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STRONG NORMALIZATION OF A TYPED LAMBDA CALCULUS FOR INTUITIONISTIC BOUNDED LINEAR-TIME TEMPORAL LOGIC

A b s t r a c t. Linear-time temporal logics (LTLs) are known to be useful for verifying concurrent systems, and a simple natural deduction framework for LTLs has been required to obtain a good computational interpretation. In this paper, a typed λ-calculus $\lambda_{\mathrm{B}[l]}$ with a Curry-Howard correspondence is introduced for an intuitionistic bounded linear-time temporal logic B$[l]$, of which the time domain is bounded by a fixed positive integer l. The strong normalization theorem for $\lambda_{\mathrm{B}[l]}$ is proved as a main result. The base logic B$[l]$ is defined as a Gentzen-type sequent calculus, and despite the restriction on the time domain, B$[l]$ can derive almost all the typical temporal axioms of LTLs. The proposed framework allows us to obtain a uniform and simple proof-theoretical treatment of both natural deduction and sequent calculus, i.e., the equivalence between them, the cut-elimination theorem, the decidability theorem, the Curry-Howard correspondence and the strong normalization theorem can be obtained uniformly.

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1. Introduction

1.1 Why do we bound the time domain?

Linear-time temporal logics (LTLs) have been studied by many researchers [8, 12, 25], since LTLs are known to be useful for verifying and specifying concurrent systems. In this paper, a logic, *intuitionistic bounded linear-time temporal logic* B[l], is introduced as a Gentzen-type sequent calculus. This logic is regarded as a modified sublogic of a *constructive temporal logic* proposed in [17].[1] Although the standard LTLs have an infinite (unbounded) time domain, i.e., the set ω of natural numbers, the logic B[l] has a *bounded time domain* which is restricted by a fixed positive integer l, i.e., the set $\omega_l := \{x \in \omega \mid x \leq l\}$. Despite the restriction on the time domain, B[l] can derive almost all the typical temporal axioms of LTLs, such as a *time induction axiom.* Moreover, B[l] allows us to obtain a uniform and simple proof-theoretical treatment of both sequent calculus and natural deduction, i.e., the equivalence between them, the cut-elimination theorem, the decidability theorem, the Curry-Howard correspondence and the strong normalization theorem can uniformly be obtained in a standard way.

Such a theoretical merit may not be obtained for the standard LTLs with the unbounded time domain, since the unbounded domain requires some infinite inference rules. Such infinite rules may not be familiar with the researchers who study implementing automated reasoning, since these rules cannot be implemented as they are. Indeed, the replacement of such infinite rules of certain proof systems by finitely rules is known as an important issue.

To restrict the time domain in LTLs is not a new idea. Such an idea was discussed in [4, 6, 7, 13]. For example, by using and introducing a bounded time domain and the notion of bounded validity in a semantics, *bounded tableaux calculi* (with temporal constraints) for propositional and first-order

[1] In [17], two constructive and bounded versions of LTL, which are extensions of intuitionistic logic and Nelson's paraconsistent logic, were introduced. Cut-free Gentzen-type sequent calculi, cut-free display calculi, Gentzen-type tree-style natural deduction systems and complete Kripke semantics were obtained for these logics. However, this framework does not fit for obtaining a strongly normalizable typed λ-calculus with a Curry-Howard correspondence.

LTLs were introduced by Cerrito, Mayer and Prand [6, 7]. It is also known that to restrict the time domain is a technique to obtain a decidable or efficient fragment of LTLs [13]. Restricting the time domain implies not only some purely theoretical merits discussed above, but also some practical merits for describing temporal databases and planning specifications [6, 7], and for implementing an efficient model checking algorithm called a *bounded model checking* [4]. Such practical merits are due to the fact that there are problems in computer science and artificial intelligence where only a finite fragment of the time sequence is of interest [6].

1.2 Why do we use intuitionistic logic as a base logic?

In classical logic, the law of excluded middle $\alpha \vee \neg\alpha$ is valid. This means that the information which is represented by classical logic is *complete information.* Such a situation representing complete information is plausible in mathematics world handling eternal truth, but the same situation is not valid in our real world. We wish to explore the consequences of *partial (or incomplete) information* about computer and information systems, and then we are desirable to have a logic which allows us to handle partial information [29]. For this motivation, intuitionistic logic rather than classical logic is needed as a base logic for temporal reasoning. Indeed, *intuitionistic (or constructive) modal and temporal logics* have been studied by many researchers. *Constructive concurrent dynamic logic* by Wijesekera and Nerode [29] is an example of such logics. The present paper's approach is regarded as one of the approaches dealing with partial information in temporal reasoning. Although a classical version of B[l] can similarly be considered, a partial information handling, a simple computational interpretation by natural deduction and a simple Curry-Howard correspondence cannot be obtained for such a classical version. This is the reason why we adopt intuitionistic logic as a base logic.

1.3 Sequent calculus

Sequent calculi for LTLs have been introduced and studied by many researchers [1, 15, 16, 18, 23, 24, 26, 27]. A sequent calculus LT_ω for an until-free version of LTLs was introduced by Kawai, and the cut-elimination and completeness theorems for this calculus were proved [18]. It was also shown in [18] that (the first-order) LT_ω is equivalent to *Kröger's infinitary temporal logic* [20, 26]. A *2-sequent calculus* $2S\omega$ for an until-free version of LTLs, which is a natural extension of the usual sequent calculus, was introduced by Baratella and Masini, and the cut-elimination and completeness theorems for this calculus were proved based on an analogy between $2S\omega$ and Peano arithmetic endowed with ω-rule [1]. A direct syntactic equivalence between Kawai's LT_ω and Baratella-Masini's $2S\omega$ was shown by introducing the translation functions that preserve cut-free proofs of these calculi [15]. Moreover, an embedding from LT_ω into a sequent calculus for *infinitary logic* is presented in [16].

In the present paper, an intuitionistic and bounded version $B[l]$ of LT_ω, which has an embedding into intuitionistic logic rather than infinitary logic, is studied. Although LT_ω characterizes the Hilbert-style axiom scheme for the temporal operators G (globally) and X (next): $G\alpha \leftrightarrow (\alpha \wedge X\alpha \wedge X^2\alpha \wedge \cdots \infty)$ where $X^i\alpha$ means $\overbrace{XX\cdots X}^{i}\alpha$, the logic $B[l]$ characterizes the Hilbert-style axiom scheme: $G\alpha \leftrightarrow (\alpha \wedge X\alpha \wedge X^2\alpha \wedge \cdots \wedge X^l\alpha)$, which is regarded as a finite approximation of the original one. Then, the following very informal correspondences are useful to understand these systems: $G\alpha$ in LT_ω corresponds to the infinite conjunction $\bigwedge_{j=0}^{\infty} X^j\alpha$ in infinitary logic, and $G\alpha$ in $B[l]$ corresponds to the finite conjunction $\bigwedge_{j=0}^{l} X^j\alpha$ in intuitionistic logic.

1.4 Natural deduction

Natural deduction systems and typed λ-calculi for LTLs and related modal logics have recently been studied by many researchers [2, 3, 5, 9, 10, 19, 21, 22, 28, 30] to obtain a basis of *staged computation* in multi-level programs.

From the purely proof-theoretical point of view, a natural deduction system PNJ for an intuitionistic LTL, which is called a *logic of positions*, was introduced by Baratella and Masini, and the strong normalization theorem for PNJ was proved [2]. The system PNJ is based on the notion of *position formulas*, and has an induction inference rule concerning a time induction axiom. A proposed natural deduction system $N_{B[l]}$ in the present paper is a bit similar to a fragment of PNJ, but $N_{B[l]}$ does not use the notion of position formulas and the induction inference rule.

An indexed formula based natural deduction PLTL_{ND} for the full classical LTL with until operator was also studied by Bolotov et al. [5]. The completeness theorem for PLTL_{ND} was shown by them, but the strong normalization for it was not shown.

A labelled natural deduction system LND-$\mathrm{K}_t4.3$ for linear temporal $\mathrm{K}_t4.3$ logic was introduced by Indrzejczak [14]. This system is more similar to labelled tableau systems than to standard natural deduction. In [14], the completeness, decidability and cut-elimination theorems for LND-$\mathrm{K}_t4.3$ were shown.

From the application point of view, a typed λ-calculus λ° (with a next-time operator $\bigcirc$) for a fragment of an intuitionistic LTL was proposed by Davies [9] to discuss *multi-level binding-time analysis.* An extension MetaML of λ° with the addition of the properties of run-time generation and persistent code was introduced by Taha et al. [28]. An extension AIM (an idealized MetaML) of MetaML was developed by Moggi et al. [21], and a refinement λ^{BN} of AIM was proposed by Benaissa et al. [3].

An alternative typed λ-calculus $\lambda^{[]}$ (with an S4-type modal operator $[]$) for an intuitionistic S4-modal logic was also introduced by Davies and Pfenning [10] in order to analyse staged computation. Some type systems based on $\lambda^{[]}$ were studied by Nanevski [22] and Kim et al. [19]. A type system $\lambda^{\circ[]}$ that includes both λ° and $\lambda^{[]}$ was introduced by Yuse and Igarashi [30] to handle both *persistent code* (by $[]$) and *ephemeral code* (by $\bigcirc$).

Although the basic formulation of the proposed calculus $\lambda_{\mathrm{B}[l]}$ is different from that of $\lambda^{\circ[]}$, the calculus $\lambda_{\mathrm{B}[l]}$ includes the purely temporal logic part of $\lambda^{\circ[]}$, since the standard temporal axioms and the characteristic axiom of $\lambda^{\circ[]}$: $[]\bigcirc\alpha \leftrightarrow \bigcirc[]\alpha$ (i.e., $\mathrm{GX}\alpha \Leftrightarrow \mathrm{XG}\alpha$) are both provable in $\mathrm{B}[l]$. It is also mentioned that the essential part of $\lambda_{\mathrm{B}[l]}$ is considerably simpler than $\lambda^{\circ[]}$ and other proposals.

1.5 Summary of this paper

In Section 2, $\mathrm{B}[l]$ and its properties are discussed. Some typical examples of provable sequents in $\mathrm{B}[l]$ are addressed. The embedding theorem of $\mathrm{B}[l]$ into a sequent calculus for a fragment of intuitionistic logic is presented. By using this theorem, the cut-elimination and decidability theorems for

B[l] are shown.

In Section 3, natural deduction formulations for B[l] are introduced. First, a tree-style natural deduction system $\mathrm{N}_{\mathrm{B}[l]}$ for B[l] is introduced, and the equivalence between $\mathrm{N}_{\mathrm{B}[l]}$ and B[l] is presented. Second, a typed λ-calculus $\lambda_{\mathrm{B}[l]}$ and a type assignment system $\mathrm{T}_{\mathrm{B}[l]}$ are introduced for $\mathrm{N}_{\mathrm{B}[l]}$ in order to observe a Curry-Howard correspondence.

In Section 4, the strong normalization theorem for $\lambda_{\mathrm{B}[l]}$ is proved by using a standard method presented in the textbook [11].

In Section 5, we give some comparisons among the systems proposed in this paper, PNJ by Baratella and Masini, $\lambda^{\circ[]}$ by Yuse and Igarashi and PLTL_{ND} by Bolotov et al.

2. Sequent calculus and cut-elimination

Formulas of B[l] are constructed from countably many propositional variables, $\to$ (implication), $\wedge$ (conjunction), X (next) and G (globally). Lower-case letters $p, q, ...$ are used to denote propositional variables, Greek lower-case letters $\alpha, \beta, ...$ are used to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used to represent finite (possibly empty) sequences of formulas. For any $\sharp \in \{\mathrm{X}, \mathrm{G}\}$, an expression $\sharp\Gamma$ is used to denote the sequence $\langle \sharp\gamma \mid \gamma \in \Gamma \rangle$. Parentheses for $\wedge$ are omitted since $\wedge$ is associative. The symbol $\equiv$ is used to denote the equality of sequences of symbols. The symbol ω is used to represent the set of natural numbers. Let l be a finite fixed positive integer. Then, the symbol ω_l is used to represent the set $\{i \in \omega \mid i \leq l\}$. An expression $\mathrm{X}^i\alpha$ for any $i \in \omega$ is defined inductively by $(\mathrm{X}^0\alpha \equiv \alpha)$ and $(\mathrm{X}^{n+1}\alpha \equiv \mathrm{X}\mathrm{X}^n\alpha)$. Lower-case letters i and j are used to denote any natural numbers. An expression of the form $\Gamma \Rightarrow \gamma$ where γ is a formula is called a *sequent*. An expression $L \vdash S$ or $\vdash S$ is used to denote the fact that a sequent S is provable in a sequent calculus L. A rule R of inference is said to be *admissible* in a sequent calculus L if the following condition is satisfied: for any instance

$$\frac{S_1 \quad \cdots \quad S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$.

