REPORTS ON MATHEMATICAL LOGIC
47 (2012), 147–171
DOI:10.4467/20842589RM.12.007.0688

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PARACONSISTENCY AND CONSISTENCY UNDERSTOOD AS THE ABSENCE OF THE NEGATION OF ANY IMPLICATIVE THEOREM

A b s t r a c t. As is stated in its title, in this paper consistency is understood as the absence of the negation of any implicative theorem. Then, a series of logics adequate to this concept of consistency is defined within the context of the ternary relational semantics with a set of designated points, negation being modelled with the Routley operator. Soundness and completeness theorems are provided for each one of these logics. In some cases, strong (i.e., in respect of deducibility) soundness and completeness theorems are also proven. All logics in this paper are included in Lewis' S4. They are all paraconsistent, but none of them is relevant.

1. Introduction

As it is well-known, in Lewis' opinion, the following are two basic principles governing the concept of deducibility:

Received 28 August 2011

1. "Any proposition one chooses may be deduced from a denial of a tautological or necessary truth" ([4], pp. 250-251).

2. "Tautologies in general are derivable from any premise we please" ([4], p. 251).

The "Principle of excluded middle"

$$\text{pem. } A \vee \neg A$$

and the propositions of the form:

$$A \wedge \neg A$$

are, according to Lewis, the paradigms of tautologies and contradictory propositions (i.e., the denials of tautologies), respectively (cf. [4], Chapter VIII). Therefore, by principle 1 and the "Principle of non-contradiction", i.e.,

3. pnc. $\neg(A \wedge \neg A)$.

We have:

4. $(A \wedge \neg A) \to B$

Consequently, none of Lewis' logics is paraconsistent (cf. Definition 1.6 below).

Now, in [10], a series of paraconsistent logics included in Lewis' S4 are defined. These logics are adequate to consistency understood as the absence of the negation of any theorem. The aim of this paper is to build up a restriction of this series by defining a spectrum of logics adequate to consistency understood as the absence of the negation of any *implicative* theorem. A consequence of this concept of consistency is that principles 1 and 2, which are not valid here, are restricted in the following form:

1′. Any proposition one chooses may be deduced from the denial of any tautological (or necessary) conditional.

2′. Tautological conditionals are derivable from any premise we please.

Therefore, 4 is unprovable; moreover, the following rule is unprovable:

5. $A \wedge \neg A \vdash B$

Consequently, all logics in this paper are paraconsistent logics (cf. Definition 1.6 below).

In order to develop these notions, we need some definitions.

Let L be a propositional language with at least the connectives $\rightarrow$ (conditional), $\wedge$ (conjunction) and $\neg$ (negation), and S be a logic defined on L. The concept of an S-theory (a theory built upon S) is defined as follows:

Definition 1.1. T is an *S-theory* iff T is closed under adjunction and S-entailment; that is, iff (i) if $A \in T$ and $B \in T$, then $A \wedge B \in T$, and (ii) if $A \rightarrow B$ is a theorem of S and $A \in T$, then $B \in T$.

In [8] two senses of a so-called weak-consistency are defined:

Definition 1.2 (Weak consistency, first sense)**.** A theory T is *w1-inconsistent* (weak inconsistent in the first sense) iff for some theorem A of S, $\neg A \in T$. A theory is *w1-consistent* —weak consistent in the first sense— iff it is not w1-inconsistent.

Definition 1.3 (Weak consistency, second sense)**.** A theory T is *w2-inconsistent* (weak inconsistent in the second sense) iff for some theorem $\neg A$ of S $A \in T$. A theory is *w2-consistent* —weak consistent in the second sense— iff it is not w2-inconsistent.

Let us now define a third sense of "weak consistency", which is the concept of consistency the title of this paper refers to: consistency understood as the absence of the negation of any implicative theorem.

Definition 1.4 (Weak consistency, third sense)**.** A theory T is *w3-inconsistent* (weak inconsistent in the third sense) iff for some theorem $A \rightarrow B$ of S $\neg(A \rightarrow B) \in T$. A theory is *w3-consistent* —weak consistent in the third sense— iff it is not w3-inconsistent.

