REPORTS ON MATHEMATICAL LOGIC
49 (2014), 99–117
DOI:10.4467/20842589RM.14.006.2276

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COMPLEXITY OF COVER-PRESERVING EMBEDDINGS OF BIPARTITE ORDERS INTO BOOLEAN LATTICES

A b s t r a c t. We study the problem of deciding, whether a given partial order is embeddable into two consecutive layers of a Boolean lattice. Employing an equivalent condition for such embeddability similar to the one given by J. Mittas and K. Reuter [5], we prove that the decision problem is NP-complete by showing a polynomial-time reduction from the not-all-equal variant of the Satisfiability problem.

1. Introduction

We study the problem of deciding, whether a given partial order of height one is a subdiagram of a Boolean lattice. Questions of embeddability are important for compression of graph-like data structures: if a structure to be compressed is embeddable into some highly-regular structure (of possibly

Received 27 March 2014

different kind), the embedding can be used to transfer the encoding of data into a realm of structures which can be compressed better or faster. To demonstrate this, consider a partial order $(P, \leq)$ with n elements (and thus having up to $\Theta(n^2)$ edges and requiring possibly $\Theta(n^2)$ bits to encode). If P is embeddable into (i.e., isomorphic to a subdiagram of) layers k and $k+1$ of a Boolean lattice, we can employ this fact to provide a space-efficient encoding of P. We encode each connected component of P separately. We first encode the embedding of one of its maximal elements x (requiring $O(k \log m)$ bits, where m is the size of the component being encoded). Then follows a tree of shortest paths from x to other elements, with edges labelled by the set differences of the embeddings of their endpoints (each vertex requires $O(\log m)$ bits). The complete representation from which P can be easily obtained takes only $O(n \log n)$ bits in total.

Another area where embeddability plays an important role is modelling parallel computer architectures, in particular the ability of one network structure to faithfully simulate another [3]. A good simulation is said to exist when adjacent processors in the guest (simulated) network can be mapped to reasonably close processors in the host network. Here, Boolean lattices correspond to networks forming binary hypercubes, known to have excellent connectivity properties (logarithmic diameter with only logarithmic node degree).

The problem of deciding whether a partial order is embeddable into a hypercube has been shown to be NP-complete [2]. However, the embeddings considered did not have to be cover-preserving. With the cover-preservation requirement added, the problem was successfully tackled by M. Wild [4] with the use of projectivities. His method seems to fail for orders of height one—the particular variant we study here. This variant was analyzed by J. Mitas and K. Reuter [5]. They showed, that embeddability cannot be characterized by a finite family of forbidden suborders. Moreover they formulated an equivalent condition for embeddability, given in terms of edge coloring of the comparability graph.

In this paper, we first provide a formal definition of an admissible coloring, together with some intuitions behind it (Section 2). Then we show, that the existence of such coloring characterizes exactly the embeddable orders (Section 3). Finally, we prove, that the embeddability problem is NP-complete, by showing a polynomial-time reduction from a variant of Satisfiability problem.

2. The Coloring

Let us start by formally defining the problem:

Definition 2.1 (Embeddability problem)**.** We are given a partial order $(P, \leq)$.
We want to know if there exist: a set X, an integer k and a function

$$f : P \to \binom{X}{k} \cup \binom{X}{k+1}$$

such, that $p \leq q$ iff $f(p) \subseteq f(q)$ whenever $p, q \in P$.
The symbol $\binom{X}{k}$ is used here for the family of all k-element subsets of the set X.

Following J. Mitas and K. Reuter, we will characterize embeddable orders by the properties of their comparability graph, i.e., a graph whose vertices are the elements of the set X, and there is an edge between u and v iff $u < v$ or $v < u$. It is obvious, that for an embedding to exist, the order must have height of no more than one. In other words, its comparability graph must be bipartite. It is also clear, that if we can find an embedding for each connected component of P, then we can create one for the entire order. Both checking whether a graph is bipartite and finding connected components are linear time problems. Therefore, for the rest of this paper, we assume that the order given is bipartite and connected. Furthermore, if there are at least two elements in the order, each element must be *either* minimal *or* maximal (it cannot be both).

Consider any suborder of two consecutive layers of a Boolean lattice, i.e., a subset of $\binom{X}{k} \cup \binom{X}{k+1}$ for some X and k, ordered by inclusion. Whenever there is an edge $\{u, v\}$ in the comparability graph, its endpoints must be comparable subsets of X, of cardinality k and $k+1$. Therefore, they must differ by a unique element of X, which we call the **color** of $\{u, v\}$. Let us briefly analyze the properties of such coloring.

Imagine a path $\Gamma = x_0 x_1 \ldots x_n$, beginning at a maximal vertex x_0 (the case for x_0 being minimal is symmetric). The path alternates between maximal vertices x_{2i} of cardinality $k + 1$ and minimal vertices x_{2i+1} of cardinality k. Walking along the path, we therefore alternate between removing and adding the edge colors to the set associated with the current vertex. Note the following properties of such a walk:

- If we encounter the same color α multiple times along the path, the actions undertaken with α must alternate between addition and removal (an element removed from the set cannot be removed again until it has been added back). As the actions are tied to the parity of the distance from x_0 to the edge considered, the edges of color α must alternate between even and odd distances from x_0.