Definition 2.1 (B[l]). Let l be a fixed finite positive integer. The initial sequents of B[l] are of the form: for any propositional variable p,

$$\mathrm{X}^i p \Rightarrow \mathrm{X}^i p.$$

The structural rules of B[l] are of the form:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \gamma}{\Gamma, \Sigma \Rightarrow \gamma} \text{ (cut)} \qquad \frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} \text{ (we)}$$

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} \text{ (co)} \qquad \frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Sigma \Rightarrow \gamma} \text{ (ex).}$$

The logical inference rules of B[l] are of the form: for any $k \in \omega_l$ and any positive integer m,

$$\frac{\Gamma \Rightarrow \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Sigma \Rightarrow \gamma}{\mathrm{X}^i(\alpha{\to}\beta), \Gamma, \Sigma \Rightarrow \gamma} \text{ (}{\to}\text{left)} \qquad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha{\to}\beta)} \text{ (}{\to}\text{right)}$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \gamma}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow \gamma} \text{ (}\wedge\text{left1)} \qquad \frac{\mathrm{X}^i\beta, \Gamma \Rightarrow \gamma}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow \gamma} \text{ (}\wedge\text{left2)}$$

$$\frac{\Gamma \Rightarrow \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha \wedge \beta)} \text{ (}\wedge\text{right)}$$

$$\frac{\mathrm{X}^l\alpha, \Gamma \Rightarrow \gamma}{\mathrm{X}^{l+m}\alpha, \Gamma \Rightarrow \gamma} \text{ (Xleft)} \qquad \frac{\Gamma \Rightarrow \mathrm{X}^l\alpha}{\Gamma \Rightarrow \mathrm{X}^{l+m}\alpha} \text{ (Xright)}$$

$$\frac{\mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \gamma}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow \gamma} \text{ (Gleft)} \qquad \frac{\{\ \Gamma \Rightarrow \mathrm{X}^{i+j}\alpha\ \}_{j \in \omega_l}}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha} \text{ (Gright).}$$

Definition 2.2 (LJ). A sequent calculus LJ for the $\{\to, \wedge\}$-fragment of intuitionistic logic is obtained from B[l] by deleting (Xleft), (Xright), (Gleft), (Gright), and replacing X^i by X^0. The modified inference rules for LJ by replacing i by 0 are denoted by labelling "LJ" in superscript, e.g., ($\to$left^{LJ}).

It is noted that (Gright) has $l+1$ (i.e., finite) premises, e.g., in the case $l = 3$, (Gright) has four premises:

$$\frac{\Gamma \Rightarrow \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \mathrm{X}^{i+1}\alpha \quad \Gamma \Rightarrow \mathrm{X}^{i+2}\alpha \quad \Gamma \Rightarrow \mathrm{X}^{i+3}\alpha}{\Gamma \Rightarrow \mathrm{X}^i\mathrm{G}\alpha} \text{ (Gright).}$$

In (Gleft), the number k is bounded by l. Then, B[l] has the Hilbert-style axiom scheme $\mathrm{G}\alpha \leftrightarrow (\alpha \wedge \mathrm{X}\alpha \wedge \mathrm{X}^2\alpha \wedge \cdots \wedge \mathrm{X}^l\alpha)$. By (Xleft) and (Xright), the nest of the outermost occurrence of X in a formula can be bounded by l. Indeed, (Xleft) and (Xright) correspond to the Hilbert-style axiom scheme $\mathrm{X}^{l+m}\alpha \leftrightarrow \mathrm{X}^l\alpha$.

It is remarked that for any formula α, the sequent of the form $\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha$ is provable in B[l]. This can be shown by induction on α. Thus, the sequents of the form $\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha$ can also be regarded as initial sequents.

It is remarked that B[l] is regarded as an intuitionistic and bounded version of Kawai's sequent calculus LT_ω for linear-time temporal logic [18]. LT_ω has no l-bounded rules such as (Xleft/right), and use ω instead of ω_l.

It is remarked that B[l] is just a logic parameterized by a fixed concrete positive integer l. Thus, before the detailed discussion, we have to fix B[l] as a concrete logic such as B[5]. Indeed, for example, B[2] is different from B[1]: $p \wedge \mathrm{X}p \Rightarrow \mathrm{G}p$ is provable in B[1], but it is not provable in B[2]. The unprovability of sequents is guaranteed by the cut-elimination theorem (Theorem 2.8), which will be proved in this section.

Proposition 2.3. *Let m and n be distinct fixed positive integers. The logics* B[m] *and* B[n] *are not theorem-equivalent.*

Proof. By Theorem 2.8. □

An expression $\alpha \Leftrightarrow \beta$ means the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$.

Proposition 2.4. *The following sequents are provable in* B[l]*: for any formulas α, β, any $i \in \omega$ and any positive integer m,*

1. $\mathrm{X}^i(\alpha \circ \beta) \Leftrightarrow \mathrm{X}^i\alpha \circ \mathrm{X}^i\beta$ *where* $\circ \in \{\rightarrow, \wedge\}$,
2. $\mathrm{X}^i\mathrm{G}\alpha \Leftrightarrow \mathrm{G}\mathrm{X}^i\alpha$,
3. $\mathrm{G}\alpha \Rightarrow \mathrm{X}\alpha$,
4. $\mathrm{G}\alpha \Rightarrow \mathrm{X}\mathrm{G}\alpha$,
5. $\mathrm{G}\alpha \Rightarrow \mathrm{G}\mathrm{G}\alpha$.
6. $\alpha, \mathrm{G}(\alpha \rightarrow \mathrm{X}\alpha) \Rightarrow \mathrm{G}\alpha$ *(time induction),*
7. $\mathrm{X}^{l+m}\alpha \Leftrightarrow \mathrm{X}^l\alpha$ *(bounded next-time),*

8. $\mathrm{G}\alpha \Leftrightarrow \bigwedge_{0\le j\le l} \mathrm{X}^j\alpha$ *(bounded globally).*

Proof. We show some cases.

(4):

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \mathrm{X}\alpha \Rightarrow \mathrm{X}\alpha\end{array}}{\mathrm{G}\alpha \Rightarrow \mathrm{X}\alpha}\,(\mathrm{Gleft}) \quad \dfrac{\begin{array}{c}\vdots\\ \mathrm{X}^2\alpha \Rightarrow \mathrm{X}^2\alpha\end{array}}{\mathrm{G}\alpha \Rightarrow \mathrm{X}^2\alpha}\,(\mathrm{Gleft}) \quad \cdots \quad \dfrac{\begin{array}{c}\vdots\\ \mathrm{X}^{l+1}\alpha \Rightarrow \mathrm{X}^{l+1}\alpha\end{array}}{\mathrm{G}\alpha \Rightarrow \mathrm{X}^{l+1}\alpha}\,(\mathrm{Gleft})}{\mathrm{G}\alpha \Rightarrow \mathrm{XG}\alpha}\,(\mathrm{Gright}).$$

(5):

$$\dfrac{\mathrm{G}\alpha \Rightarrow \mathrm{G}\alpha \quad \begin{array}{c}\vdots\\ \mathrm{G}\alpha \Rightarrow \mathrm{XG}\alpha\end{array} \quad \begin{array}{c}\vdots\\ \mathrm{G}\alpha \Rightarrow \mathrm{X}^2\mathrm{G}\alpha\end{array} \quad \cdots \quad \begin{array}{c}\vdots\\ \mathrm{G}\alpha \Rightarrow \mathrm{X}^l\mathrm{G}\alpha\end{array}}{\mathrm{G}\alpha \Rightarrow \mathrm{GG}\alpha}\,(\mathrm{Gright})$$

where $\vdash \mathrm{G}\alpha \Rightarrow \mathrm{X}^i\mathrm{G}\alpha$ for any $i \in \omega_l$ can be shown in a similar way as in (4).

(6): In the following proofs, the applications of (ex) are omitted.

$$\dfrac{\begin{array}{c}\vdots\\ \{\alpha, \mathrm{G}(\alpha{\to}\mathrm{X}\alpha) \Rightarrow \mathrm{X}^k\alpha\}_{k\in\omega_l}\end{array}}{\alpha, \mathrm{G}(\alpha{\to}\mathrm{X}\alpha) \Rightarrow \mathrm{G}\alpha}\,(\mathrm{Gright})$$

where $\vdash \alpha, \mathrm{G}(\alpha{\to}\mathrm{X}\alpha) \Rightarrow \mathrm{X}^k\alpha$ for any $k \in \omega_l$ is shown by mathematical induction on k as follows: the base step is obvious, and the induction step can be shown by

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \text{ind.hyp.}\\ \alpha, \mathrm{G}(\alpha{\to}\mathrm{X}\alpha) \Rightarrow \mathrm{X}^k\alpha\end{array} \quad \mathrm{X}^{k+1}\alpha \Rightarrow \mathrm{X}^{k+1}\alpha}{\alpha, \mathrm{G}(\alpha{\to}\mathrm{X}\alpha), \mathrm{X}^k(\alpha{\to}\mathrm{X}\alpha) \Rightarrow \mathrm{X}^{k+1}\alpha}\,({\to}\mathrm{left})}{\alpha, \mathrm{G}(\alpha{\to}\mathrm{X}\alpha), \mathrm{G}(\alpha{\to}\mathrm{X}\alpha) \Rightarrow \mathrm{X}^{k+1}\alpha}\,(\mathrm{Gleft})}{\alpha, \mathrm{G}(\alpha{\to}\mathrm{X}\alpha) \Rightarrow \mathrm{X}^{k+1}\alpha}\,(\mathrm{co}).$$

(7):

$$\dfrac{\begin{array}{c}\vdots\\ \mathrm{X}^l\alpha \Rightarrow \mathrm{X}^l\alpha\end{array}}{\mathrm{X}^{l+m}\alpha \Rightarrow \mathrm{X}^l\alpha}\,(\mathrm{Xleft}) \qquad \dfrac{\begin{array}{c}\vdots\\ \mathrm{X}^l\alpha \Rightarrow \mathrm{X}^l\alpha\end{array}}{\mathrm{X}^l\alpha \Rightarrow \mathrm{X}^{l+m}\alpha}\,(\mathrm{Xright}).$$

(8):

$$\dfrac{\begin{array}{c}\vdots\\ \{\ \mathrm{X}^j\alpha \Rightarrow \mathrm{X}^j\alpha\ \}_{0\le j\le l}\end{array}}{\begin{array}{c}\{\ \mathrm{G}\alpha \Rightarrow \mathrm{X}^j\alpha\ \}_{0\le j\le l}\\ \vdots\ (\wedge\text{right})\\ \mathrm{G}\alpha \Rightarrow \bigwedge_{0\le j\le l}\mathrm{X}^j\alpha\end{array}}\ (\text{Gleft}) \qquad \dfrac{\begin{array}{c}\vdots\\ \{\ \mathrm{X}^k\alpha \Rightarrow \mathrm{X}^k\alpha\ \}_{k\in\omega_l}\\ \vdots\ (\wedge\text{left}1,2)\\ \{\ \bigwedge_{0\le j\le l}\mathrm{X}^j\alpha \Rightarrow \mathrm{X}^k\alpha\ \}_{k\in\omega_l}\end{array}}{\bigwedge_{0\le j\le l}\mathrm{X}^j\alpha \Rightarrow \mathrm{G}\alpha}\ (\text{Gright}).$$

□

Proposition 2.5. *The following rule is admissible in cut-free* B$[l]$*:*

$$\frac{\Gamma \Rightarrow \gamma}{\mathrm{X}\Gamma \Rightarrow \mathrm{X}\gamma}\ (\text{Xregu}).$$

Proof. By induction on the proofs P of $\Gamma \Rightarrow \gamma$ in cut-free B$[l]$. We distinguish the cases according to the last inference of P. We show some cases.