The aim of this paper is to define a series of logics adequate to this sense of consistency in the ternary relational semantics with a set of designated points. The logics in this spectrum will include the result of extending Routley and Meyer's well-known basic logic B (cf., e.g., [12]) with the characteristic S4 axiom

$$B \rightarrow (A \rightarrow A)$$

Therefore, no logic in this paper is a relevant logic. From another perspective, all logics in the paper will be included in what from an intuitive point of view can be described as Lewis' S4 minus the ECQ ("E contradictione quodlibet") axiom

$$(A \wedge \neg A) \rightarrow B$$

On the other hand, these logics are said to be adequate to the concept of consistency in Definition 1.4 in the sense that the completeness proof can be carried out only if consistency is understood as stated in this definition. If consistency is understood in the standard sense (i.e., as the absence of any contradiction), or as in Definition 1.2 or Definition 1.3, the completeness proof would fail, at least in the present semantical context, i.e., the ternary relational semantics with a set of designated points.

Let us now consider the following definition:

Definition 1.5 (Absolute consistency)**.** A theory T is *a-inconsistent* (inconsistent in an absolute sense) iff T is trivial, i.e., iff every wff belongs to T. A theory is *a-consistent* —consistent in an absolute sense— iff it is not a-inconsistent.

It will be proved (cf. Proposition 4.5) that any theory built upon any of the logics contemplated in this paper is w3-consistent iff it is a-consistent.

Next, we turn to a brief discussion on the concept of paraconsistency.

Let $\vDash$ be a relation of consequence, be it defined either semantically or proof-theoretically. As is known, the standard concept of paraconsistency can be defined as follows (cf. [7]):

Definition 1.6 (Standard concept of paraconsistency)**.** A logic S is said to be *sc-paraconsistent* (paraconsistent in the standard sense) iff the rule

$$A \wedge \neg A \vDash B$$

is not derivable in S.

On the other hand, let us take into account the following definition of consistency.

Definition 1.7 (Standard concept of consistency)**.** A theory T is *sc-inconsistent* (inconsistent in the standard sense) iff $A \wedge \neg A \in T$ for some wff A. A theory is *sc-consistent* —consistent in the standard sense— iff it is not sc-inconsistent.

There can be no doubt whatsoever that the concept of sc-consistency is at the base of the concept of sc-paraconsistency.

In [9], the concepts of paraconsistency corresponding to the concepts of w1-consistency and w2-consistency are defined like this (where the symbol $\vDash$ is used as in [7] quoted above):

Definition 1.8 (w1-paraconsistency)**.** A logic S is *w1-paraconsistent* iff the rule

$$\text{If} \quad \vDash A, \text{ then } \neg A \vDash B$$

is not derivable in S.

Definition 1.9 (w2-paraconsistency)**.** A logic S is *w2-paraconsistent* iff the rule

$$\text{If} \quad \vDash \neg A, \text{ then } A \vDash B$$

is not provable in S.

Obviously, if the axiom of weak double negation

$$A \to \neg\neg A$$

is provable in S, then S is w1-paraconsistent iff it is w2-paraconsistent. Given that this axiom is provable in all logics in this paper, w1-paraconsistency and w2-paraconsistency will be considered as synonymous, as being co-extensive concepts in the context of the present paper. However, as regards the relationship between w1-paraconsistency and sc-paraconsistency, we remark that the two concepts are by no means equivalent. Consider for instance Łukasiewicz logics, Ł. These logics can be considered as sc-paraconsistent. Actually, they could belong to one of the groups in which paraconsistent logics are customarily classified (cf. [7]). But they are not w1-paraconsistent because the EFQ ("E falso quodlibet") axioms

$$\neg A \to (A \to B)$$

and

$$A \to (\neg A \to B)$$

are provable in Łukasiewicz's logics. We also remark that Ł logics are adequate to w1-consistency in the sense explained above.

Something similar occurs in the case of the logics in this paper. Consider the following definition:

Definition 1.10 (w3-paraconsistency)**.** A logic S is *w3-paraconsistent* iff the rule

$$\text{If} \quad \vDash A \to B, \text{ then } \neg(A \to B) \vDash C$$

is not derivable in S.

