mathbb{B}_q, \exists)$ embeds into $(\mathbb{B}_k, \exists')$ as a subalgebra if and only if there exists a surjective map $\pi : \mathbb{A}_k \to \mathbb{A}_q$ such that for each $j = 1 \dots s$ there is some i, $1 \leq i \leq m$ with $\pi(D_j) = C_i$. Given π, let $I_i = \{j : \pi(D_j) = C_i\}$. Then clearly, $\{I_i : 1 \leq i \leq m\}$ is a partition of $\{1, \dots, s\}$ and $\delta_j = |D_j| \geq |C_i| = \alpha_i$ whenever $j \in I_i$. Therefore, (ii) holds.

(ii) $\Rightarrow$ (i): Take any $j \in \{1, \dots s\}$ and suppose that $j \in I_i$ for some (unique) i with $1 \leq i \leq m$. Since $\delta_j \geq \alpha_i$, we may choose a surjection $\pi_j : D_j \to C_i$. Now, $\pi = \bigcup_{j=1}^{s} \pi_j$ is a map with the required properties. $\Box$

In what follows we denote with $F_{(a_{ij})_{n\times m}}$ the finite simple $\mathbf{Df}_2$-algebra A such that $(a_{ij})_{n\times m} \in F(A)$.

Lemma 3.2. *The following conditions are equivalent:*

(i) $F_{(a_{ij})_{n\times m}} \in \mathbf{S}(F_{(b_{ij})_{r\times s}})$,

(ii) *there exists a partition* $\{I_{ij} : 1 \le i \le n, 1 \le j \le m\}$ *of* $\{1,\dots,r\} \times \{1,\dots,s\}$ *such that for all* i *(*$1 \le i \le m$*) and all* j *(*$1 \le j \le n$*),* $a_{ij} \le b_{uv}$ *whenever* $(u,v) \in I_{ij}$*.*

Proof. Let $A = (\mathbb{B}_q, \exists_1, \exists_2)$, $B = (\mathbb{B}_k, \exists_1', \exists_2')$ be two finite $\mathbf{Df}_2$-algebras, $\mathcal{P}_1 = \{C_1,\dots,C_n\}$ and $\mathcal{P}_2 = \{D_1,\dots,D_m\}$ the partitions of $\mathbb{A}_q$ associated to $\exists_1$ and $\exists_2$ respectively, $\mathcal{P}_1' = \{C_1',\dots,C_r'\}$ and $\mathcal{P}_2 = \{D_1',\dots,D_s'\}$ the partitions of $\mathbb{A}_k$ associated to $\exists_1'$ and $\exists_2'$ respectively. Let $H_{ij} = C_i \cap D_j$ for $1 \le i \le n$ and $1 \le j \le m$ and let $H_{ij}' = C_i' \cap D_j'$ for $1 \le i \le r$ and $1 \le j \le s$. Suppose that $N(A) = (a_{ij})_{n\times m}$ and $N(B) = (b_{ij})_{r\times s}$

(i) $\Rightarrow$ (ii): Let $\pi : \mathbb{A}_k \to \mathbb{A}_q$ be a surjection and let $I_{ij} = \{(u,v) : \pi(H_{uv}') = H_{ij}\}$. It is clear that $\{I_{ij} : 1 \le i \le n, 1 \le j \le m\}$ is a partition of $\{1,\dots,r\} \times \{1,\dots,s\}$.

(ii) $\Rightarrow$ (i): Let $(u,v) \in \{1,\dots,r\} \times \{1,\dots,s\}$ such that $(u,v) \in I_{ij}$. Since $a_{ij} \le b_{uv}$ we can choose an onto map $\pi_{ij} : H_{uv}' \to H_{ij}$. Then, $\pi = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \pi_{ij}$ is a function from $\mathbb{A}_k$ onto $\mathbb{A}_q$. □

Remark 3.3. $\{\bigcup_{j=1}^{m} I_{ij} : 1 \le i \le n\}$ is a partition of $\{1,\dots,r\}$, $\{\bigcup_{i=1}^{n} I_{ij} : 1 \le j \le m\}$ is a partition of $\{1,\dots,s\}$.

The following Lemma will be useful in what follows.