Case (Gleft): The last inference of P is of the form:

$$\frac{\mathrm{X}^{i+k}\alpha, \Delta \Rightarrow \gamma}{\mathrm{X}^i\mathrm{G}\alpha, \Delta \Rightarrow \gamma}\ (\text{Gleft}).$$

By induction hypothesis, we obtain:

$$\dfrac{\begin{array}{c}\vdots\\ \mathrm{X}\mathrm{X}^{i+k}\alpha, \mathrm{X}\Delta \Rightarrow \mathrm{X}\gamma\end{array}}{\mathrm{X}\mathrm{X}^i\mathrm{G}\alpha, \mathrm{X}\Delta \Rightarrow \mathrm{X}\gamma}\ (\text{Gleft}).$$

Case ($\rightarrow$left): The last inference of P is of the form:

$$\frac{\Pi \Rightarrow \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Delta \Rightarrow \gamma}{\mathrm{X}^i(\alpha\rightarrow\beta), \Pi, \Delta \Rightarrow \gamma}\ (\rightarrow\text{left}).$$

By induction hypothesis, we obtain:

$$\dfrac{\begin{array}{c}\vdots\\ \mathrm{X}\Pi \Rightarrow \mathrm{X}\mathrm{X}^i\alpha\end{array} \quad \begin{array}{c}\vdots\\ \mathrm{X}\mathrm{X}^i\beta, \mathrm{X}\Delta \Rightarrow \mathrm{X}\gamma\end{array}}{\mathrm{X}\mathrm{X}^i(\alpha\rightarrow\beta), \mathrm{X}\Pi, \mathrm{X}\Delta \Rightarrow \mathrm{X}\gamma}\ (\rightarrow\text{left}).$$

□

An expression like $\bigwedge\{\alpha_i \mid i \in \omega_l\}$ where $\{\alpha_i \mid i \in \omega_l\}$ is a multiset means $\alpha_0 \wedge \alpha_1 \wedge \cdots \wedge \alpha_l$. For example, $\bigwedge\{\alpha, \alpha, \beta\}$ means $\alpha \wedge \alpha \wedge \beta$.

The following definition of the mapping f is regarded as a finite analogue of the definition of the mapping of Kawai's LT_ω into infinitary logic [16].

Definition 2.6. We fix a countable non-empty set Φ of propositional variables, and define the sets $\Phi_i := \{p_i \mid p \in \Phi\}$ $(1 \leq i \in \omega)$ and $\Phi_0 := \Phi$ of propositional variables. The language $\mathcal{L}_{\mathrm{B}[l]}$ of $\mathrm{B}[l]$ is defined by using Φ, $\rightarrow, \wedge$, X and G. The language $\mathcal{L}_{\mathrm{LJ}}$ of LJ is defined by using $\bigcup_{i \in \omega} \Phi_i$, $\rightarrow$ and $\wedge$.

A mapping f from $\mathcal{L}_{\mathrm{B}[l]}$ to $\mathcal{L}_{\mathrm{LJ}}$ is defined by the following: for any $i \in \omega$ and any positive integer m,

1. $f(\mathrm{X}^i p) := p_i \in \Phi_i$ for any $p \in \Phi$ (especially, $f(p) := p \in \Phi_0$),
2. $f(\mathrm{X}^i(\alpha \circ \beta)) := f(\mathrm{X}^i\alpha) \circ f(\mathrm{X}^i\beta)$ where $\circ \in \{\rightarrow, \wedge\}$,
3. $f(\mathrm{X}^{l+m}\alpha) := f(\mathrm{X}^l\alpha)$,
4. $f(\mathrm{X}^i\mathrm{G}\alpha) := \bigwedge\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega_l\}$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$.

Strictly speaking, the mapping f is strongly dependent on the time bound l, i.e., f should be denoted as f_l. Indeed, $f_3(\mathrm{G}p)$ and $f_5(\mathrm{G}p)$ are different. But, for the sake of brevity, a simple expression f will be used in the following.

Theorem 2.7 (Embedding). *Let Γ be a sequence of formulas in $\mathcal{L}_{\mathrm{B}[l]}$, γ be a formula in $\mathcal{L}_{\mathrm{B}[l]}$, and f be the mapping defined in Definition 2.6.*

1. $\mathrm{B}[l] \vdash \Gamma \Rightarrow \gamma$ *iff* $\mathrm{LJ} \vdash f(\Gamma) \Rightarrow f(\gamma)$.

2. $\mathrm{B}[l] - (\mathrm{cut}) \vdash \Gamma \Rightarrow \gamma$ *iff* $\mathrm{LJ} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\gamma)$.

Proof. Since the case (2) can be obtained as a subproof of the case (1), we show only (1).

• ($\Longrightarrow$): By induction on a proof P of $\Gamma \Rightarrow \gamma$ in $\mathrm{B}[l]$. We distinguish the cases according to the last inference of P, and show some cases.

Case ($\mathrm{X}^i p \Rightarrow \mathrm{X}^i p$). The last inference of P is of the form: $\mathrm{X}^i p \Rightarrow \mathrm{X}^i p$. In this case, we obtain $\mathrm{LJ} \vdash f(\mathrm{X}^i p) \Rightarrow f(\mathrm{X}^i p)$, i.e., $\mathrm{LJ} \vdash p_i \Rightarrow p_i$ $(p_i \in \Phi_i)$.

Case (Xleft). The last inference of P is of the form:

$$\frac{\mathrm{X}^{l}\alpha, \Gamma \Rightarrow \gamma}{\mathrm{X}^{l+m}\alpha, \Gamma \Rightarrow \gamma} \text{ (Xleft)}.$$

By the hypothesis of induction, we have $\mathrm{LJ} \vdash f(\mathrm{X}^{l}\alpha), f(\Gamma) \Rightarrow f(\gamma)$, and $f(\mathrm{X}^{l}\alpha) = f(\mathrm{X}^{l+m}\alpha)$. Thus, we obtain $\mathrm{LJ} \vdash f(\mathrm{X}^{l+m}\alpha), f(\Gamma) \Rightarrow f(\gamma)$.

Case (Gleft). The last inference of P is of the form:

$$\frac{\mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \gamma}{\mathrm{X}^{i}\mathrm{G}\alpha, \Gamma \Rightarrow \gamma} \text{ (Gleft)}.$$

By the hypothesis of induction, we have $\mathrm{LJ} \vdash f(\mathrm{X}^{i+k}\alpha), f(\Gamma) \Rightarrow f(\gamma)$, and hence obtain:

$$\begin{array}{c} \vdots \\ f(\mathrm{X}^{i+k}\alpha), f(\Gamma) \Rightarrow f(\gamma) \\ \vdots \ (\wedge\text{left}^{LJ}) \\ \bigwedge\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega_l\}, f(\Gamma) \Rightarrow f(\gamma) \end{array}$$

where $\bigwedge\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega_l\} = f(\mathrm{X}^{i}\mathrm{G}\alpha)$, and $f(\mathrm{X}^{i+k}\alpha)$ is in the multiset $\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega_l\}$. It is remarked that the case $i > l$ is also included in this proof. In such a case, $f(\mathrm{X}^{i+k}\alpha)$ and $\bigwedge\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega_l\}$ mean $f(\mathrm{X}^{l}\alpha)$ and $\overbrace{f(\mathrm{X}^{l}\alpha) \wedge f(\mathrm{X}^{l}\alpha) \wedge \cdots \wedge f(\mathrm{X}^{l}\alpha)}^{l}$, respectively.

Case (Gright). The last inference of P is of the form:

$$\frac{\{\ \Gamma \Rightarrow \mathrm{X}^{i+j}\alpha\ \}_{j \in \omega_l}}{\Gamma \Rightarrow \mathrm{X}^{i}\mathrm{G}\alpha} \text{ (Gright)}.$$

By the hypothesis of induction, we have $\mathrm{LJ} \vdash f(\Gamma) \Rightarrow f(\mathrm{X}^{i+j}\alpha)$ for all $j \in \omega_l$. Let Φ be the multiset $\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega_l\}$. We obtain

$$\begin{array}{c} \vdots \\ \{\ f(\Gamma) \Rightarrow f(\mathrm{X}^{i+j}\alpha)\ \}_{f(\mathrm{X}^{i+j}\alpha) \in \Phi} \\ \vdots \ (\wedge\text{right}^{LJ}) \\ f(\Gamma) \Rightarrow \bigwedge \Phi \end{array}$$

where $\bigwedge \Phi = f(\mathrm{X}^{i}\mathrm{G}\alpha)$.

• ($\Longleftarrow$) : By induction on the proofs Q of $f(\Gamma) \Rightarrow f(\gamma)$ in LJ. We distinguish the cases according to the last inference of Q, and show some cases.

Case (cut). The last inference of Q is of the form:

$$\frac{f(\Gamma_1) \Rightarrow \beta \quad \beta, f(\Gamma_2) \Rightarrow f(\gamma)}{f(\Gamma_1), f(\Gamma_2) \Rightarrow f(\gamma)} \text{ (cut).}$$

Since β is in $\mathcal{L}_{\mathrm{LJ}}$, we have the fact $\beta = f(\beta)$. This fact can be shown by induction on β. Then, by induction hypothesis, we have: $\mathrm{B}[l] \vdash \Gamma_1 \Rightarrow \beta$ and $\mathrm{B}[l] \vdash \beta, \Gamma_2 \Rightarrow \gamma$. We then obtain the required fact: $\mathrm{B}[l] \vdash \Gamma_1, \Gamma_2 \Rightarrow \gamma$ by using (cut) in $\mathrm{B}[l]$.

Case ($\wedge$right^{LJ}). The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\mathrm{X}^i\alpha) \quad f(\Gamma) \Rightarrow f(\mathrm{X}^i\beta)}{f(\Gamma) \Rightarrow f(\mathrm{X}^i(\alpha \wedge \beta))} \ (\wedge\text{right}^{LJ})$$

where $f(\mathrm{X}^i(\alpha \wedge \beta)) = f(\mathrm{X}^i\alpha) \wedge f(\mathrm{X}^i\beta)$. By the hypothesis of induction, we have $\mathrm{B}[l] \vdash \Gamma \Rightarrow \mathrm{X}^i\alpha$ and $\mathrm{B}[l] \vdash \Gamma \Rightarrow \mathrm{X}^i\beta$. Then we obtain

$$\frac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \mathrm{X}^i\alpha\end{matrix} \quad \begin{matrix}\vdots \\ \Gamma \Rightarrow \mathrm{X}^i\beta\end{matrix}}{\Gamma \Rightarrow \mathrm{X}^i(\alpha \wedge \beta)} \ (\wedge\text{right}).$$

□

Theorem 2.8 (Cut-elimination)**.** *The rule* (cut) *is admissible in cut-free* $\mathrm{B}[l]$.

Proof. Suppose $\mathrm{B}[l] \vdash \Gamma \Rightarrow \gamma$. Then, we have $\mathrm{LJ} \vdash f(\Gamma) \Rightarrow f(\gamma)$ by Theorem 2.7 (1), and hence $\mathrm{LJ} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\gamma)$ by the well-known cut-elimination theorem for LJ. By Theorem 2.7 (2), we obtain $\mathrm{B}[l] - (\mathrm{cut})$ $\vdash \Gamma \Rightarrow \gamma$. □

Theorem 2.9 (Decidability)**.** $\mathrm{B}[l]$ *is decidable.*

Proof. By Theorem 2.7, the provability of $\mathrm{B}[l]$ can be transformed into that of LJ. Since LJ is decidable, $\mathrm{B}[l]$ is also decidable. □

3. Natural deduction

3.1 $\mathbf{N}_{\mathrm{B}[l]}$

Definition 3.1 ($\mathbf{N}_{\mathrm{B}[l]}$)**.** The inference rules of $\mathrm{N}_{\mathrm{B}[l]}$ are of the form: for any $k \in \omega_l$ and any positive integer m,

$$\cfrac{\begin{matrix}[\mathrm{X}^i\alpha] \\ \vdots \\ \mathrm{X}^i\beta\end{matrix}}{\mathrm{X}^i(\alpha{\to}\beta)}\ ({\to}I) \qquad \frac{\mathrm{X}^i(\alpha{\to}\beta) \quad \mathrm{X}^i\alpha}{\mathrm{X}^i\beta}\ ({\to}E)$$

$$\frac{\mathrm{X}^i\alpha \quad \mathrm{X}^i\beta}{\mathrm{X}^i(\alpha\wedge\beta)}\ (\wedge I) \qquad \frac{\mathrm{X}^i(\alpha\wedge\beta)}{\mathrm{X}^i\alpha}\ (\wedge E1) \qquad \frac{\mathrm{X}^i(\alpha\wedge\beta)}{\mathrm{X}^i\beta}\ (\wedge E2)$$

$$\frac{\mathrm{X}^l\alpha}{\mathrm{X}^{l+m}\alpha}\ (\mathrm{X}I) \qquad \frac{\mathrm{X}^{l+m}\alpha}{\mathrm{X}^l\alpha}\ (\mathrm{X}E)$$

$$\frac{\{\ \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega_l}}{\mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{G}I) \qquad \frac{\mathrm{X}^i\mathrm{G}\alpha}{\mathrm{X}^{i+k}\alpha}\ (\mathrm{G}E).$$

Any proofs constructed only on an assumption are considered to be axioms.