It will be shown that all logics in this paper are sc-paraconsistent but that none of them is w3-paraconsistent. That is, *they are not* paraconsistent *in respect of* the concept of consistency to which they are adequate.

The structure of the paper is as follows. In §2, the logic B_{K2} is defined. In §3, a semantics for B_{K2} is provided and soundness of this logic is proved. In §4-5, it is shown that B_{K2} is complete in respect of the semantics introduced in §2. In §6, some extensions of B_{K2} are considered and soundness and completeness are proved. Finally, in §7, it is proved that all logics in the paper are sc-paraconsistent but that none of them is w3-paraconsistent. Strong soundness and completeness of some of the logics here studied are proved. Negation will be modelled with the Routley operator, which is adequate to interpreting a De Morgan negation (on the origin of this type of modelling introduced by the Polish logicians in the fifties of the past century see [2], sec 3.4). Knowledge of the Routley-Meyer semantics for relevant logics will be presupposed.

2. The logic B_{K2}

As is known, Routley and Meyer's basic positive logic B_+ can be axiomatized as follows (cf. [1], [11] or [12])

Axioms

A1. $A \to A$

A2. $(A \wedge B) \to A \quad / \quad (A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4. $A \to (A \vee B) \quad / \quad B \to (A \vee B)$

A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

Rules

$$\begin{aligned} \textit{Modus ponens (MP)}: &\ (\vdash A \to B \ \& \vdash A) \Rightarrow \vdash B \\ \textit{Adjunction (Adj)}: &\ (\vdash A \ \& \vdash B) \Rightarrow \vdash A \wedge B \\ \textit{Suffixing (Suf)}: &\ \vdash A \to B \Rightarrow \vdash (B \to C) \to (A \to C) \\ \textit{Prefixing (Pref)}: &\ \vdash B \to C \Rightarrow \vdash (A \to B) \to (A \to C) \end{aligned}$$

Routley and Meyer's basic logic B is then the result of adding to B_+ the following axioms

A7. $A \to \neg\neg A$

A8. $\neg\neg A \to A$

and the rule

Contraposition (Con): $\vdash A \to B \Rightarrow \vdash \neg B \to \neg A$

Remark 2.1. We note that Meyer and Routley's original B also contained the principle of excluded middle $A \vee \neg A$ together with the truth constant t and the binary connective fusion $\circ$ (cf. [6]).

The following are some theorems and rules of B (a proof for each one of them is sketched to their right):

T1. $\vdash A \to \neg B \Rightarrow \vdash B \to \neg A$	Con, A7
T2. $\vdash \neg A \to B \Rightarrow \vdash \neg B \to A$	Con, A8
T3. $\vdash \neg A \to \neg B \Rightarrow \vdash B \to A$	A7, T2
T4. $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$	Con, T2
T5. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$	Con, T1

By adding the following axiom

A9. $B \to (A \to A)$

to B, we obtain an axiomatization of the logic we will name B_{K2} where, in addition to T1-T5, we have:

T6. $\vdash A \to B \Rightarrow \vdash C \to (A \to B)$	A9
T7. $(A \to B) \to [A \to (A \wedge B)]$	T6
T8. $\vdash A \to B \Rightarrow \vdash \neg(A \to B) \to C$	T2, T6

Notice that A1 is not independent within this axiom system and that T6 is a restricted (to implicative theorems) form of rule K

$$\text{K. } \vdash A \Rightarrow \vdash B \to A$$

which is not derivable in any of the logics in this paper (cf. Appendix).

Remark 2.2. As in [8], the logic B_{K+}, which is the result of adding the K rule to B_+, was defined, the present logic (which contains a restricted form of K) is labelled "B_{K2}" to distinguish it from B_{K+} and its extensions.