- If we ever come back to x_0, i.e., end up with the same set, we must have seen each color an even number of times. Moreover, this is an if-and-only-if condition.

- If the endpoints x_0, x_n of Γ are connected with an edge, there must be exactly one color which has been present on Γ an odd number of times (exactly the color of the edge $\{x_0x_n\}$). This condition is also an if-and-only-if, because if we have two sets whose symmetric difference is a singleton, one of them must be a subset of the other.

It turns out, that the above set of properties is, in a sense, complete: we will show that if a partial order can be colored so as to satisfy them, it must be embeddable.

Let us start with formally stating the required properties of the coloring. We consider an undirected, connected, bipartite graph $G = (V, E)$, $E \subseteq \binom{V}{2}$ and its edge-coloring $c : E \to \Omega$.

We denote the set of all paths by E^*:

$$E^* = \{x_0x_1 \dots x_n : \forall i \ \{x_i, x_{i+1}\} \in E\}$$

For each path $\Gamma = x_0x_1 \dots x_n \in E^*$, we define its induced color set as the set of colors that appear on Γ an odd number of times (here, $\div$ denotes the symmetric difference of sets; individual elements of Ω are promoted to singleton sets as needed):

$$c(\Gamma) = c(\{x_0, x_1\}) \div c(\{x_1, x_2\}) \div \dots \div c(\{x_{n-1}, x_n\})$$

We begin with the weakest of the required properties:

Definition 2.2 (Consistent coloring)**.** We say that the edge coloring c of a graph G is **consistent** iff for every cycle $\Gamma \in E^*$ we have $c(\Gamma) = \emptyset$

If the coloring c is consistent, then for any two vertices $u, v \in V$ and two paths Γ, Δ from u to v we must have $c(\Gamma) = c(\Delta)$—otherwise by joining these paths we would get a cycle with a nonempty induced color set. Therefore, for a consistent coloring of a connected graph we can define the **color distance** of vertices $u, v \in V$ as:

$$c(u,v) = c(\Gamma), \text{ where } \Gamma \text{ is any path that joins } u \text{ and } v$$

Stating the two if-and-only-if properties is now simple:

Definition 2.3 (Totally consistent coloring)**.** A consistent edge coloring c of G is **totally consistent** iff for $u, v \in V$:

$$\begin{aligned} |c(u,v)| = 0 &\text{ implies } u = v \\ |c(u,v)| = 1 &\text{ implies } \{u,v\} \in E \end{aligned}$$

Defining the remaining property requires a bit more work: we need some means of relating the parity of the length of the path to the parity of the number of times a specific color α appears on it. For connected graphs it is enough to consider only half of the cases, for example for those paths Γ where α appears an odd number of times, i.e., $\alpha \in c(\Gamma)$.

For each vertex $v \in V$ we define the **odd** and **even** subsets of the color set:

$$\begin{aligned} O(v) = \{\alpha \in \Omega : \;&\alpha \text{ is the color of the last edge of some} \\ &\text{odd-length path } \Gamma \text{ starting in } v \text{ and } \alpha \in c(\Gamma)\} \\ E(v) = \{\alpha \in \Omega : \;&\alpha \text{ is the color of the last edge of some} \\ &\text{even-length path } \Gamma \text{ starting in } v \text{ and } \alpha \in c(\Gamma)\} \end{aligned}$$

Note that the above definitions try to capture the elements which must be present in and absent from the set associated with the vertex v (with one of the two interpretations depending on whether v is maximal or minimal). The full property of the coloring can now be stated as follows:

Definition 2.4 (Admissible coloring)**.** An edge coloring c is **admissible**[1] iff it is totally consistent and $O(v) \cap E(v) = \emptyset$ for each $v \in V$.

[1]Our definition of admissible coloring is equivalent to the one used by J. Mitas and K. Reuter [5]

3. A Characterization

We now turn to the formal proof of the characterization of embeddable orders. Our argument is a slight modification of the one in [5], employing our definition of admissible coloring.

Theorem 3.1 (Characterization of embeddable orders)**.** *A connected, bipartite partial order $(P, \leq)$ is embeddable into two consecutive layers of some Boolean lattice if and only if there exists an admissible edge coloring of its comparability graph.*

Proof. Let $(P, \leq)$ be an embeddable order, and this fact be witnessed by X, k and f. We define a coloring c by putting

$$c(\{u, v\}) = f(u) \div f(v),$$

for each edge $u < v \in P$.
Then, for each path $x_0 x_1 \dots x_n \in E^*$ we have

$$\begin{aligned} c(x_0 x_1 \dots x_n) &= c(\{x_0, x_1\}) \div \dots \div c(\{x_{n-1}, x_n\}) \\ &= f(x_0) \div f(x_1) \div f(x_1) \div \dots \div f(x_{n-1}) \div f(x_{n-1}) \div f(x_n) \\ &= f(x_0) \div f(x_n), \end{aligned}$$

which depends exclusively on the endpoints x_0 and x_n. Thus, the coloring is consistent, and we have $c(u, v) = f(u) \div f(v)$ for all $u, v \in P$. Consequently,

$$\begin{aligned} |c(u, v)| = 0 &\implies f(u) = f(v) \\ &\implies u = v, \end{aligned}$$

and

$$\begin{aligned} |c(u, v)| = 1 &\implies |\{f(u) \div f(v)\}| = 1 \\ &\implies f(u) \subset f(v) \text{ or } f(v) \subset f(u) \\ &\implies u < v \text{ or } v < u. \end{aligned}$$

This means, that the coloring c is totally consistent.