Lemma 3.4. *Let* $A = (\mathbb{B}_q, \exists_1, \exists_2)$, $B = (\mathbb{B}_k, \exists_1', \exists_2')$ *be two finite* $\mathbf{Df}_2$*-algebras,* $\mathcal{P}_1 = \{C_1,\dots,C_n\}$ *and* $\mathcal{P}_2 = \{D_1,\dots,D_m\}$ *the partitions of* $\mathbb{A}_q$ *associated to* $\exists_1$ *and* $\exists_2$ *respectively,* $\mathcal{P}_1' = \{C_1',\dots,C_r'\}$ *and* $\mathcal{P}_2' = \{D_1',\dots,D_s'\}$ *the partitions of* $\mathbb{A}_k$ *associated to* $\exists_1'$ *and* $\exists_2'$ *respectively. Then, if* $h : \mathbb{B}_k \to \mathbb{B}_q$ *is an arbitrary fucntion, the following conditions are equivalent:*

(i) h *is a* $\mathbf{Df}_2$*-epimorphism,*

(ii) $f_h : \mathbb{A}_q \to \mathbb{A}_k$ *defined by* $f_h(a) = b$ *iff* $h(b) = a$*,* $a \in \mathbb{A}_q$ *and* $b \in \mathbb{A}_k$ *verifies that:*

   (a) f_h *is one-to-one,*

   (b) for all $C \in \mathcal{P}_1$ *and all* $D \in \mathcal{P}_2$ *there are* $C' \in \mathcal{P}_1'$ *and* $D' \in \mathcal{P}_2'$ *such that (*b_1*)* $f_h(C) = C'$*, (*b_2*)* $f_h(D) = D'$*.*

Proof. Since h is a $\mathbf{Df}_2$-epimorphism, we have that, in particular, h is a monadic epimorphism from $(\mathbb{B}_q, \exists_i)$ onto $(\mathbb{B}_k, \exists'_i)$ for $i = 1, 2$. So, for Lemma 3.1, condition (ii) (b) is fulfilled. On the other hand, if $f_h : \mathbb{A}_q \to \mathbb{A}_k$ verifies conditions (ii) (a) and (ii) (b) then, taking into acount Lemma 3.1, we have that h is monadic epimorphism from $(\mathbb{B}_q, \exists_i)$ onto $(\mathbb{B}_k, \exists'_i)$ for $i = 1, 2$ and therefore a $\mathbf{Df}_2$-epimorphism. □

Remark 3.5. (i) Let $A = (\mathbb{B}_q, \exists_1, \exists_2)$, $B = (\mathbb{B}_k, \exists'_1, \exists'_2)$ be two finite $\mathbf{Df}_2$-algebras as in the lemma just above. If A is an homomorphic image of B then, $q \leq k$, $n \leq r$ and $m \leq s$.

(ii) If $f_h : \mathbb{A}_q \to \mathbb{A}_k$ fulfilling conditions (ii) (a) and (ii) (b) of Lemma 3.4, then for all $C \in \mathcal{P}_1$ and all $D \in \mathcal{P}_2$ we have that $f_h(C \cap D) = f_h(C) \cap f_h(D)$. Besides, $|C| = |f_h(C)|$, $|D| = |f_h(D)|$ and $|C \cap D| = |f_h(C) \cap f_h(D)|$.

Lemma 3.6. *The following conditions are equivalent:*

(i) $F_{(a_{ij})_{n\times m}} \in \mathbf{H}(F_{(b_{ij})_{r\times s}})$,

(ii) $n = r$, $m = s$ *and* $F((a_{ij})_{n\times m}) = F((b_{ij})_{r\times s})$

Proof. Let $A = (\mathbb{B}_q, \exists_1, \exists_2)$, $B = (\mathbb{B}_k, \exists'_1, \exists'_2)$ be two simple finite $\mathbf{Df}_2$-algebras, $\mathcal{P}_1 = \{C_1, \ldots, C_n\}$ and $\mathcal{P}_2 = \{D_1, \ldots, D_m\}$ the partitions of $\mathbb{A}_q$ associated to $\exists_1$ and $\exists_2$ respectively, $\mathcal{P}'_1 = \{C'_1, \ldots, C'_r\}$ and $\mathcal{P}_2 = \{D'_1, \ldots, D'_s\}$ the partitions of $\mathbb{A}_k$ associated to $\exists'_1$ and $\exists'_2$ respectively. Let $H_{ij} = C_i \cap D_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ and let $H'_{ij} = C'_i \cap D'_j$ for $1 \leq i \leq r$ and $1 \leq j \leq s$. Suppose that $N(A) = (a_{ij})_{n\times m}$ and $N(B) = (b_{ij})_{r\times s}$.