3. A semantics for B_{K2}

Next, we provide a semantics for B_{K2}:

Definition 3.1. A B_{K2}-*model* is a structure $\langle K, O, R, *, \vDash\rangle$ where O is a subset of K, R is a ternary relation on K, and $*$ a unary operation on K subject to the following definitions and postulates for all a, b, c, $d \in K$:

d1. $a \leq b =_{\text{df}} (\exists x \in O)Rxab$

d2. $a = b =_{\text{df}} a \leq b \ \& \ b \leq a$

P1. $a \leq a$

P2. $(a \leq b \ \& \ Rbcd) \Rightarrow Racd$

P3. $Rabc \Rightarrow b \leq c$

P4. $a = a**$

P5. $a \leq b \Rightarrow b* \leq a*$

Finally, $\vDash$ is a relation from K to the formulas of the propositional language such that the following conditions are satisfied for all propositional variables p, wff A, B and $a \in K$

(i). $(a \leq b \ \& \ a \vDash p) \Rightarrow b \vDash p$

(ii). $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$

(iii). $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$

(iv). $a \vDash A \to B$ iff for all b, $c \in K$ $(Rabc \ \& \ b \vDash A) \Rightarrow c \vDash B$

(v). $a \vDash \neg A$ iff $a* \nvDash A$

Validity is defined as follows:

Definition 3.2. A formula A is $\mathrm{B_{K2}}$-*valid* ($\vDash_{\mathrm{B_{K2}}} A$) iff $a \vDash A$ for all $a \in O$ in all models.

Then, it is proved:

Theorem 3.3 (Soundness of $\mathrm{B_{K2}}$)**.** *If* $\vdash_{B_{K2}} A$*, then* $\vDash_{B_{K2}} A$.

Proof. It is left to the reader (cf., e.g. [12]). The only difference with the proof of soundness for B concerns the validity of A9, which is valid by P3. □

4. Completeness of $\mathbf{B_{K2}}$ I. The canonical model. W3-consistency

We begin by recalling some definitions (cf. [1] or [12]).

Definition 4.1.

1. A $\mathrm{B_{K2}}$-*theory* is a set of formulas closed under adjunction and provable $\mathrm{B_{K2}}$-entailment. That is, a is a $\mathrm{B_{K2}}$-theory if whenever $A, B \in a$, then $A \wedge B \in a$; and if whenever $A \to B$ is a theorem of $\mathrm{B_{K2}}$ and $A \in a$, then $B \in a$ (cf. Definition 1.1).
2. A $\mathrm{B_{K2}}$-theory a is *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$.
3. A $\mathrm{B_{K2}}$-theory is *regular* iff all theorems of $\mathrm{B_{K2}}$ belong to it.
4. A $\mathrm{B_{K2}}$-theory is *empty* iff no wff belongs to it.
5. A $\mathrm{B_{K2}}$-theory is *w3-consistent* iff for no theorem $A \to B$ of $\mathrm{B_{K2}}$, $\neg(A \to B) \in a$ (cf. Definition 1.4).
6. A $\mathrm{B_{K2}}$-theory is *a-consistent* iff it is not trivial (cf. Definition 1.5).

We can now define the $\mathrm{B_{K2}}$ canonical model:

Definition 4.2. Let K^T be the set of all $\mathrm{B_{K2}}$-theories and R^T be defined on K^T as follows: for all $a, b, c \in K^T$ and wff A, B, $R^T abc$ iff $(A \to B \in a \quad \& \quad A \in b) \Rightarrow B \in c$. Now, let K^C be the set of all non-empty, prime, w3-consistent theories, and O^C the set of all regular, prime, w3-consistent theories. On the other hand, let R^C be the restriction of R^T to K^C and $*^C$ be defined on K^C as follows: for any $a \in K^C$, $a*^C = \{A \mid \neg A \notin a\}$. Finally, $\vDash^C$ is defined as follows: for any $a \in K^C$, $a \vDash^C A$ iff $A \in a$. Then, the $\mathrm{B_{K2}}$-*canonical model* is the structure $\langle K^C, O^C, R^C, *^C, \vDash^C \rangle$.

Remark 4.3. What distinguishes the $\mathrm{B_{K2}}$ canonical model from those for standard relevant logics is just one important fact: in the latter, members in K^C need not be non-empty or regular or consistent in any sense of the term, and, furthermore, members in O^C are not necessarily consistent in any sense of the term. On the other hand, the $\mathrm{B_{K2}}$ canonical model and canonical models in [10] are distinguished by the fact that members in O^C are w1-consistent in the latter.

Next, we will provide a lemma on non-empty theories and an easy proposition stating the equivalence between w3-consistency and a-consistency in the context of K^T.