Now take a vertex $v \in P$ with $O(v) \cap E(v) \neq \emptyset$. Then for $\alpha \in O(v) \cap E(v)$ there are paths $v x_1 \dots x_n$ and $v y_1 \dots y_m$ with

$$\begin{aligned} c(\{x_{n-1}, x_n\}) = \alpha \in c(v x_1 \dots x_n) \text{ and } 2 \nmid n, \\ c(\{y_{m-1}, y_m\}) = \alpha \in c(v y_1 \dots y_m) \text{ and } 2 \mid m. \end{aligned}$$

Since $\alpha \in c(vx_1 \dots x_n) \cap c(vy_1 \dots y_m)$, we have

$$\alpha \notin c(x_n x_{n-1} \dots x_1 v y_1 \dots y_m) = f(x_n) \div f(y_m).$$

Hence,

$$\begin{aligned} x_n \text{ is maximal in } P &\iff \alpha \in f(x_n) \\ &\iff \alpha \in f(y_m) \\ &\iff y_m \text{ is maximal in } P. \end{aligned}$$

Therefore x_n and y_m must both be maximal or minimal elements of P. But this is not possible, as the length of the path $x_n x_{n-1} \dots x_1 v y_1 \dots y_m$ is odd. We know then that our assumption that $O(v) \cap E(v) \neq \emptyset$ must have been false, and thus the coloring c is admissible.

We now turn into the other direction of the proof and assume that we are given an admissible edge coloring $c : E \to \Omega$ of P's comparability graph. We then define

$$\begin{aligned} X &= \Omega, \\ f(v) &= \begin{cases} E(v), & \text{if } v \text{ is minimal in } P, \\ O(v), & \text{if } v \text{ is maximal in } P. \end{cases} \end{aligned}$$

Now for $u, v \in P$, $u < v$, $\alpha = c(\{u, v\})$ we have:

$$\begin{aligned} &u \text{ is minimal in } P \text{ and } \alpha \notin f(u) = E(u), \\ &v \text{ is maximal in } P \text{ and } \alpha \in f(v) = O(v). \end{aligned}$$

Let us consider another color $\beta \neq \alpha$. We can see, that if $\beta \in f(u) = E(u)$ is witnessed by an even length path $ux_1 \dots x_n \in E^*$ with

$$c(\{x_{n-1}, x_n\}) = \beta \in c(ux_1 \dots x_n),$$

then the augmented path $vux_1 \dots x_n \in E^*$ has odd length and witnesses the fact that $\beta \in O(v) = f(v)$. Analogously, one can show that each $\beta \neq \alpha$ that lies in $f(v)$ must be in $f(u)$. This yields $f(v) = f(u) \cup c(\{u, v\})$ and, finally, $|f(v)| = |f(u)| + 1$.

The comparability graph is connected, and therefore there exists k such that

$$|f(v)| = \begin{cases} k, & \text{if } v \text{ is minimal in } P, \\ k+1, & \text{if } v \text{ is maximal in } P. \end{cases}$$

We now show that $\forall u, v \in P : f(u) \div f(v) = c(u,v)$. This follows by induction on the distance between u and v (remember, that for $u < v$, we have $f(u) \div f(v) = \{c(\{u,v\})\}$).

Hence, by total consistency of c, we get

$$\begin{aligned} f(u) \subseteq f(v) &\implies |\,(f(u) \div f(v))\,| \leq 1 \\ &\implies |c(u,v)| \leq 1 \\ &\implies \{u,v\} \in E \text{ or } u = v \\ &\implies u \leq v, \end{aligned}$$

which means that f is a correct embedding of P into $\binom{X}{k} \cup \binom{X}{k+1}$. $\square$

4. Main Result

Having shown that embeddability is equivalent to the existence of an admissible coloring, we know that checking embeddability is equivalent to checking colorability. This allows us to prove that both these problems are NP-complete, by showing one of them to be in NP, and the other to be NP-hard. Interestingly, we were not able to find any direct proof of either the hardness of embeddability, or the solvability of colorability—for the latter, note that even *verifying* whether a given coloring is admissible seems to require checking all paths in the graph, and there are exponentially many of them.

We begin with the simple part:

Lemma 4.1. *The embeddability problem is in NP.*

Proof. We are given a partially ordered set $(P, \leq)$ with n elements. If it is an embeddable one, there must exist a proper embedding with the space X having no more than n elements. Thus, a description of proper embedding needs no more than $O(n^2 \log n)$ bits. Checking that the function described is indeed a correct embedding is trivially a polynomial time problem. From the above it follows that the embeddability problem is polynomially verifiable, and thus it belongs to the class NP. $\square$

We will employ a standard technique of proving NP-hardness, by presenting a polynomial-time reduction from a known hard problem. Here, we have chosen the following variant of Satisfiability:

Definition 4.2 (NAESAT)**.** NAESAT ("not-all-equal Satisfiability") is a decision problem, with an instance being a logical formula having the form:

$$C^1 \wedge C^2 \wedge \ldots \wedge C^k$$

built over a finite set of variables $X = \{x_1 \ldots x_n\}$, with each clause C^i constructed from exactly three literals (variables or their negations) as

$$(l_1^i \vee l_2^i \vee l_3^i) \wedge \neg(l_1^i \wedge l_2^i \wedge l_3^i).$$

All literals in a clause must come from distinct variables.
The question is: is the given formula satisfiable?