(i) $\Rightarrow$ (ii): Let A be an homomorphic image of B and $f : \mathbb{A}_q \to \mathbb{A}_k$ fulfilling conditions (ii) (a) and (ii) (b) of Lemma 3.4. We know by Remark 3.5 (i) that $n \leq r$ and suppose that $n < r$. Let $\sigma \in S_r$ such that $f(C_i) = C'_{\sigma(i)}$ for all $1 \leq i \leq n$ and that $\mathcal{P}_1 = \{C'_{\sigma(i)} : 1 \leq i \leq r\}$. Let $D_j \in \mathcal{P}_2$ and $D'_j \in \mathcal{P}'_2$ such that $f(D_j) = D'_j$. Since $\mathcal{P}'_1$ is a partition $\mathbb{A}_k$ we can assert that $D'_j = \bigcup\limits_{i=1}^{r} (C'_{\sigma(i)} \cap D'_j)$ and this is a disjoint union. Therefore, $|D'_j| = \sum\limits_{i=1}^{r} |C'_{\sigma(i)} \cap D'_j|$.

On the other hand, $D_j = \bigcup_{i=1}^{n} (C_i \cap D_j)$ and $|D_j| = \sum_{i=1}^{n} |C_i \cap D_j|$. By Remark 3.5 (ii), $\sum_{i=1}^{n} |C_i \cap D_j| = \sum_{i=1}^{r} |C'_{\sigma(i)} \cap D'_j|$. But $\sum_{i=1}^{n} |C_i \cap D_j| = \sum_{i=1}^{n} |f(C_i \cap D_j)| = \sum_{i=1}^{n} |f(C_i) \cap f(D_j)| = \sum_{i=1}^{n} |C'_{\sigma(i)} \cap D'_j| = \sum_{i=1}^{n} |C'_{\sigma(i)} \cap D'_j| + \sum_{i=n+1}^{r} |C'_{\sigma(i)} \cap D'_j|$. Therefore, $\sum_{i=n+1}^{r} |C'_{\sigma(i)} \cap D'_j| = 0$. A contradiction since $|C'_{\sigma(i)} \cap D'_j| > 0$. Analogously it is shown that $m = s$ and $q = k$.

(ii) $\Rightarrow$ (i): If A and B are two simple finite $\mathbf{Df}_2$-algebras which verify condition (ii) then is clear that $A \simeq B$. □

Corollary 3.7. $\mathbf{H}(\mathbf{S}(F_{(b_{ij})_{r\times s}}) = \mathbf{S}(F_{(b_{ij})_{r\times s}})$

Now, we are going to characterise the poset $\mathrm{Si}_{fin}(\mathbf{Df}_2)$. Recall that the order relation is given by $A \preceq B$ if and only if $A \in \mathbf{H}(\mathbf{S}(B))$ for all $A, B \in \mathrm{Si}_{fin}(\mathbf{Df}_2)$.

Let $\mathcal{M} = \bigcup_{k=1}^{\infty} \mathcal{M}_k$ and consider the equivalence relation $\sim$ over $\mathcal{M}$ defined as in eq. (4).