The terminologies of the standard natural deduction system are used. The notions of proof (of $\mathrm{N}_{\mathrm{B}[l]}$), open and discharged assumptions of proof, and end-formula of proof are defined as usual. A formula α is said to be provable in $\mathrm{N}_{\mathrm{B}[l]}$ if there exists a proof of $\mathrm{N}_{\mathrm{B}[l]}$ with no open assumption whose end-formula is α.

Let P be a proof. Then, the expression $\mathrm{oa}(P)$ denotes the set of open assumptions of P, and the expression $\mathrm{end}(P)$ denotes the end-formula of P.

Proposition 3.2 (Equivalence between $\mathbf{B}[l]$ and $\mathbf{N}_{\mathrm{B}[l]}$)**.** *Let Γ be a sequence of formulas, and $\{\Gamma\}$ be the set of formulas in Γ.*

1. *If P is a proof in $\mathrm{N}_{\mathrm{B}[l]}$ such that $\mathrm{oa}(P) = \{\Gamma\}$ and $\mathrm{end}(P) = \beta$, then the sequent $\Gamma \Rightarrow \beta$ is provable in $\mathrm{B}[l]$.*

2. *If a sequent $\Gamma \Rightarrow \beta$ is provable in $\mathrm{B}[l]$, then there is a proof Q in $\mathrm{N}_{\mathrm{B}[l]}$ which satisfies $\mathrm{oa}(Q) = \{\Gamma\}$ and $\mathrm{end}(Q) = \beta$.*

Although the reduction relation on the set of proofs of $\mathrm{N}_{\mathrm{B}[l]}$ can naturally be defined and the strong normalization theorem for $\mathrm{N}_{\mathrm{B}[l]}$ can also be shown, such a discussion is omitted since the strong normalization theorem will be proved for the corresponding typed λ-calculus $\lambda_{\mathrm{B}[l]}$. By the Curry-Howard correspondence, the strong normalization theorem for $\mathrm{N}_{\mathrm{B}[l]}$ is derived from that of $\lambda_{\mathrm{B}[l]}$.

3.2 $\lambda_{\mathrm{B}[l]}$

Terms are constructed from variables, a λ-abstraction λ, an application operator $\cdot$ (it is always omitted), a pairing function $\langle , \rangle$, an $(l+1)$-ary pairing function $\langle , ..., \rangle$, projection functions $\pi_1, \pi_2, ..., \pi_{l+1}$, and two new constructors ι, ι^{-1} concerning X, called *time-bounded functions*. The intended meaning of ι and ι^{-1} can be presented as the equations: $(\iota^{-1}(\iota M^{\mathrm{X}^l\alpha})^{\mathrm{X}^{l+m}\alpha})^{\mathrm{X}^l\alpha} = M^{\mathrm{X}^l\alpha}$ and $(\iota(\iota^{-1}M^{\mathrm{X}^{l+m}\alpha})^{\mathrm{X}^l\alpha})^{\mathrm{X}^{l+m}\alpha} = M^{\mathrm{X}^{l+m}\alpha}$, which are the analogues of the equations with respect to $\langle , \rangle$ and π_i: $(\pi_1\langle M^\alpha, N^\beta\rangle^{\alpha\wedge\beta})^\alpha = M^\alpha$ and $\langle(\pi_1 M^{\alpha\wedge\beta})^\alpha, (\pi_2 M^{\alpha\wedge\beta})^\beta\rangle^{\alpha\wedge\beta} = M^{\alpha\wedge\beta}$. *Types* are constructed from atomic types (denoted as $p, q, ...$), $\rightarrow$, $\wedge$, X and G. Variables are denoted as $x, x_n, y, ...$, untyped terms are denoted as $M, M_n, N, ...$, types are denoted as $\alpha, \beta, \gamma, ...$, and typed terms are denoted as $M^\alpha, N^\beta, L^\gamma$, ... Typed terms are sometimes denoted as $M, N, L, ...$ by omitting the types. It is assumed that in a λ-term, the same variables do not occur simultaneously as both free and bound variables. It is also assumed that in a λ-term, there are no iterated occurrences of the same bound variable x, such as $\cdots\lambda x^\alpha.(\cdots\lambda x^\alpha.(\cdots)\cdots)\cdots$. An expression $[N^\alpha/x^\alpha]M^\beta$means, in a usual sense, the substitution of N^α to a free variable x^α in M^β. For the new constructor $\iota' \in \{\iota, \iota^{-1}\}$, we also assume the condition $[N^\alpha/ x^\alpha](\iota' M^\beta)^\gamma = (\iota'[N^\alpha/x^\alpha]M^\beta)^\gamma$. To avoid the clash of bound variables by substitutions, α-conversions are occasionally assumed.

(Untyped) terms are defined as usual, and types are defined below.

Definition 3.3. Types for $\lambda_{\mathrm{B}[l]}$ are defined inductively as follows.

1. For any atomic type p, $\mathrm{X}^i p$ is a type.
2. If $\mathrm{X}^i\alpha$ and $\mathrm{X}^i\beta$ are types, then $\mathrm{X}^i(\alpha\circ\beta)$ where $\circ \in \{\rightarrow, \wedge\}$ are types.
3. If $\mathrm{X}^i\alpha$ is a type, then $\mathrm{X}^i\mathrm{G}\alpha$ is a type.

4. If $\mathrm{X}^i\alpha$ is a type, then $\mathrm{X}^{i+l}\alpha$ is a type.

Definition 3.4. The *degree* $d(\alpha)$ of a type α is defined as follows.

1. $d(\mathrm{X}^i p) = i + 1$ for any atomic type p.
2. $d(\mathrm{X}^i(\alpha \circ \beta)) = i + d(\alpha) + d(\beta)$ where $\circ \in \{\rightarrow, \wedge\}$.
3. $d(\mathrm{X}^i\mathrm{G}\alpha) = i + l + 1 + d(\alpha)$.

It is remarked that $d(\mathrm{X}^l\alpha) < d(\mathrm{G}\alpha)$ holds, and this fact is critical to show a key lemma.

Definition 3.5. Let m be an arbitrary positive integer. Typed λ-terms for $\lambda_{\mathrm{B}[l]}$ are inductively defined as follows.

1. if $x^{\mathrm{X}^i\alpha}$ is a typed variable, then it is a typed λ-term.
2. if $x^{\mathrm{X}^i\alpha}$ and $M^{\mathrm{X}^i\beta}$ are typed λ-terms, then $(\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\rightarrow\beta)}$ is a typed λ-term.
3. if $M^{\mathrm{X}^i(\alpha\rightarrow\beta)}$ and $N^{\mathrm{X}^i\alpha}$ are typed λ-terms, then $(M^{\mathrm{X}^i(\alpha\rightarrow\beta)}N^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta}$ is a typed λ-term.
4. if $M^{\mathrm{X}^i\alpha}$ and $N^{\mathrm{X}^i\beta}$ are typed λ-terms, then $\langle M^{\mathrm{X}^i\alpha}, N^{\mathrm{X}^i\beta}\rangle^{\mathrm{X}^i(\alpha\wedge\beta)}$ is a typed λ-term.
5. if $M^{\mathrm{X}^i(\alpha\wedge\beta)}$ is a typed λ-term, then $(\pi_1 M^{\mathrm{X}^i(\alpha\wedge\beta)})^{\mathrm{X}^i\alpha}$ and $(\pi_2 M^{\mathrm{X}^i(\alpha\wedge\beta)})^{\mathrm{X}^i\beta}$ are typed λ-terms.
6. if $M_0^{\mathrm{X}^i\alpha}$, $M_1^{\mathrm{X}^{i+1}\alpha}$, ..., $M_l^{\mathrm{X}^{i+l}\alpha}$ are typed λ-terms, then $\langle M_0^{\mathrm{X}^i\alpha}, M_1^{\mathrm{X}^{i+1}\alpha}, ... M_l^{\mathrm{X}^{i+l}\alpha}\rangle^{\mathrm{X}^i\mathrm{G}\alpha}$ is a typed λ-term.
7. if $M^{\mathrm{X}^i\mathrm{G}\alpha}$ is a typed λ-term, then $(\pi_1 M^{\mathrm{X}^i\mathrm{G}\alpha})^{\mathrm{X}^i\alpha}, \ldots, (\pi_{l+1} M^{\mathrm{X}^i\mathrm{G}\alpha})^{\mathrm{X}^{i+l}\alpha}$ are typed λ-terms.
8. if $M^{\mathrm{X}^l\alpha}$ is a typed λ-term, then $(\iota M^{\mathrm{X}^l\alpha})^{\mathrm{X}^{l+m}\alpha}$ is a typed λ-term.
9. if $M^{\mathrm{X}^{l+m}\alpha}$ is a typed λ-term, then $(\iota^{-1} M^{\mathrm{X}^{l+m}\alpha})^{\mathrm{X}^l\alpha}$ is a typed λ-term.

Definition 3.6 ($\lambda_{\mathrm{B}[l]}$). The typed λ-calculus $\lambda_{\mathrm{B}[l]}$ is defined by reductions for the typed λ-terms defined in Definition 3.5. In the following, the transformation process from the left hand side of $\succ$ to the right hand side of $\succ$ is called a *reduction*, and the term of the left hand side of $\succ$ is called a *redex*.

1. $((\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\to\beta)}N^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta} \succ [N^{\mathrm{X}^i\alpha}/x^{\mathrm{X}^i\alpha}]M^{\mathrm{X}^i\beta}$.
2. $(\pi_1\langle M^{\mathrm{X}^i\alpha}, N^{\mathrm{X}^i\beta}\rangle^{\mathrm{X}^i(\alpha\wedge\beta)})^{\mathrm{X}^i\alpha} \succ M^{\mathrm{X}^i\alpha}$.
3. $(\pi_2\langle M^{\mathrm{X}^i\alpha}, N^{\mathrm{X}^i\beta}\rangle^{\mathrm{X}^i(\alpha\wedge\beta)})^{\mathrm{X}^i\beta} \succ N^{\mathrm{X}^i\beta}$.
4. $(\pi_k\langle M_0^{\mathrm{X}^i\alpha}, M_1^{\mathrm{X}^{i+1}\alpha}, ..., M_l^{\mathrm{X}^{i+l}\alpha}\rangle^{\mathrm{X}^i\mathrm{G}\alpha})^{\mathrm{X}^{i+k-1}\alpha} \succ M_{k-1}^{\mathrm{X}^{i+k-1}\alpha}$ with $1 \le k \in \omega_{l+1}$.
5. $(\iota^{-1}(\iota M^{\mathrm{X}^l\alpha})^{\mathrm{X}^{l+m}\alpha})^{\mathrm{X}^l\alpha} \succ M^{\mathrm{X}^l\alpha}$ with $1 \le m \in \omega$.
6. if $M \succ N$, then $\lambda x.M \succ \lambda x.N$, $ML \succ NL$, $LM \succ LN$, $\langle M, L\rangle \succ \langle N, L\rangle$, $\langle L, M\rangle \succ \langle L, N\rangle$, $\langle ..., M, ...\rangle \succ \langle ..., N, ...\rangle$, $\pi_1 M \succ \pi_1 N$, $\pi_2 M \succ \pi_2 N$, $\pi_k M \succ \pi_k N$ with $2 < k \in \omega_{l+1}$, $\iota M \succ \iota N$ and $\iota^{-1}M \succ \iota^{-1}N$.

In the next section, we will prove the strong normalization theorem for $\lambda_{\mathrm{B}[l]}$. Since the framework for $\lambda_{\mathrm{B}[l]}$ is strongly dependent on the time bound l, the method for strong normalization proof is not adapted for the unbounded (infinite-time) version. Such a version is required to use some infinitely long λ-terms. Thus, it is unknown whether the strong normalization theorem for such an unbounded version holds or not. This problem is remained as an open question.

3.3 $\mathbf{T}_{\mathrm{B}[l]}$

The precise definition of typed λ-terms for $\mathrm{T}_{\mathrm{B}[l]}$ is omitted, since it can be obtained analogously w.r.t. $\lambda_{\mathrm{B}[l]}$.