It has been proven by T.J. Shaefer [1], that the NAESAT problem is NP-complete.

Our main contribution is the following theorem:

Theorem 4.3. *The NAESAT problem reduces polynomially to the problem of existence of admissible coloring.*

We will explicitly provide the reduction—an algorithm constructing a graph for a given instance of NAESAT. As is typical for reductions from Satisfiability, the resulting graph will consist of **gadgets** representing individual variables and clauses, joined in such a way as to guarantee the equivalence between the satisfiability of the formula and the colorability of the whole graph.

The definition of admissible coloring is, however, non-local: it talks about the properties of the whole graph, not of individual vertices and edges (or their small neighborhoods). Therefore, to make the construction modular and the proof manageable, we first provide the following technical lemma:

Lemma 4.4 (Gluing lemma)**.** *Let (X, E_X) and (Y, E_Y) be two disjoint, connected, bipartite graphs with admissible edge colorings $c_X : E_X \to \Omega_X$ and $c_Y : E_Y \to \Omega_Y$. Suppose that the vertices $x_1, x_2, \ldots, x_n \in X$ and $y_1, y_2, \ldots, y_n \in Y$ satisfy the following conditions:*

$$\forall i : c_X(x_1, x_i) = c_Y(y_1, y_i), \tag{1}$$
$$\forall i : O_X(x_i) \cap E_Y(y_i) = E_X(x_i) \cap O_Y(y_i) = \emptyset \tag{2}$$

Let $Z = \{z_1, \dots, z_n\}$ *be a set of fresh vertices* ($Z \cap (X \cup Y) = \emptyset$)*. Then for the graph* (V, E)*, where*

$$\begin{aligned} V &= X \cup Y \cup Z, \\ E &= E_X \cup E_Y \cup \{\{x_i, z_i\}, \{y_i, z_i\} : i = 1 \dots n\}, \end{aligned}$$

there exists an admissible edge-coloring $c : E \to \Omega$*, such that*

$$\begin{aligned} \Omega &= \Omega_X \cup \Omega_Y \cup \{\phi, \psi\} \text{ with } (\phi, \psi \notin \Omega_X \cup \Omega_Y), \\ c\,|_{E_X} &= c_X, \\ c\,|_{E_Y} &= c_Y. \end{aligned}$$

Proof. First, from (1) we can easily deduce, that $c_X(x_i, x_j) = c_Y(y_i, y_j)$ for all i, j. Then we define c as:

$$\begin{aligned} c(\{u,v\}) &= c_X(\{u,v\}) \text{ for } u, v \in X \\ c(\{u,v\}) &= c_Y(\{u,v\}) \text{ for } u, v \in Y \\ c(\{x_i, z_i\}) &= \begin{cases} \phi, & \text{if } |c(x_1, x_i)| \text{ is even} \\ \psi, & \text{if } |c(x_1, x_i)| \text{ is odd} \end{cases} \\ c(\{y_i, z_i\}) &= \begin{cases} \psi, & \text{if } |c(x_1, x_i)| \text{ is even} \\ \phi, & \text{if } |c(x_1, x_i)| \text{ is odd} \end{cases} \end{aligned}$$

By definition, c extends both c_X and c_Y. It remains to show that it is admissible.

Let us consider a cycle $\Gamma = v_0 v_1 \dots v_{m-1} v_0 \in E^*$. Without loss of generality we can assume that $v_0 \in X$ (X and Y play symmetric roles in the construction, and no cycle can be completely contained in Z). We induct on the number of the points in $\Gamma \cap Z$ to show that $c(\Gamma) = \emptyset$. If there is no such point, then $\Gamma \subseteq X$. The colorings c and c_X coincide on X, and therefore $c(\Gamma) = \emptyset$. If there is such a fresh point, say z_i, in Γ, then there are two possibilities:

$$\begin{aligned} \Gamma &= v_0 \dots x_i z_i x_i \dots v_0, \text{ or} \\ \Gamma &= v_0 \dots x_i z_i y_i \dots y_j z_j x_j \dots v_0 \\ &\quad \text{(with } i \text{ and } j \text{ possibly equal).} \end{aligned}$$

In the first case, we modify Γ by removing z_i and one of the x_is. We obtain a cycle, having obviously the same induced color set, with one fewer point from Z in it.

In the second case, we modify Γ by replacing the fragment $z_i y_i \dots y_j z_j$ by an arbitrary path connecting x_i to x_j in X (it exists, because X is connected). Knowing that $c_X(x_i, x_j) = c_Y(y_i, y_j)$ and $c(x_i z_i y_i) = c(y_j z_j x_j) = \{\phi, \psi\}$, we see that the modified cycle has the same induced color set as Γ. And, again, it uses fewer points from Z. Having considered both cases, we know that the induced color set of any cycle in V is empty. Thus, c is consistent.