Lemma 4.4. *Let a be a non-empty member in K^T. Then, a contains every implicative theorem of B_{K2}.*

Proof. Let $A \to B$ be a theorem and $C \in a$. Then, $A \to B \in a$ by T6. □

Proposition 4.5. *Let $a \in K^T$. Then, a is w3-inconsistent iff a is a-inconsistent.*

Proof. (1) From right to left, it is immediate. (b) In the inverse direction it is immediate by T8. □

Given Proposition 4.5, w3-consistency and a-consistency can be considered as synonymous terms in the context of the present paper. Having reached this conclusion, another question arise which merits to be discussed here briefly. What is the relationship between w3-consistency and sc-consistency (i.e., consistency understood in the standard sense of the term —cf. Definition 1.6) in the context of the present paper?

Let a be a $\mathrm{B_{K2}}$-theory. It is clear that a can be sc-consistent but w3-inconsistent, or just the other way around, that is, it can be w3-consistent and sc-inconsistent. Nevertheless, let us suppose that a is a regular member in K^T. Then, if a is sc-consistent, a is obviously w3-consistent. The converse, however, does not hold: a may contain all theorems of B_{K2} and some contradiction but not necessarily the negation of an implicative theorem.

As regards the relationship between w1-consistency and w3-consistency, it is evident that the former entails the latter, but not conversely. And this is the case for regular members in K^T. So, this is also the case for any member in K^T.

We shall return to this important matter later on in Remark 4.8 and in the last section of this paper. We will now prove an important lemma on w3-consistency and the relation R^T, but first let us consider the following:

Lemma 4.6. *Let a, b be non-empty elements in K^T. The set $x = \{B \mid \exists A[A \in b \ \& \ A \to B \in a]\}$ is a non-empty theory such that $R^T abx$.*

Proof. It is easy to prove that x is a theory. Moreover, x is non-empty: let $A \in b$; by A1 and Lemma 4.4, $A \to A \in a$. So, $A \in x$. Finally, $R^T abx$ is immediate by definition of R^T. (Cf. Definition 4.2.) □

Suppose that a and b in Lemma 4.6 are w3-consistent. It is clear that the theory x in this lemma cannot generally be proved w3-consistent. However, it is proved:

Lemma 4.7. *Let a, b be non-empty elements in K^T and c a w3-consistent member in K^T such that $R^T abc$. Then, a and b are w3-consistent.*

Proof. (1) Suppose a is w3-inconsistent. Now, let $C \in b$ and $A \to B$ be a theorem. As a is a-inconsistent (cf. Proposition 4.5), every wff belongs to it and, then, $C \to \neg(A \to B) \in a$. So, $\neg(A \to B) \in c$, contradicting the w3-consistency of c.

(2) Suppose b is w3-inconsistent. Then, $\neg(A \to B) \in b$, $A \to B$ being a theorem. As a is non-empty, $\neg(A \to B) \to \neg(A \to A) \in a$ by Lemma 4.4, T6 and Con. So, $\neg(A \to A) \in c$, which is impossible. □

Remark 4.8. Suppose that c in Lemma 4.6 is sc-consistent. Then, a and b cannot be shown to be sc-consistent. In order to prove a and b sc-consistent, the following restriction of the ECQ axiom

$$(A \wedge \neg A) \to (B \to C)$$

or in fact the weaker formula

$$(A \wedge \neg A) \to [B \to (A \wedge \neg A)]$$

is needed. But these axioms are not derivable in any of the logics in this paper (cf. Appendix). On the other hand, Lemma 4.7 plays an essential role in the completeness proof to follow. Actually, its usage is crucial when proving Lemmas 5.5 and 5.6 and Proposition 6.1, for instance. It is then in this sense that B_{K2} (in fact, all its extensions in §6) are said to be adequate to w3-consistency (cf. §1). However, these logics would not be adequate to sc-consistency.