Definition 3.7 ($\mathbf{T}_{\mathrm{B}[l]}$). The typing rules of $\mathrm{T}_{\mathrm{B}[l]}$ are of the form: for any $k \in \omega_l$ and any positive integer m,

$$\dfrac{\begin{array}{c}[x : \mathrm{X}^i\alpha]\\ \vdots \\ M : \mathrm{X}^i\beta\end{array}}{\lambda x.M : \mathrm{X}^i(\alpha\to\beta)}\ (\to I^T) \qquad \dfrac{M : \mathrm{X}^i(\alpha\to\beta) \quad N : \mathrm{X}^i\alpha}{MN : \mathrm{X}^i\beta}\ (\to E^T)$$

$$\frac{M : \mathrm{X}^i\alpha \quad N : \mathrm{X}^i\beta}{\langle M, N\rangle : \mathrm{X}^i(\alpha \wedge \beta)}\ (\wedge I^T) \quad \frac{M : \mathrm{X}^i(\alpha \wedge \beta)}{\pi_1 M : \mathrm{X}^i\alpha}\ (\wedge E1^T) \quad \frac{M : \mathrm{X}^i(\alpha \wedge \beta)}{\pi_2 M : \mathrm{X}^i\beta}\ (\wedge E2^T)$$

$$\frac{M : \mathrm{X}^l\alpha}{\iota M : \mathrm{X}^{l+m}\alpha}\ (\mathrm{X}I^T) \quad \frac{M : \mathrm{X}^{l+m}\alpha}{\iota^{-1} M : \mathrm{X}^l\alpha}\ (\mathrm{X}E^T)$$

$$\frac{\{\ M_j : \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega_l}}{\langle M_0, M_1, ..., M_l\rangle : \mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{G}I^T) \qquad \frac{M : \mathrm{X}^i\mathrm{G}\alpha}{\pi_{k+1} M : \mathrm{X}^{i+k}\alpha}\ (\mathrm{G}E^T).$$

Any proofs constructed only on an assumption $(x : \alpha)$ are considered to be axioms.

4. Strong normalization

The following proof of the strong normalization theorem for $\lambda_{\mathrm{B}[l]}$ is based on the technique presented in the textbook [11]. All the definitions and lemmas presented below are thus similar to the definitions and lemmas presented in the book [11] for a simply typed λ-calculus.

Definition 4.1. A typed λ-term is said to be *normal* if it contains no redex. A sequence $M_0^\alpha, M_1^\alpha, \cdots$ of typed λ-terms is called a *reduction sequence* if it satisfies the following conditions (1) $M_i^\alpha \succ M_{i+1}^\alpha$ for all $0 \leq i$ and (2) the last typed λ-term in the sequence is normal if the sequence is finite. A typed λ-term M^α is called *strongly normalizable* if each reduction sequence starting from M^α is terminated.

We now start to prove the strong normalization theorem for $\lambda_{\mathrm{B}[l]}$. The proof is similar to that for the simply typed λ-calculus with the conjunction type, $\lambda^{\rightarrow\wedge}$, because the behaviors of the new constructors ι and ι^{-1} are similar to that of the pairing function $\langle , \rangle$ and the projection function π_1, respectively.

In the following, SN means the set of all strongly normalizable typed λ-terms for $\lambda_{\mathrm{B}[l]}$, and TERM means the set of all typed λ-terms for $\lambda_{\mathrm{B}[l]}$. In order to show TERM $\subseteq$ SN, i.e., the strong normalization theorem for $\lambda_{\mathrm{B}[l]}$, we will define the set RED of reducible terms, and will show TERM $\subseteq$ RED $\subseteq$ SN. First, we will show RED $\subseteq$ SN by induction on the degree of a type (Definition 3.4), and second, will show TERM $\subseteq$ RED by induction on the construction of a term.

Definition 4.2. The set RED_γ of reducible terms of type γ (for $\lambda_{\mathrm{B}[l]}$) is defined by induction on the type γ as follows.

1. $M^{\mathrm{X}^i p} \in \mathrm{RED}_{\mathrm{X}^i p}$ if and only if $M^{\mathrm{X}^i p} \in \mathrm{SN}$, for any atomic type p.
2. $M^{\mathrm{X}^i(\alpha\to\beta)} \in \mathrm{RED}_{\mathrm{X}^i(\alpha\to\beta)}$ if and only if $\forall N^{\mathrm{X}^i\alpha} \in \mathrm{RED}_{\mathrm{X}^i\alpha}$ $[(M^{\mathrm{X}^i(\alpha\to\beta)} N^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}]$.
3. $M^{\mathrm{X}^i(\alpha\wedge\beta)} \in \mathrm{RED}_{\mathrm{X}^i(\alpha\wedge\beta)}$ if and only if $(\pi_1 M^{\mathrm{X}^i(\alpha\wedge\beta)})^{\mathrm{X}^i\alpha} \in \mathrm{RED}_{\mathrm{X}^i\alpha}$ and $(\pi_2 M^{\mathrm{X}^i(\alpha\wedge\beta)})^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$.
4. $M^{\mathrm{X}^i\mathrm{G}\alpha} \in \mathrm{RED}_{\mathrm{X}^i\mathrm{G}\alpha}$ if and only if $(\pi_k M^{\mathrm{X}^i\mathrm{G}\alpha})^{\mathrm{X}^{i+k-1}\alpha} \in \mathrm{RED}_{\mathrm{X}^{i+k-1}\alpha}$ for all k with $1 \leq k \in \omega_{l+1}$.
5. $M^{\mathrm{X}^{l+m}\alpha} \in \mathrm{RED}_{\mathrm{X}^{l+m}\alpha}$ if and only if $(\iota^{-1} M^{\mathrm{X}^{l+m}\alpha})^{\mathrm{X}^l\alpha} \in \mathrm{RED}_{\mathrm{X}^l\alpha}$, for any positive integer m.

Definition 4.3. A typed λ-term M^α for $\lambda_{\mathrm{B}[l]}$ is said to be *neutral* if M is one of the forms x, NP, $\pi_1 N$, $\pi_2 N$, $\pi_k N$ with $2 < k \in \omega_{l+1}$, and $\iota^{-1}N$.

If $M^\alpha \in \mathrm{SN}$, then an expression $v(M^\alpha)$ means the least number which bounds the length of every reduction sequence begining with M^α.

The following lemma has the same statements as those in [11], but the proof is rather different: The division of cases for induction is generalized with the addition of the expression X^i.

Lemma 4.4. *For all typed λ-term M^α for $\lambda_{\mathrm{B}[l]}$, M^α satisfies the following four conditions.*

(CR1) *if $M^\alpha \in \mathrm{RED}_\alpha$, then $M^\alpha \in \mathrm{SN}$.*

(CR2) *if $M^\alpha \in \mathrm{RED}_\alpha$ and $M^\alpha \succ N^\alpha$, then $N^\alpha \in \mathrm{RED}_\alpha$.*

(CR3) *if M^α is neutral, then $\forall N^\alpha$ [if $M^\alpha \succ N^\alpha$ and $N^\alpha \in \mathrm{RED}_\alpha$, then $M^\alpha \in \mathrm{RED}_\alpha$].*

(CR4) *if M^α is neutral and normal, then $M^\alpha \in \mathrm{RED}_\alpha$. It is remarked that* (CR4) *is a special case of* (CR3).

Proof. By induction on the degree $d(\alpha)$ of the type α. We consider the cases for induction: Case $\alpha \equiv \mathrm{X}^i p$ for any atomic type p, case $\alpha \equiv \mathrm{X}^i(\beta{\to}\gamma)$, case $\alpha \equiv \mathrm{X}^i(\beta \wedge \gamma)$, case $\alpha \equiv \mathrm{X}^i\mathrm{G}\beta$, and case $\alpha \equiv \mathrm{X}^m\mathrm{X}^l\beta$ where m is a positive integer. In this case division, all the cases for the forms of types are covered. In these cases, i in X^i can be 0, and hence these cases include the cases for $\lambda^{\to\wedge}$. A special case is the case $\alpha \equiv \mathrm{X}^m\mathrm{X}^l\beta$ (m: positive integer), where the $m = 0$ case is included in the other cases. This special case is for the given positive integer l. We now show some cases below.

• Case ($\alpha \equiv \mathrm{X}^i p$ for any atomic type p).

(CR1): Obvious by the definition of $\mathrm{RED}_{\mathrm{X}^i p}$.

(CR2): Suppose that $M^{\mathrm{X}^i p} \in \mathrm{RED}_{\mathrm{X}^i p}$ and $M^{\mathrm{X}^i p} \succ N^{\mathrm{X}^i p}$. By the definition of RED, we have $M^{\mathrm{X}^i p} \in \mathrm{SN}$. Thus, we also have $N^{\mathrm{X}^i p} \in \mathrm{SN}$. Therefore we obtain $N^{\mathrm{X}^i p} \in \mathrm{RED}_{\mathrm{X}^i p}$ by the definition of RED.

(CR3): Suppose that for any neutral $M^{\mathrm{X}^i p}$ and any $N^{\mathrm{X}^i p}$, we have $M^{\mathrm{X}^i p} \succ N^{\mathrm{X}^i p}$ and $N^{\mathrm{X}^i p} \in \mathrm{RED}_{\mathrm{X}^i p}$. Then, we have $N^{\mathrm{X}^i p} \in \mathrm{SN}$ by the definition of RED. This means that any terms one step fron $M^{\mathrm{X}^i p}$ is in SN. Thus, we have $M^{\mathrm{X}^i p} \in \mathrm{SN}$, and hence $M^{\mathrm{X}^i p} \in \mathrm{RED}_{\mathrm{X}^i p}$ by the definition of RED.

• Case ($\alpha \equiv \mathrm{X}^i(\beta{\to}\gamma)$).

(CR1): Suppose $M^{\mathrm{X}^i(\beta\to\gamma)} \in \mathrm{RED}_{\mathrm{X}^i(\beta\to\gamma)}$ and take $x^{\mathrm{X}^i\beta}$. Then, we have the fact that $(M^{\mathrm{X}^i(\beta\to\gamma)}x^{\mathrm{X}^i\beta})^{\mathrm{X}^i\gamma}$ and $x^{\mathrm{X}^i\beta}$ have (CR1–4) by the induction hypothesis with $d(\mathrm{X}^i(\beta{\to}\gamma)) > d(X^i\gamma)$ and $d(\mathrm{X}^i(\beta{\to}\gamma)) > d(\mathrm{X}^i\beta)$. By (CR4), we have $x^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$, and by the definition of RED, we have $(M^{\mathrm{X}^i(\beta\to\gamma)}x^{\mathrm{X}^i\beta})^{\mathrm{X}^i\gamma} \in \mathrm{RED}_{\mathrm{X}^i\gamma}$. By (CR1), we obtain $(M^{\mathrm{X}^i(\beta\to\gamma)}x^{\mathrm{X}^i\beta})^{\mathrm{X}^i\gamma} \in \mathrm{SN}$. If there is an infinite reduction sequence starting from $M^{\mathrm{X}^i(\beta\to\gamma)}$, then there is also an inifinite reduction sequence starting from $(M^{\mathrm{X}^i(\beta\to\gamma)}x^{\mathrm{X}^i\beta})^{\mathrm{X}^i\gamma}$. This is a contradiction. Therefore $M^{\mathrm{X}^i(\beta\to\gamma)} \in \mathrm{SN}$.

(CR2): Suppose that $M^{\mathrm{X}^i(\beta\to\gamma)} \in \mathrm{RED}_{\mathrm{X}^i(\beta\to\gamma)}$ and $M^{\mathrm{X}^i(\beta\to\gamma)} \succ N^{\mathrm{X}^i(\beta\to\gamma)}$. Then, for any $L^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$, we have the fact that $L^{\mathrm{X}^i\beta}$ and $(M^{\mathrm{X}^i(\beta\to\gamma)}L^{\mathrm{X}^i\beta})^{\mathrm{X}^i\gamma}$ have (CR1–4) by the induction hypothesis with $d(\mathrm{X}^i(\beta{\to}\gamma)) > d(\mathrm{X}^i\beta)$ and $d(\mathrm{X}^i(\beta{\to}\gamma)) > d(\mathrm{X}^i\gamma)$. By the definition of RED, we obtain $(M^{X^i(\beta\to\gamma)}L^{X^i\beta})^{X^i\gamma} \in \mathrm{RED}_{X^i\gamma}$. Then, we have $(M^{X^i(\beta\to\gamma)}L^{X^i\beta})^{X^i\gamma} \succ (N^{X^i(\beta\to\gamma)}L^{X^i\beta})^{X^i\gamma}$, and hence $(N^{X^i(\beta\to\gamma)}L^{X^i\beta})^{X^i\gamma} \in \mathrm{RED}_{X^i\gamma}$ by (CR2). Therefore we obtain $N^{X^i(\beta\to\gamma)} \in \mathrm{RED}_{X^i(\beta\to\gamma)}$ by the definition of RED.