Given vertices $u \in X$ and $v \in Y$, we know that there is a path Γ in X from u to x_1, and a path Δ in Y from y_1 to v. Thus the path $\Sigma = \Delta z_1 \Gamma$ connects u to v and $\phi, \psi \in c(\Sigma)$. Therefore $|c(u, v)| \geq 2$.

Consider two vertices $u, v \in V$, with $c(u, v) = \emptyset$. Because of the previous fact, they cannot belong to both X and Y. If they both lie in X, then $u = v$, because c_X is totally consistent. If $u \in X$ and $v = z_i \in Z$, then there is a path $u \dots x_i v$ which contains exactly one edge ($\{x_i, v\}$) with color ϕ or ψ, and therefore $c(u, v)$ cannot be empty. The remaining possibility is that $u, v \in Z$, say $u = z_i$, $v = z_j$. Then we can form a path $z_i x_i \dots x_j z_j$, such that $x_i \dots x_j$ lies totally in X. The only edges with colors ϕ or ψ on this path are $\{z_i, x_i\}$ and $\{x_j, z_j\}$. Since $c(u, v)$ is empty, both these edges have the same color. Then we know that $c(x_i, x_j) = c(z_i, z_j) = \emptyset$. Since c_X is totally consistent, $x_i = x_j$ and, finally, $u = z_i = z_j = v$.

Now suppose that $c(u, v) = \{\alpha\}$. Again, u and v cannot belong to both X and Y. If they both lie in X, then $\{u, v\} \in E_X \subset E$ (because c_X is totally consistent). If $u \in X$ and $v = z_i$ then, by creating the path $u \dots x_i z_i$ (with $u \dots x_i$ contained in X) we see that $\alpha = c(\{x_i, z_i\})$. Then $c(u, x_i) = \emptyset$ and $u = x_i$, which gives $\{u, v\} = \{x_i, z_i\} \in E$. In the last case we have $u = z_i$, $v = z_j$. Again, we form a path $z_i x_i \dots x_j z_j$. The path contains exactly two edges with colors from the set $\{\phi, \psi\}$. But they cannot have different colors, because we assumed that $|c(u, v)| = 1$. Therefore $c(\{x_i, z_i\}) = c(\{x_j, z_j\})$ and, what follows from the definition of c, $|c(x_1, x_i)|$ and $|c(x_1, x_j)|$ are of the same parity. But then 2 divides $|c(x_i, x_j)| = |c(z_i, z_j)| = |c(u, v)| = 1$. The resulting contradiction proves that c is totally consistent.

To show that c is admissible, consider a vertex $v \in V$ with $O(v) \cap E(v) \neq \emptyset$. Then there exists $\alpha \in O(v) \cap E(v)$, which has to satisfy:

$$\exists v = u_0 u_1 \dots u_m \in E^* : c(\{u_{m-1}, u_m\}) = \alpha \in c(v_0 v_1 \dots v_m) \wedge 2 \nmid m,$$
$$\exists v = w_0 w_1 \dots w_k \in E^* : c(\{w_{k-1}, w_k\}) = \alpha \in c(w_0 w_1 \dots w_k) \wedge 2 \mid k.$$

By joining these two paths we get the path $\Gamma = u_m \dots u_1 v w_1 \dots w_k$ with the following properties:

$$\text{first and last edge on } \Gamma \text{ have color } \alpha \tag{3}$$

$$\alpha \notin c(\Gamma) \tag{4}$$

$$|c(\Gamma)| \text{ is odd} \tag{5}$$

Because at least one path with these properties exists, pick a shortest one, say Δ. If α occurs on Δ more than twice, then it must (by (4)) occur at least 4 times. Looking at three subpaths, between the first and the second, the second and the third, and between the third and the last occurrence of α, we see that at least one of them enjoys the properties (3)-(5). Minimality condition we put on Δ gives that the only occurrences of α on Δ are on the first and the last edge.

Let Δ go through $v_0 v_1 \dots v_m$ ($2 \nmid m$) with $c(\{v_0, v_1\}) = c(\{v_{m-1}, v_m\}) = \alpha$. We will prove by induction, that for $i = 1, \dots, m-1$ we have

$$\begin{aligned} v_i \in X &\implies (2 \mid i \iff \alpha \in E_X(v_i)), \\ v_i \in Y &\implies (2 \mid i \iff \alpha \in E_Y(v_i)). \end{aligned} \tag{6}$$

The case of $i = 1$ is trivial. Suppose that the above holds for $1, 2, \dots, i$ and that $v_{i+1} \in X$. If $v_i \in X$, then augmenting the path witnessing $\alpha \in E_X(v_i)$ (or $O_X(v_i)$) by the edge $\{v_i, v_{i+1}\}$ gives a witness for $\alpha \in O_X(v_{i+1})$ (or $E_X(v_{i+1})$, respectively). Otherwise $v_i \in Z$. Then if $v_{i-1} \in X$ then $v_{i+1} = v_{i-1}$ and (6) follows. The last possibility is that $v_{i-1} \in Y$. But then there is a j with $v_{i-1} = y_j, v_i = z_j, v_{i+1} = x_j$, and we conclude (6) using (2).

If $v_{m-1} \in X$, then from the above induction we have $\alpha \in E_X(v_{m-1})$ (because $2 \mid m-1$). But of course $\alpha \in O_X(v_{m-1})$, because the edge $\{v_{m-1}, v_m\}$ has color α. This contradicts our assumption that c_X is an admissible coloring.