Now, we define on $M/\sim$ the order relation

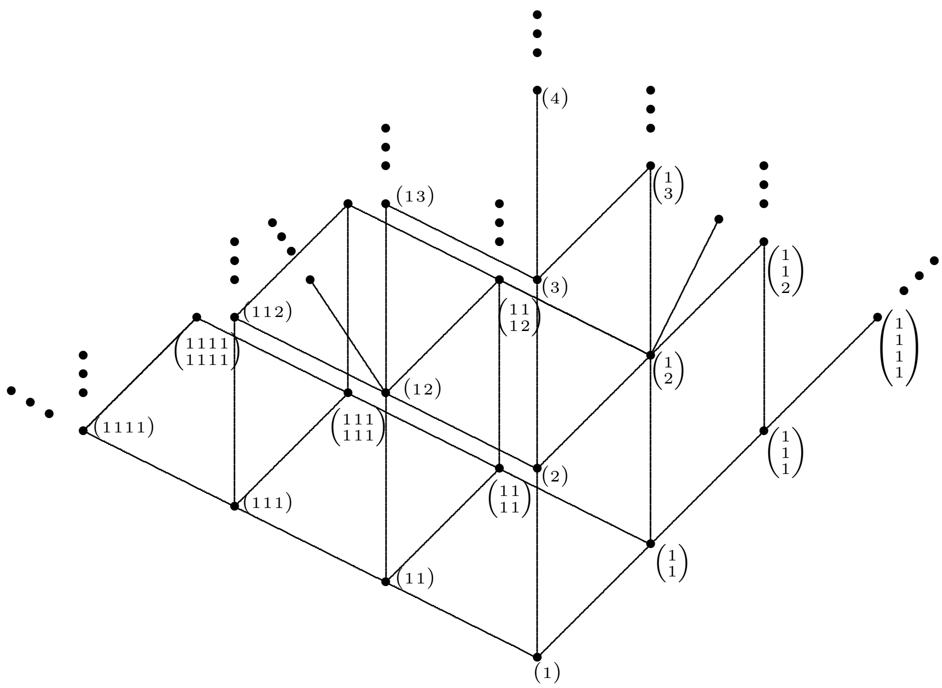
$$\overline{(a_{ij})_{n\times m}} \preceq \overline{(b_{ij})_{r\times s}} \text{ if and only if } (a_{ij})_{n\times m} \text{ and } (b_{ij})_{r\times s}$$

$$\text{verify the condition Lemma 3.2 (ii)}$$

Taking into account all what was stated above we can assert that:

Lemma 3.8. $\mathrm{Si}_{fin}(\mathbf{Df}_2)$ *and* $(M/\sim, \preceq)$ *are isomorphic posets.*

Let A be the finite simple $\mathbf{Df}_2$-algebra such that $(a_{ij})_{n\times m} \in F(A)$. We shall represent the class of A in $M/\sim$ by the only matrix $(\delta_{ij}) \in F(A)$ whose existence is guaranted by Proposition 2.7. The following is the Hasse diagram of the lower part of $(M/\sim, \preceq)$.



Finally, taking into account both Davey's well-known results, and the fact that every proper subvariety of $\mathbf{Df}_2$ is locally finite, we may assert that:

Theorem 3.9. $\Lambda(\mathrm{Df}_2)$ *is a completely distributive lattice and is isomorphic to* $\mathcal{O}((M/\sim, \preceq))$.

References

[1] N. Bezhanishvili, *Varieties of two-dimensional cylindric algebras. Part I: Diagonal-free case*, Algebra Universalis, **48**, 1 (2002), pp. 11–42.

[2] B. Davey, *On the lattice of subvarieties.* Houston J. Math. **5**(1979), pp. 183 – 192.

[3] M. Figallo, *Finite diagonal–free two–dimensional cylindric algebras*, Log. J. IGPL. **12**, 6 (2004), pp. 509–523.

[4] M. Figallo, Una Contribución sobre las álgebras cilíndricas libres de elementos diagonales de dimensión dos, Ms. Dissertation. Universidad Nacional del Sur, 2005.

[5] L. Henkin, D. Monk and A. Tarski, Cylindric Algebras, Parts I & II, North-Holland, 1971 & 1985.

[6] B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand., **21** (1967), pp. 110–121.

[7] L. Monteiro, M. Abad, S. Savini, J. Sewald and M. Zander, *Subalgebras of a finite monadic Boolean Algebra*, Reports on Mathematical Logic, **40** (2006), pp. 199–206.

[8] W. Taylor, *The fine spectrum of a variety.* Algebra Universalis **5** no. 2 (1975), pp. 263–303.

Departamento de Matemática
Universidad Nacional del Sur
8000 Bahía Blanca, Argentina.

figallomartin@gmail.com