(CR3): Suppose that $M^{X^i(\beta\to\gamma)}$ is neutral and $M^{X^i(\beta\to\gamma)} \succ N^{X^i(\beta\to\gamma)}$ for any $N^{X^i(\beta\to\gamma)} \in \mathrm{RED}_{X^i(\beta\to\gamma)}$. We take an arbitrary $P^{X^i\beta} \in \mathrm{RED}_{X^i\beta}$. By the hypothesis of induction with $d(X^i(\beta\to\gamma)) > d(X^i\beta)$ and $d(X^i(\beta\to\gamma)) > d(X^i\gamma)$, we have the fact that $P^{X^i\beta}$ and $(M^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma}$ have (CR1–4). It is sufficient to show $(M^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma} \in \mathrm{RED}_{X^i\gamma}$, because this derives the required fact $M^{X^i(\beta\to\gamma)} \in \mathrm{RED}_{X^i\gamma}$ by the definition of RED. We thus show $(M^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma} \in \mathrm{RED}_{X^i\gamma}$ in the following. First, we consider the case that $(M^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma}$ is normal. In this case, since $(M^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma}$ is neutral, we obtain the required fact $(M^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma} \in \mathrm{RED}_{X^i\gamma}$ by (CR4). Next, we consider the case that $(M^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma}$ is not normal. In this case, we can consider two cases $(M^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma} \succ (N^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma}$ and $(M^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma} \succ (M^{X^i(\beta\to\gamma)}P'^{X^i\beta})^{X^i\gamma}$ with $P^{X^i\beta} \succ P'^{X^i\beta}$, because $M^{X^i(\beta\to\gamma)}$ is neutral. Then, in order to use (CR3), we will show the (CR3)-assumption (*): $(N^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma} \in \mathrm{RED}_{X^i\gamma}$ and $(M^{X^i(\beta\to\gamma)}P'^{X^i\beta})^{X^i\gamma} \in \mathrm{RED}_{X^i\gamma}$, by induction on $v(P^{X^i\beta})$. (Case $v(P^{\beta}) = 0$): In this case, we only have $(M^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma} \succ (N^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma}$. By the definition of RED, we obtain $(N^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma} \in \mathrm{RED}_{X^i\gamma}$. (Case $v(P^{X^i\beta}) \neq 0$): The case $(M^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma} \succ (N^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma}$ is the same as the case discussed just above. We consider the case $(M^{X^i(\beta\to\gamma)}P^{X^i\beta})^{X^i\gamma} \succ (M^{X^i(\beta\to\gamma)}P'^{X^i\beta})^{X^i\gamma}$. By (CR2), we obtain $P'^{X^i\beta} \in \mathrm{RED}_{X^i\beta}$. By the hypothesis of (main) induction, we have the fact that $P'^{X^i\beta}$ has (CR1–4). Thus, we obtain $P'^{X^i\beta} \in \mathrm{SN}$ by (CR1). Now, we have $v(P^{\mathrm{X}^i\beta}) > v(P'^{\mathrm{X}^i\beta})$. Thus, we obtain $(M^{\mathrm{X}^i(\beta\to\gamma)}P'^{\mathrm{X}^i\beta})^{\mathrm{X}^i\gamma} \in \mathrm{RED}_{\mathrm{X}^i\gamma}$ by the hypothesis of induction w.r.t. $v(P^{\mathrm{X}^i\beta})$. Thus, we obtain the fact (*), and hence obtain the required fact $(M^{\mathrm{X}^i(\beta\to\gamma)}P^{\mathrm{X}^i\beta})^{\mathrm{X}^i\gamma} \in \mathrm{RED}_{\mathrm{X}^i\gamma}$ by (CR3).

• Case ($\alpha \equiv \mathrm{X}^i\mathrm{G}\beta$).

(CR1): Suppose $M^{\mathrm{X}^i\mathrm{G}\beta} \in \mathrm{RED}_{\mathrm{X}^i\mathrm{G}\beta}$. Then, by the definition of RED, $(\pi_k M^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta} \in \mathrm{RED}_{\mathrm{X}^{i+k-1}\beta}$ for all k with $1 \leq k \in \omega_{l+1}$. We have $k-1 \leq l$ and $d(\mathrm{X}^{i+k-1}\beta) < d(\mathrm{X}^i\mathrm{G}\beta)$. Hence we can apply the induction hypothesis of (CR1), and obtain $(\pi_k M^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta} \in \mathrm{SN}$. Moreover, we have $v((\pi_k M^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta}) \geq v(M^{\mathrm{X}^i\mathrm{G}\beta})$, because from any reduction sequence $M^{\mathrm{X}^i\mathrm{G}\beta} \succ M_1^{\mathrm{X}^i\mathrm{G}\beta} \succ M_2^{\mathrm{X}^i\mathrm{G}\beta} \succ \cdots$, one can construct a reduction sequence $(\pi_k M^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta} \succ (\pi_k M_1^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta} \succ (\pi_k M_2^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta} \succ \cdots$. So $v(M^{\mathrm{X}^i\mathrm{G}\beta})$ is finite, and hence $M^{\mathrm{X}^i\mathrm{G}\beta} \in \mathrm{SN}$.

(CR2): Suppose $M^{\mathrm{X}^i\mathrm{G}\beta} \succ N^{\mathrm{X}^i\mathrm{G}\beta}$. Then, $(\pi_k M^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta} \succ (\pi_k N^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta}$ for all k with $1 \leq k \in \omega_{l+1}$. By the hypothesis, we have $M^{\mathrm{X}^i\mathrm{G}\beta} \in \mathrm{RED}_{\mathrm{X}^i\mathrm{G}\beta}$, and hence $(\pi_k M^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta} \in \mathrm{RED}_{\mathrm{X}^{i+k-1}\beta}$ by the definition of RED. We have $k-1 \leq l$ and $d(\mathrm{X}^{i+k-1}\beta) < d(\mathrm{X}^i\mathrm{G}\beta)$. Hence we can apply the induction hypothesis of (CR2), and then obtain $(\pi_k N^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta} \in \mathrm{RED}_{\mathrm{X}^{i+k-1}\beta}$. Thus, $N^{\mathrm{X}^i\mathrm{G}\beta} \in \mathrm{RED}_{\mathrm{X}^i\mathrm{G}\beta}$ by the definition of RED.

(CR3): Let $M^{\mathrm{X}^i\mathrm{G}\beta}$ is neutral and suppose all the $N^{\mathrm{X}^i\mathrm{G}\beta}$ such that $M^{\mathrm{X}^i\mathrm{G}\beta} \succ N^{\mathrm{X}^i\mathrm{G}\beta} \in \mathrm{RED}_{\mathrm{X}^i\mathrm{G}\beta}$. Since $M^{\mathrm{X}^i\mathrm{G}\beta}$ is neutral, $(\pi_k M^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta}$ for all k with $1 \leq k \in \omega_{l+1}$ cannot itself be a redex. Thus, we obtain $(\pi_k M^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta} \succ (\pi_k N^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta}$ and $(\pi_k N^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta} \in \mathrm{RED}_{\mathrm{X}^{i+k-1}\beta}$, because of the hypothesis $N^{\mathrm{X}^i\mathrm{G}\beta} \in \mathrm{RED}_{\mathrm{X}^{i+k-1}\beta}$ and the definition of RED. We have that $(\pi_k M^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta}$ is neutral and all the typed λ-terms one step from $(\pi_k M^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta}$ are in $\mathrm{RED}_{\mathrm{X}^{i+k-1}\beta}$, and that $k-1 \leq l$ and $d(\mathrm{X}^{i+k-1}\beta) < d(\mathrm{X}^i\mathrm{G}\beta)$. Thus, we can apply the induction hypothesis of (CR3), and obtain $(\pi_k M^{\mathrm{X}^i\mathrm{G}\beta})^{\mathrm{X}^{i+k-1}\beta} \in \mathrm{RED}_{\mathrm{X}^{i+k-1}\beta}$. Therefore, we obtain $M^{\mathrm{X}^i\mathrm{G}\beta} \in$

$\mathrm{RED}_{\mathrm{X}^{i+k-1}\beta}$ by the definition of RED.

• Case ($\alpha \equiv \mathrm{X}^m\mathrm{X}^l\beta \equiv \mathrm{X}^{l+m}\beta$ with a positive integer m).

(CR1): Suppose $M^{\mathrm{X}^{l+m}\beta} \in \mathrm{RED}_{\mathrm{X}^{l+m}\beta}$. Then, $(\iota^{-1}M^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta} \in \mathrm{RED}_{\mathrm{X}^l\beta}$ by the definition of RED. By the induction hypothesis of (CR1), we obtain $(\iota^{-1}M^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta} \in \mathrm{SN}$. Moreover, we have $v((\iota^{-1}M^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta}) \geq v(M^{\mathrm{X}^{l+m}\beta})$, because from any reduction sequence $M^{\mathrm{X}^{l+m}\beta} \succ M_1^{\mathrm{X}^{l+m}\beta} \succ M_2^{\mathrm{X}^{l+m}\beta} \succ \cdots$, one can construct a reduction sequence $(\iota^{-1}M^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta} \succ (\iota^{-1}M_1^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta} \succ (\iota^{-1}M_2^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta} \succ \cdots$. So $v(M^{\mathrm{X}^{l+m}\beta})$ is finite, and hence $M^{\mathrm{X}^{l+m}\beta} \in \mathrm{SN}$.

(CR2): Suppose $M^{\mathrm{X}^{l+m}\beta} \succ N^{\mathrm{X}^{l+m}\beta}$. Then, $(\iota^{-1}M^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta} \succ (\iota^{-1}N^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta}$. By the hypothesis, we have $M^{\mathrm{X}^{l+m}\beta} \in \mathrm{RED}_{\mathrm{X}^{l+m}\beta}$, and hence $(\iota^{-1}M^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta} \in \mathrm{RED}_{\mathrm{X}^l\beta}$ by the definition of RED. By the induction hypothesis of (CR2), we obtain $(\iota^{-1}N^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta} \in \mathrm{RED}_{\mathrm{X}^l\beta}$, and hence $N^{\mathrm{X}^{l+m}\beta} \in \mathrm{RED}_{\mathrm{X}^{l+m}\beta}$.

(CR3): Let $M^{\mathrm{X}^{l+m}\beta}$ be neutral and suppose all the $N^{\mathrm{X}^{l+m}\beta}$ such that $M^{\mathrm{X}^{l+m}\beta} \succ N^{\mathrm{X}^{l+m}\beta} \in \mathrm{RED}_{\mathrm{X}^{l+m}\beta}$. Since $M^{\mathrm{X}^{l+m}\beta}$ is neutral, $(\iota^{-1}M^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta}$ cannot itself be a redex. Thus, we obtain $(\iota^{-1}M^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta} \succ (\iota^{-1}N^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta}$ and $(\iota^{-1}N^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta} \in \mathrm{RED}_{\mathrm{X}^l\beta}$ because of the hypothesis $N^{\mathrm{X}^{l+m}\beta} \in \mathrm{RED}_{\mathrm{X}^{l+m}\beta}$ and the definition of RED. Since $(\iota^{-1}M^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta}$ is neutral and all the typed λ-terms one step from $(\iota^{-1}M^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta}$ are in $\mathrm{RED}_{\mathrm{X}^l\beta}$, we can apply the induction hypothesis of (CR3), and obtain $(\iota^{-1}M^{\mathrm{X}^{l+m}\beta})^{\mathrm{X}^l\beta} \in \mathrm{RED}_{\mathrm{X}^l\beta}$. Therefore we obtain $M^{\mathrm{X}^{l+m}\beta} \in \mathrm{RED}_{\mathrm{X}^{l+m}\beta}$ by the definition of RED.

□

By (CR1) of Lemma 4.4, we have RED ⊆ SN.

Using (CR1) – (CR4) in Lemma 4.4, we can prove Lemma 4.5. This lemma is regarded as a generalization of the corresponding reducibility lemma presented in [11], and is for showing Lemma 4.6. Some statements of Lemma 4.5 reflect the forms of reductions of $\lambda_{\mathrm{B}[l]}$. The statements 1–3 are the same as those of the corresponding reducibility lemma for $\lambda^{\rightarrow\wedge}$ if i is 0 . The proof of Lemma 4.5 is also similar to the proof of the reducibility lemma. By deleting X^i in the proofs of 1–3 in Lemma 4.5, we can obtain the same proofs as those for $\lambda^{\rightarrow\wedge}$.

Lemma 4.5. *The following conditions hold for* $\lambda_{\mathrm{B}[l]}$.

1. *If* $x^{\mathrm{X}^i\alpha}$ *is a typed variable, then* $x^{\mathrm{X}^i\alpha} \in \mathrm{RED}_{\mathrm{X}^i\alpha}$.