The case when $v_{m-1} \in Y$ is analogous.

If $v_{m-1} \in Z$, then there is a j such that $v_{m-1} = z_j$ and (without loss of generality) $v_{m-2} = x_j$. Thus, $\alpha \in \{\phi, \psi\}$, and therefore either v_0 or v_1 is in Z. In both cases the parity of the length of the path connecting it to z_j and the setting of α (either ϕ or ψ) contradict the definition of c on E_Z.

Having considered all the cases, we know that a situation when $O(v) \cap E(v) \neq \emptyset$ is impossible, which means that c is admissible and ends the proof. $\square$

Equipped with this construction tool, we proceed to the proof of Theorem 4.3.

Proof. We are given an instance of the NAESAT problem with n variables and k clauses, as in Definition 4.2. We will show a method of constructing an undirected, connected, bipartite graph $G = (V, E)$, dependent only on this instance. Along with the construction, we will show its admissible edge-coloring, under the assumption that the formula is satisfiable. Later on, we will show that if an admissible coloring exists, the formula must be satisfiable.

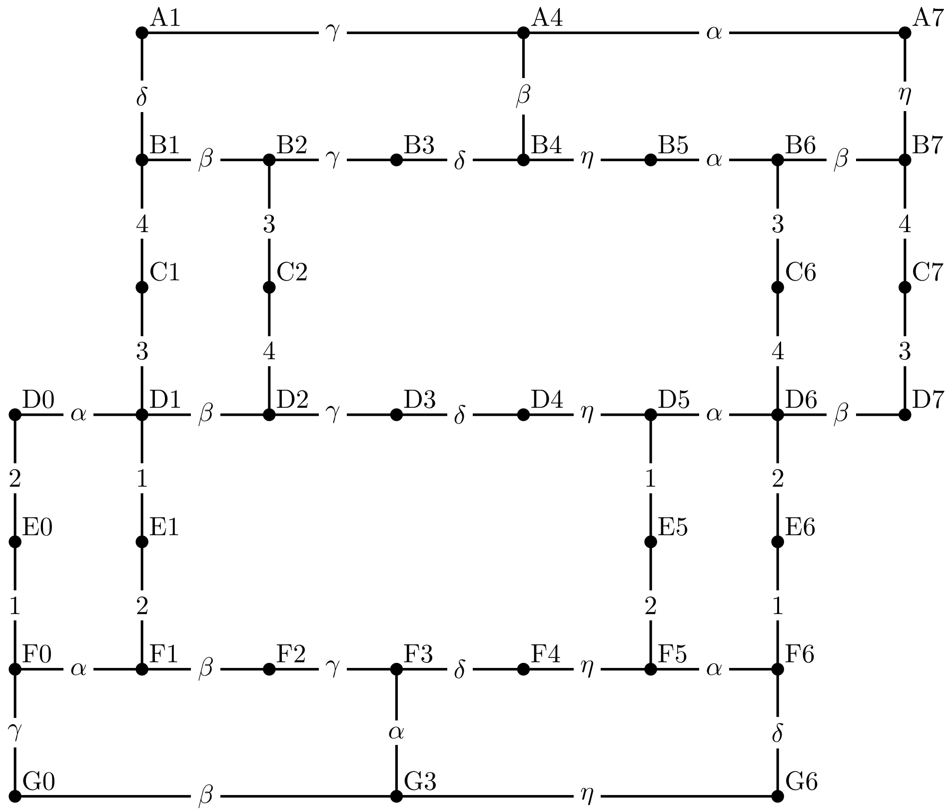
First let us set up some naming conventions:

The construction will employ three types of gadgets: an **initializer**, **selectors**, capturing the assignment of truth values to variables, and **validators**, ensuring satisfiability of particular clauses. Each type will receive its one letter abbreviation (I, S and V, respectively). Multiple gadgets of the same type will be numbered using superscripts, and specific points of a gadget will be denoted by subscripts. Thus, by S^2_{A3} we will denote the point $A3$ of the second selector.

For color names we will use Greek letters and integers. Colors named with integers will be local to a single gadget (i.e., color 2 of each gadget is different). Greek letters will denote global colors.

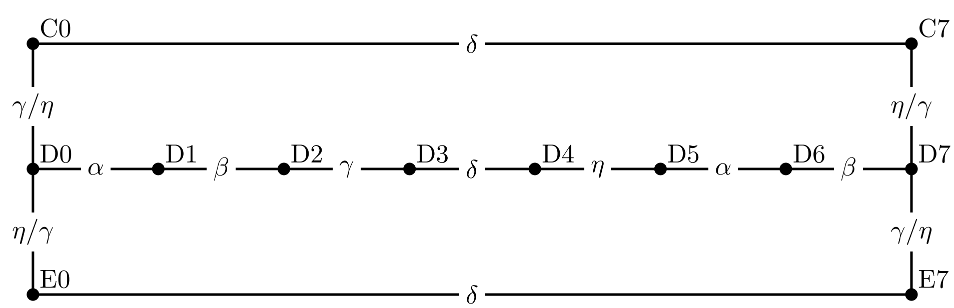
We will say that we make a bridge between points $x_1 \dots x_m$ and $y_1 \dots y_m$ when we add fresh points $z_1 \dots z_m$ and create edges $\{x_i, z_i\}, \{z_i, y_i\}$ (this is exactly the operation used to connect the graphs in the gluing lemma).

We begin the construction with static (i.e., independent from the given NAESAT formula) **initializer**:



Rows A and B form an obviously admissibly-colored subgraph, and so do rows F and G. Coloring of row D is trivially admissible as well. It is easy to check, that the assumptions of gluing lemma hold for point sets $(D1, D2, D6, D7)$ and $(B1, B2, B6, B7)$, as well as for point sets $(D0, D1, D5, D6)$ and $(F0, F1, F5, F6)$. Thus, the presented coloring of the whole gadget is admissible.

The next gadget type is the **selector**:



The coloring shown can be made admissible by setting either

$$c(C0, D0) = c(D7, E7) = \gamma,$$
$$c(D0, E0) = c(C7, D7) = \eta,$$

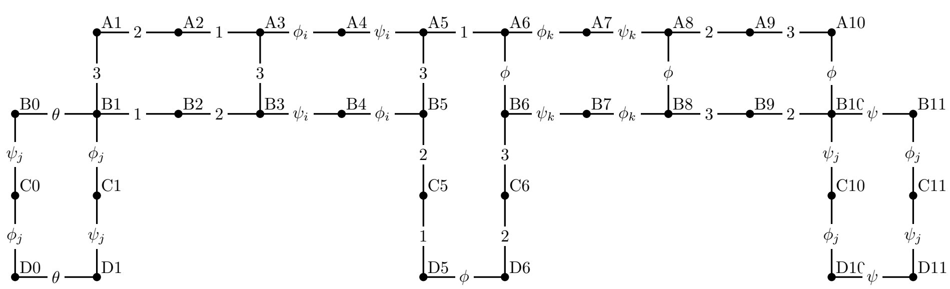
or

$$c(C0, D0) = c(D7, E7) = \eta,$$
$$c(D0, E0) = c(C7, D7) = \gamma.$$

We create n selectors (one for each variable), and for each of them we create a bridge between points $S^i_{D0} \dots S^i_{D7}$ and $I_{D0} \dots I_{D7}$. Given the valuation $v : X \to \{0, 1\}$ of variables that satisfies the formula, we set $c(S^i_{C0} S^i_{D0})$ to γ iff $v(x_i) = 1$.

We now iteratively use the gluing lemma to prove, that the graph constructed so far is admissibly colorable. It is not hard to see, that all assumptions of the lemma hold regardless of the choice between γ and η made above. Application of the lemma to the i-th selector creates two fresh colors, say ϕ_i and ψ_i, used only on the bridge between the initializer and this selector.

Now we proceed to our final gadget—the **validator**:



Assuming that $\phi, \psi, \theta \in \{\gamma, \eta\}$, $\phi \neq \psi$, and that the six colors $\phi_i, \psi_i, \phi_j, \psi_j, \phi_k, \psi_k$ are pairwise different, the coloring shown is admissible (we have checked it using a computer).

We create a validator V^t for the t-th clause in the given formula. The clause contains 3 distinct literals l^t_1, l^t_2 and l^t_3. Denote by $var(l)$ the index of variable used by literal l and let $i = var(l^t_1)$, $j = var(l^t_2)$ and $k = var(l^t_3)$. We create a bridge between vertices V^t_{D0}, V^t_{D1}, V^t_{D6}, V^t_{D5}, V^t_{D10},

V^t_{D11} and S^i_{D0}, S^i_{C0}, S^j_{D0}, S^j_{C0}, S^k_{D0}, S^k_{C0}, replacing S_{C0} with S_{E0} whenever the corresponding variable appears negated in the clause.

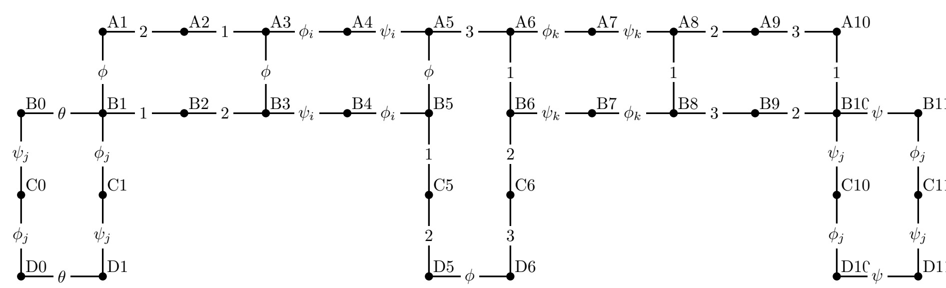
Since v satisfies the formula, we know that in the clause there are two literals with different values. We determine the colors ϕ, ψ and θ (chosen from $\{\gamma, \eta\}$) depending on the valuation, by setting:

$$\theta = \begin{cases} \gamma, & \text{if } v(l^t_1) = 1 \\ \eta & \text{otherwise} \end{cases}$$

$$\phi = \begin{cases} \gamma, & \text{if } v(l^t_2) = 1 \\ \eta & \text{otherwise} \end{cases}$$

$$\psi = \begin{cases} \gamma, & \text{if } v(l^t_3) = 1 \\ \eta & \text{otherwise} \end{cases}$$

If $v(l^t_2) \neq v(l^t_3)$, then $\phi \neq \psi$ and the coloring is admissible. If $v(l^t_2) = v(l^t_3)$, we must have $v(l^t_1) \neq v(l^t_2)$. We then create a mirror image of the coloring shown above (admissible, because $\theta \neq \phi$):



In both cases, not only the obtained coloring is admissible, but it also fulfills all assumptions of the gluing lemma (on the bridge between the validator and the three selectors). Again, using the gluing lemma iteratively, we deduce that the entire created graph is admissibly colorable.