2. *For any* $M^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$ *and any* $N^{\mathrm{X}^i\alpha} \in \mathrm{RED}_{\mathrm{X}^i\alpha}$, *if* $[N^{\mathrm{X}^i\alpha}/x^{\mathrm{X}^i\alpha}]M^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$, *then* $(\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\rightarrow\beta)} \in \mathrm{RED}_{\mathrm{X}^i(\alpha\rightarrow\beta)}$.

3. *If* $M^{\mathrm{X}^i\alpha} \in \mathrm{RED}_{\mathrm{X}^i\alpha}$ *and* $N^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$, *then* $\langle M^{\mathrm{X}^i\alpha}, N^{\mathrm{X}^i\beta}\rangle^{\mathrm{X}^i(\alpha\wedge\beta)} \in \mathrm{RED}_{\mathrm{X}^i(\alpha\wedge\beta)}$.

4. *If* $M_0^{\mathrm{X}^i\alpha} \in \mathrm{RED}_{\mathrm{X}^i\alpha}$, $M_1^{\mathrm{X}^{i+1}\alpha} \in \mathrm{RED}_{\mathrm{X}^{i+1}\alpha}$, ..., $M_l^{\mathrm{X}^{i+l}\alpha} \in \mathrm{RED}_{\mathrm{X}^{i+l}\alpha}$, *then* $\langle M_0^{\mathrm{X}^i\alpha}, ..., M_l^{\mathrm{X}^{i+l}\alpha}\rangle^{\mathrm{X}^i\mathrm{G}\alpha} \in \mathrm{RED}_{\mathrm{X}^i\mathrm{G}\alpha}$.

5. *If* $M^{\mathrm{X}^l\alpha} \in \mathrm{RED}_{\mathrm{X}^l\alpha}$, *then* $(\iota M^{\mathrm{X}^l\alpha})^{\mathrm{X}^{l+m}\alpha} \in \mathrm{RED}_{\mathrm{X}^{l+m}\alpha}$ *for any positive integer* m.

Proof. (1) is obvious by (CR4). (2) and (3) are similar to the reducibility lemmas in [11]. (4) and (5) are similar to (3). We show only (2) and (5) below.

• (2). Suppose that for any $M^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$ and any $N^{\mathrm{X}^i\alpha} \in \mathrm{RED}_{\mathrm{X}^i\alpha}$ with $[N^{\mathrm{X}^i\alpha}/x^{\mathrm{X}^i\alpha}]M^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$. Then, it is sufficient to show (*): $((\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\rightarrow\beta)}N^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$, because (*) and the definition of RED derives the required fact $(\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\rightarrow\beta)} \in \mathrm{RED}_{\mathrm{X}^i(\alpha\rightarrow\beta)}$. In order to show (*), since we have that $((\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\rightarrow\beta)}N^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta}$: neutral, $((\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\rightarrow\beta)}N^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta} \succ [N^{\mathrm{X}^i\alpha}/x^{\mathrm{X}^i\alpha}]M^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$ and (CR3), it is sufficient to show the assumption of (CR3) as (**): for any $L^{\mathrm{X}^i\beta}$, if $((\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\rightarrow\beta)}N^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta} \succ L^{\mathrm{X}^i\beta}$, then $L^{\mathrm{X}^i\beta} \in$

$\mathrm{RED}_{\mathrm{X}^i\beta}$. We have $M^{\mathrm{X}^i\beta}, N^{\mathrm{X}^i\alpha} \in \mathrm{SN}$ by (CR1) since $M^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$ and $N^{\mathrm{X}^i\alpha} \in \mathrm{RED}_{\mathrm{X}^i\alpha}$. We thus show (**) by induction on $v(M^{\mathrm{X}^i\beta}) + v(N^{\mathrm{X}^i\alpha})$ as follows. (Case $v(M^{\mathrm{X}^i\beta}) + v(N^{\mathrm{X}^i\alpha}) = 0$): We have $((\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\to\beta)}N^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta} \succ [N^{\mathrm{X}^i\alpha}/x^{\mathrm{X}^i\alpha}]M^{\mathrm{X}^i\beta}$. By the hypothesis, we obtain $[N^{\mathrm{X}^i\alpha}/x^{\mathrm{X}^i\alpha}]M^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$. (Case $v(M^{\mathrm{X}^i\beta}) + v(N^{\mathrm{X}^i\alpha}) \neq 0$): In this case, we consider the following cases:

(a): $((\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\to\beta)}N^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta} \succ [N^{\mathrm{X}^i\alpha}/x^{\mathrm{X}^i\alpha}]M^{\mathrm{X}^i\beta}$.

(b):

$$((\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\to\beta)}N^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta} \succ ((\lambda x^{\mathrm{X}^i\alpha}.M'^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\to\beta)}N^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta}$$

with $M^{\mathrm{X}^i\beta} \succ M'^{\mathrm{X}^i\beta}$.

(c):

$$((\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\to\beta)}N^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta} \succ ((\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\to\beta)}N'^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta}$$

with $N^{\mathrm{X}^i\alpha} \succ N'^{\mathrm{X}^i\alpha}$.

For the case (a), we have $[N^{\mathrm{X}^i\alpha}/x^{\mathrm{X}^i\alpha}]M^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$ by the hypothesis. We then consider the case (b). Since $M^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$, we have $M'^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$ by (CR2). By (CR1), we have $M^{\mathrm{X}^i\beta} \in \mathrm{SN}$, and hence $M'^{\mathrm{X}^i\beta} \in \mathrm{SN}$. Obviously we have $v(M^{\mathrm{X}^i\beta}) > v(M'^{\mathrm{X}^i\beta})$. Thus, by the hypothesis of induction, we obtain $((\lambda x^{\mathrm{X}^i\alpha}.M'^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\to\beta)}N^{\mathrm{X}^i\alpha})^{\mathrm{X}^i\beta} \in \mathrm{RED}_{\mathrm{X}^i\beta}$. The case (c) is similar to (b). Therefore, we obtain (**), and hence obtain (*) and the required fact $(\lambda x^{\mathrm{X}^i\alpha}.M^{\mathrm{X}^i\beta})^{\mathrm{X}^i(\alpha\to\beta)} \in \mathrm{RED}_{\mathrm{X}^i(\alpha\to\beta)}$.

- (5). Suppose $M^{\mathrm{X}^l\alpha} \in \mathrm{RED}_{\mathrm{X}^l\alpha}$. We will show $(\iota M^{\mathrm{X}^l\alpha})^{\mathrm{X}^{l+m}\alpha} \in \mathrm{RED}_{\mathrm{X}^{l+m}\alpha}$, i.e., it is enough to show $(\iota^{-1}(\iota M^{\mathrm{X}^l\alpha})^{\mathrm{X}^{l+m}\alpha})^{\mathrm{X}^l\alpha} \in \mathrm{RED}_{\mathrm{X}^l\alpha}$. Because of (CR1) and the hypothesis, we have $M^{\mathrm{X}^l\alpha} \in \mathrm{SN}$. Thus, we can consider $v(M^{\mathrm{X}^l\alpha})$. In the following, we show $(\iota^{-1}(\iota M^{\mathrm{X}^l\alpha})^{\mathrm{X}^{l+m}\alpha})^{\mathrm{X}^l\alpha} \in \mathrm{RED}_{\mathrm{X}^l\alpha}$ by induction on $v(M^{\mathrm{X}^l\alpha})$. This typed λ-term converts (1) $M^{\mathrm{X}^l\alpha}$ or (2) $(\iota^{-1}(\iota N^{\mathrm{X}^l\alpha})^{\mathrm{X}^{l+m}\alpha})^{\mathrm{X}^l\alpha}$ where $M^{\mathrm{X}^l\alpha} \succ N^{\mathrm{X}^l\alpha}$. For the case (2), we obtain $N^{\mathrm{X}^l\alpha} \in \mathrm{RED}_{\mathrm{X}^l\alpha}$ by (CR2), and we have $v(M^{\mathrm{X}^l\alpha}) > v(N^{\mathrm{X}^l\alpha})$.

So we obtain $(\iota^{-1}(\iota N^{\mathrm{X}^l\alpha})^{\mathrm{X}^{l+m}\alpha})^{\mathrm{X}^l\alpha} \in \mathrm{RED}_{\mathrm{X}^l\alpha}$ by the induction hypothesis. In both cases, the neutral term $(\iota^{-1}(\iota M^{\mathrm{X}^l\alpha})^{\mathrm{X}^{l+m}\alpha})^{\mathrm{X}^l\alpha}$ converts to reducible terms only, and by (CR3), it is reducible. Therefore $(\iota M^{\mathrm{X}^l\alpha})^{\mathrm{X}^{l+m}\alpha} \in \mathrm{RED}_{\mathrm{X}^{l+m}\alpha}$.

□

An expression $[N_1^{\beta_1}/x_1^{\beta_1}, ..., N_n^{\beta_n}/x_n^{\beta_n}]M^{\alpha}$ denotes the simultaneous substitution.

Using Lemma 4.5, we can prove the following lemma, which has the same statement as that in [11].

Lemma 4.6. *Let M^{α} be a typed λ-term for $\lambda_{\mathrm{B}[l]}$. If $N_1^{\beta_1} \in \mathrm{RED}_{\beta_1}$, ... , $N_n^{\beta_n} \in \mathrm{RED}_{\beta_n}$, then $[N_1^{\beta_1}/x_1^{\beta_1}, ..., N_n^{\beta_n}/x_n^{\beta_n}]M^{\alpha} \in \mathrm{RED}_{\alpha}$.*

Proof. By induction on the construction of M. Let

$$\sigma = [N_1^{\beta_1}/x_1^{\beta_1}, ..., N_n^{\beta_n}/x_n^{\beta_n}].$$

(Case $M^{\alpha} \equiv x_i^{\beta_i}$ $(1 \leq i \leq n)$): Obvious, i.e., $\sigma x_i^{\beta_i} \equiv N_i^{\beta_i} \in \mathrm{RED}_{\beta_i}$.

(Case $M^{\alpha} \equiv x^{\alpha}$ and $x^{\alpha} \neq x_1^{\beta_1}, ..., x_n^{\beta_n}$): By Lemma 4.5 (1).

(Case $M^{\mathrm{X}^i\alpha} \equiv (\lambda x^{\mathrm{X}^i\beta}.N^{\mathrm{X}^i\gamma})^{\mathrm{X}^i(\beta\to\gamma)}$): B using Lemma 4.5 (2).

(Case $M^{\alpha} \equiv (N^{\beta}, L^{\gamma})^{\alpha}$ where $(,)$ is a pairing $\langle,\rangle$ or an application): By the hypothesis of induction, we have $\sigma N^{\beta} \in \mathrm{RED}_{\beta}$ and $\sigma L^{\gamma} \in \mathrm{RED}_{\gamma}$. We thus obtain $\sigma M^{\alpha} \equiv (\sigma N^{\beta}, \sigma L^{\gamma})^{\alpha} \in \mathrm{RED}_{\alpha}$ by Lemma 4.5 (3) or by the definition.

(Case $M^{\alpha} \equiv \langle M_0^{\alpha_0}, M_1^{\alpha_1}, ..., M_l^{\alpha_l}\rangle$): By using Lemma 4.5 (4). Similar to the case just above.

(Case $M^{\alpha} \equiv (\iota M^{\mathrm{X}^{l+m}\alpha})^{\mathrm{X}^l\alpha}$): By Lemma 4.5 (5). Similar to the case above.

(Case $M^{\alpha} \equiv (\pi M^{\beta})^{\alpha}$ where π is π_1, π_2, π_k with $2 < k \in \omega_{l+1}$ or ι^{-1}): By the hypothesis of induction, we have $\sigma M^{\beta} \in \mathrm{RED}_{\beta}$. This fact derives $(\pi\sigma M^{\beta})^{\alpha} \in \mathrm{RED}_{\alpha}$ by the definition. Therefore we obtain $\sigma(\pi M^{\beta})^{\alpha} \in \mathrm{RED}_{\alpha}$. □

Theorem 4.7 (Strong normalization)**.** *All typed λ-terms for $\lambda_{\mathrm{B}[l]}$ are strongly normalizable.*

Proof. In Lemma 4.6, taking $N_1 \equiv x_1, ..., N_n \equiv x_n$, we have $M^\alpha \in \mathrm{RED}_\alpha$ for any typed λ-term M^α for $\lambda_{\mathrm{B}[l]}$, i.e., TERM $\subseteq$ RED. Since we already have RED $\subseteq$ SN, we obtain TERM $\subseteq$ SN. $\square$

5. Comparisons

5.1 $\mathbf{N}^2_{\mathrm{B}[l]}$: An indexed natural deduction system

A new natural deduction system $\mathrm{N}^2_{\mathrm{B}[l]}$, which is analogous to the 2-sequent calculus [1], is presented below. The language of $\mathrm{N}^2_{\mathrm{B}[l]}$ and the notations used are almost the same as those of $\mathrm{N}_{\mathrm{B}[l]}$. An expression α^i (α is a formula and $i \in \omega$) is called an *indexed formula*. $\mathrm{N}^2_{\mathrm{B}[l]}$ is defined based on indexed formulas.