This finishes the description of our reduction from NAESAT problem. It also proves, that if the formula is satisfiable, then the resulting graph is admissibly colorable.

Now we turn to the other direction of the proof and assume that the resulting graph has an admissible coloring c.

We will use a few consequences of the definition of admissible coloring:

on each cycle, there is an even number of edges with each color (7)

every 3 consecutive edges have different colors (8)

on each cycle with 6 vertices, the opposite edges have the same color (9)

We first look at the initializer. By (9), in any admissible coloring the edges $I_{D0}I_{D1}$, $I_{F0}I_{F1}$, $I_{F3}I_{G3}$, $I_{F5}I_{F6}$ and $I_{D5}I_{D6}$ must have the same color. Analogously, $I_{D1}I_{D2}$, $I_{B1}I_{B2}$, $I_{A4}I_{B4}$, $I_{B6}I_{B7}$ and $I_{D6}I_{D7}$ are all of the same color. By (8) applied to edges $I_{D0}I_{D1}$ and $I_{D1}I_{D2}$, the above two colors are different, and so we denote them by α and β. Colors of the edges $I_{D2}I_{D3}$, $I_{D3}I_{D4}$ and $I_{D4}I_{D5}$ must be pairwise different and different from α and β. Let us denote them by γ, δ and η, respectively.

Now we analyze the colors of the edges in the i-th selector. Vertices $S^i_{D0} \dots S^i_{D7}$ are bridged with the initializer, and thus, by (9), the edges between them must copy exactly the colors from the initializer. Now on each of the 10-vertex cycles of the selector, the colors γ, δ and η must appear exactly twice. Due to the parity constraint on the admissible coloring, edges $S^i_{C0}S^i_{C7}$ and $S^i_{E0}S^i_{E7}$ must have color δ. Therefore there are exactly two possible colorings of the selector:

$$c(S^i_{C0}S^i_{D0}) = c(S^i_{D7}S^i_{E7}) = \gamma,$$
$$c(S^i_{D0}S^i_{E0}) = c(S^i_{C7}S^i_{D7}) = \eta,$$

or

$$c(S^i_{C0}S^i_{D0}) = c(S^i_{D7}S^i_{E7}) = \eta,$$
$$c(S^i_{D0}S^i_{E0}) = c(S^i_{C7}S^i_{D7}) = \gamma.$$

We define a validation v by setting:

$$v(x_i) = \begin{cases} 1, & \text{when } c(S^i_{C0}S^i_{D0}) = \gamma \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

Let us now move to the bridges between the selectors and the validators. In the t-th validator, each of the edges $V^t_{D0}V^t_{D1}$, $V^t_{D6}V^t_{D5}$ and $V^t_{D10}V^t_{D11}$ is bridged with the edge $S^i_{D0}S^i_{C0}$ or $S^i_{D0}S^i_{E0}$ of the appropriate selector, depending on whether the variable is negated in the particular clause. Thus, as the coloring is admissible, each of these edges has either color γ or η, and it has color γ precisely when the corresponding literal is true.

To finish the proof, assume to the contrary, that the values we have chosen for the variables do not satisfy the formula. Then they must fail to satisfy at least one of the clauses, say C^t. This means that all three literals in the clause C^t have the same value (true or false). What follows from the construction of the graph, and from (9), is that in the corresponding validator V^t the edges $V^t_{D0}V^t_{D1}$, $V^t_{D6}V^t_{D5}$ and $V^t_{D10}V^t_{D11}$ have the same color. Looking closer, we can see that the edge $V^t_{D6}V^t_{D5}$ must have the same color as either $V^t_{A5}V^t_{B5}$ or $V^t_{A6}V^t_{B6}$ (by (7), (8) and parity). In the first case, the color passes (by (9)) to $V^t_{A3}V^t_{B3}$ and later to $V^t_{A1}V^t_{B1}$. But this is impossible, as $V^t_{B0}V^t_{B1}$ also has (by (9)) the same color as $V^t_{D0}V^t_{D1}$. The other case leads to a similar conflict between $V^t_{A10}V^t_{B10}$ and $V^t_{B10}V^t_{B11}$. This shows that the valuation defined in (10) must satisfy the formula.

Up to this end we know that the graph created from a given NAE-SAT formula is admissibly colorable iff the formula is satisfiable. From the construction of the graph it is clear, that the reduction needs only polynomial time. This ends the proof that NAESAT reduces polynomially to admissible coloring problem. □

What follows from the above theorem and Lemma 4.1 is the final result of our work:

Corollary 4.5. *The problem of deciding whether a given partial order is embeddable into two consecutive layers of some Boolean lattice is NP-complete.*

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