Definition 5.1 ($\mathbf{N}^2_{\mathrm{B}[l]}$). The inference rules of $\mathrm{N}^2_{\mathrm{B}[l]}$ are of the form: for any $k \in \omega_l$ and any positive integer m,

$$\frac{\begin{array}{c}[\alpha^i]\\ \vdots \\ \beta^i\end{array}}{(\alpha\to\beta)^i}\ (\to I^2) \qquad \frac{(\alpha\to\beta)^i \quad \alpha^i}{\beta^i}\ (\to E^2)$$

$$\frac{\alpha^i \quad \beta^i}{(\alpha\wedge\beta)^i}\ (\wedge I^2) \qquad \frac{(\alpha\wedge\beta)^i}{\alpha^i}\ (\wedge E1^2) \qquad \frac{(\alpha\wedge\beta)^i}{\beta^i}\ (\wedge E2^2)$$

$$\frac{\alpha^{i+1}}{(\mathrm{X}\alpha)^i}\ (\mathrm{X}I2^2) \qquad \frac{(\mathrm{X}\alpha)^i}{\alpha^{i+1}}\ (\mathrm{X}E2^2)$$

$$\frac{\{\ \alpha^{i+j}\ \}_{j\in\omega_l}}{(\mathrm{G}\alpha)^i}\ (\mathrm{G}I^2) \qquad \frac{(\mathrm{G}\alpha)^i}{\alpha^{i+k}}\ (\mathrm{G}E^2)$$

$$\frac{\alpha^l}{\alpha^{l+m}}\ (\mathrm{X}I1^2) \quad \frac{\alpha^{l+m}}{\alpha^l}\ (\mathrm{X}E1^2).$$

Any proofs constructed only on an assumption are considered to be axioms.

Definition 5.2. Let L_1 be the set of formulas of $\mathrm{N}_{\mathrm{B}[l]}$ and L_2 be the set of indexed formulas of $\mathrm{N}^2_{\mathrm{B}[l]}$. A mapping f from L_1 to L_2 is defined by $f(\mathrm{X}^i\alpha) := \alpha^i$ for any formula α. A mapping g from L_2 to L_1 is defined by $g(\alpha^i) := \mathrm{X}^i\alpha$ for any formula α.

Proposition 5.3 (Equivalence between $\mathbf{N}^2_{\mathrm{B}[l]}$ and $\mathbf{N}_{\mathrm{B}[l]}$). *$\mathrm{N}^2_{\mathrm{B}[l]}$ and $\mathrm{N}_{\mathrm{B}[l]}$ are equivalent, that is, we have the following.*

1. *Let Γ be a set of indexed formulas and β be an indexed formula. If there is a proof P in $\mathrm{N}^2_{\mathrm{B}[l]}$ such that $\mathrm{oa}(P) = \Gamma$ and $\mathrm{end}(P) = \beta$, then there is a proof P' in $\mathrm{N}_{\mathrm{B}[l]}$ such that $\mathrm{oa}(P') = g(\Gamma)$ and $\mathrm{end}(P') = g(\beta)$.*

2. *Let Γ be a set of formulas and β be a formula. If there is a proof P in $\mathrm{N}_{\mathrm{B}[l]}$ such that $\mathrm{oa}(P) = \Gamma$ and $\mathrm{end}(P) = \beta$, then there is a proof P' in $\mathrm{N}^2_{\mathrm{B}[l]}$ such that $\mathrm{oa}(P') = f(\Gamma)$ and $\mathrm{end}(P') = f(\beta)$.*

5.2 PNJ by Baratella and Masini

We give a comparison between Baratella-Masini's PNJ [2] (for full intuitionistic LTL) and the system $\mathrm{N}^2_{\mathrm{B}[l]}$ introduced just above. The base logic $\mathrm{B}[l]$ is regarded as a sublogic of PNJ. Thus, we consider only about the $\{\to, \wedge, \mathrm{X}, \mathrm{G}\}$-fragment of PNJ. We also call it PNJ.

PNJ adopts the notion of *position formula*. A position formula is an expression of the form α^s where α is a formula and s is a *position*. The set of positions is the set of all pairs $\langle n, S\rangle$ where n is a natural number and S is a finite set of tokens from a denumerable set $T = \{x_0, x_1, ...\}$. Let $s = \langle n, S\rangle$ and $t = \langle m, T\rangle$ be positions. The following notations are used:

1. $s + t$ for $\langle n + m, S \cup T\rangle$,

2. if $T = \emptyset$, we write $s + m$ for $s + t$,

3. if $t = \langle 0, \{x\}\rangle$, we write $s + x$ for $s + t$.

Then, PNJ is defined by the following inference rules:

$$\dfrac{\begin{matrix}[\alpha^s]\\ \vdots \\ \beta^s\end{matrix}}{(\alpha{\to}\beta)^s}\ ({\to}I^s) \qquad \dfrac{(\alpha{\to}\beta)^s \quad \alpha^s}{\beta^s}\ ({\to}E^s)$$

$$\dfrac{\alpha^s \quad \beta^s}{(\alpha \wedge \beta)^s}\ (\wedge I^s) \qquad \dfrac{(\alpha \wedge \beta)^s}{\alpha^s}\ (\wedge E1^s) \qquad \dfrac{(\alpha \wedge \beta)^s}{\beta^s}\ (\wedge E2^s)$$

$$\frac{\alpha^{s+1}}{(\mathrm{X}\alpha)^s}\ (\mathrm{X}I^s) \qquad \frac{(\mathrm{X}\alpha)^s}{\alpha^{s+1}}\ (\mathrm{X}E^s) \qquad \frac{\alpha^{s+x}}{(\mathrm{G}\alpha)^s}\ (\mathrm{G}I^s) \qquad \frac{(\mathrm{G}\alpha)^s}{\alpha^{s+t}}\ (\mathrm{G}E^s)$$

$$\frac{\alpha^s \quad \begin{array}{c}[\alpha^{s+x}]\\ \vdots \\ \alpha^{s+\langle 1,x\rangle}\end{array}}{\alpha^{s+t}}\ (\mathrm{IND})$$

where the token x in (IND) does not occur in s or in any of the assumptions on which $\alpha^{s+\langle 1,x\rangle}$ depends, with the exception of the discharged assumptions α^{s+x}.

The inference rules without $\{(GI^s), (GE^s), (\mathrm{IND})\}$ are just the same as $\mathrm{N}^2_{\mathrm{B}[l]}$ where the position s is replaced by an index i. The differences are that PNJ uses (IND) and does not use any infinite or many premises rules like (GI) and (GI^2).

5.3 $\lambda^{\circ[]}$ by Yuse and Igarashi

As mentioned before, $\lambda^{\circ[]}$ [30] is an extension (or integration) of Davies' λ° [9] and Davies-Pfenning's $\lambda^{[]}$ [10]. The direct comparison between $\lambda^{\circ[]}$ and the systems proposed in this paper cannot be obtained, since $\lambda^{\circ[]}$ is based on the different framework of the *hypothetical judgments*. We thus give a comparison between their base logics (i.e., Hilbert-style axiomatizations).

Before to compare the systems, we present some inference rules for $\lambda^{\circ[]}$ as examples. The type judgment is of the form $\Delta;\Gamma \vdash^n M : \alpha$ where Δ and Γ are persistent context and ordinary context, respectively, and n denotes the time (or stage). The following are examples of the inference rules with respect to X and G:

$$\frac{\Delta;\Gamma \vdash^{n+1} M : \alpha}{\Delta;\Gamma \vdash^{n} \text{next } M : \mathrm{X}\alpha} \qquad \frac{\Delta;\Gamma \vdash^{n} M : \mathrm{X}\alpha}{\Delta;\Gamma \vdash^{n+1} \text{prev } M : \alpha} \qquad \frac{\Delta;\cdot \vdash^{n} M : \alpha}{\Delta;\Gamma \vdash^{n} \text{box } M : \mathrm{G}\alpha}$$

$$\frac{\Delta;\Gamma \vdash^{n+i} M : \mathrm{G}\alpha \quad \Delta, u ::^{n+i} \alpha;\Gamma \vdash^{n} N : \beta \quad (i \geq 0)}{\Delta;\Gamma \vdash^{n} \text{let box } u\ =_i\ M\ inN\beta}\ .$$

In [30], Yuse and Igarashi state that the corresponding base logic for $\lambda^{\circ[]}$ includes the following axiom schemes and inference rules:

1. $\mathrm{G}(\alpha\to\beta)\to(\mathrm{G}\alpha\to G\beta)$,

2. $\mathrm{X}(\alpha{\to}\beta) \leftrightarrow (\mathrm{X}\alpha{\to}X\beta)$,
3. $\mathrm{G}\alpha{\to}\alpha$,
4. $\mathrm{G}\alpha{\to}\mathrm{GG}\alpha$,
5. $\mathrm{G}\alpha{\to}\mathrm{X}\alpha$,
6. $\mathrm{GX}\alpha \leftrightarrow \mathrm{XG}\alpha$,

$$\frac{\alpha{\to}\beta \quad \alpha}{\beta}\ (\mathrm{mp}) \qquad \frac{\alpha}{\mathrm{G}\alpha}\ (\mathrm{Gness}).$$

The characteristic axiom schemes of $\lambda^{\circ[]}$ are 2 and 6. The logic B[l] has all the axiom schemes and inference rules displayed above, and B[l] also has the time induction axiom $\alpha{\to}(\mathrm{G}(\alpha{\to}\mathrm{X}\alpha){\to}\mathrm{G}\alpha)$ and the inference rule of the form

$$\frac{\alpha}{\mathrm{X}\alpha}\ (\mathrm{Xness}).$$

Thus, B[l] is strictly stronger than the base logic of $\lambda^{\circ[]}$.

5.4 PLTL$_{ND}$ by Bolotov et al.

We give a comparison between the natural deduction system PLTL$_{ND}$ by Bolotov et al. [5] and the systems presented in this paper. Although PLTL$_{ND}$ is a full classical system with the until operator, we consider only the $\{\to, \wedge, \mathrm{X}, \mathrm{G}\}$-fragment. We also call it the same name PLTL$_{ND}$.

PLTL$_{ND}$ uses a labelled formula of the form $i : \alpha$, which is similar to the indexed formula α^i of $\mathrm{N}^2_{\mathrm{B}[l]}$. PLTL$_{ND}$ includes the following inference rules:

$$\frac{[i:\alpha] \quad i:\beta}{i:\alpha{\to}\beta}\ ({\to}I^b) \qquad \frac{i:\alpha{\to}\beta \quad i:\alpha}{i:\beta}\ ({\to}E^b)$$

$$\frac{i:\alpha \quad i:\beta}{i:\alpha\wedge\beta}\ (\wedge I^b) \qquad \frac{i:\alpha\wedge\beta}{i:\alpha}\ (\wedge E1^b) \qquad \frac{i:\alpha\wedge\beta}{i:\beta}\ (\wedge E2^b)$$

$$\frac{i^c:\alpha \quad Next(i,i^c)}{i:\mathrm{X}\alpha}\ (\mathrm{X}I^b) \qquad \frac{i:\mathrm{X}\alpha}{i^c:\alpha}\ (\mathrm{X}E^b)$$

$$\frac{j:\alpha \quad [i\le j]}{i:\mathrm{G}\alpha}\ (\mathrm{G}I^b) \qquad \frac{j:\mathrm{G}\alpha \quad [i\le j]}{j:\alpha}\ (\mathrm{G}E^b)$$

$$\frac{i : \alpha \quad [i \leq j] \quad j : \alpha \to \mathrm{X}\alpha}{i : \mathrm{G}\alpha}\ (\mathrm{IND}^b)$$

where $\cdot^c$, $Next$ and $\leq$ are certain operators and relation (the precise definitions are omitted here), and the rules $(\mathrm{G}E^b)$, $(\mathrm{X}E^b)$ and (IND^b) have some conditions, e.g., in $(\mathrm{G}E^b)$, $i \leq j$ must be the most recent assumption, applying the rule on the step n of the proof, we discard $i \leq j$ and all formulas until the step n.

The use of the operations $\cdot^c$ and $Next$ and the rule (IND^b) is different from our framework.

As a consequence of the comparisons, the advantage of the proposed framework is regarded as the simple setting of the systems and the natural correspondence between sequent calculus and natural deduction.

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