5. Completeness of $\mathbf{B_{K2}}$ II. $*^C$-theories. The canonical model is in fact a model

We first prove a trivial lemma:

Lemma 5.1. *The following hold for any wff A and a, $b \in K^C$:*

1. $\neg A \in a*^C$ *iff* $A \notin a$
2. $a *^C *^C = a$
3. $a*^C \subseteq b*^C \Leftrightarrow b \subseteq a$

Proof. 1 follows by A7 and A8; 2 and 3 follow by 1 and definitions. □

Next, it is proved that $*^C$ is an operation on K^C.

Proposition 5.2. *$*^C$ is an operation on K^C.*

Proof. Let $a \in K^C$. By Con, T4 and T5 $a*^C$ is a prime theory (cf. [12]). Next, we prove that $a*^C$is non-empty and w3-consistent. (1) Suppose $a*^C$ is empty and let $A \to B$ be a theorem. Then, $A \to B \notin a*^C$. So, $\neg(A \to B) \in a$, contradicting the w3-consistency of a. (2) Suppose $a*^C$ is w3-inconsistent. Then, $\neg(A \to B) \in a*^C$, $A \to B$ being a theorem. So, $A \to B \notin a$ (Lemma 5.1(1)), which is impossible by Lemma 4.4. □

In order to demonstrate that the canonical model is actually a model, we shall start by proving three auxiliary lemmas:

Lemma 5.3. *Let a be a non-empty member in K^T and A a wff such that $A \notin a$. Then, there is some x in K^C such that $a \subseteq x$ and $A \notin x$.*

Proof. Build up a prime non-empty theory x such that $a \subseteq x$ and $A \notin x$ (cf. [12]). This theory is a-consistent. So, it is w3-consistent (Proposition 4.5). Therefore, $x \in K^C$. □

From Lemma 5.3, we obtain the following corollary:

Corollary 5.4. *Let A be a wff such that $\nvdash_{B_{K2}} A$. Then, there is some x in O^C such that $B_{K2} \subseteq x$ and $A \notin x$.*

Lemma 5.5. *Let a, b be non-empty elements in K^T and $c \in K^C$ such that $R^T abc$. Then, there are x, $y \in K^C$ such that $a \subseteq x$, $b \subseteq y$, $R^T xbc$ and $R^T ayc$.*

Proof. Build up prime non-empty theories x and y such that $R^T xbc$ and $R^T ayc$ (cf. [12]). By lemma 4.7, x and y are, in addition, w3-consistent. □

Lemma 5.6. *For any a, $b \in K^C$, $a \leq^C b$ iff $a \subseteq b$.*

Proof. From left to right, it is immediate. So, suppose $a \subseteq b$. Clearly, $R^T \mathrm{B}_{\mathrm{K2}} aa$. Then, by Lemma 5.5 there is some (regular) y in K^C such that $\mathrm{B}_{\mathrm{K2}} \subseteq y$ and $R^C yaa$. As y is regular, $y \in O^C$. By hypothesis, $R^C yab$. Then, $a \leq^C b$ by d1. □

We can now proceed to prove the following proposition:

Proposition 5.7. *The canonical model is in fact a model.*

Proof. We have to prove:

1. The set O^C is not empty.
2. Clauses (i)-(v) are satisfied by the canonical model.
3. Postulates P1-P5 hold in the canonical model.

1. 1 is immediate by Corollary 5.4.
2. Clause (i) is immediate by Lemma 5.6; clauses (ii), (iii) and (v), and (iv) from left to right are proved as in the semantics for B (cf., e.g., [12]). So, let us prove clause (iv) from right to left. Suppose for wff A, B and $a \in K^C$, $A \to B \notin a$. We prove that there are x, $y \in K^C$ such that $R^C axy$, $A \in x$ and $B \notin y$. The sets $z = \{C \mid\vdash_{\mathrm{B}_{\mathrm{K2}}}$

$A \to C\}$, $u = \{C \mid \exists D[D \to C \in a \ \ \& \ \ D \in z]\}$ are theories such that $R^T azu$. Now, z is w3-consistent: if it is not, then $B \in z$ (Proposition 4.5), i.e., $\vdash A \to B$, and then, $A \to B \in a$ (Lemma 4.4), contradicting the hypothesis. Moreover, $A \in z$ (by A1). So, u is non-empty (Lemma 4.6). On the other hand, $B \notin u$ (if $B \in u$, then $A \to B \in a$, contradicting the hypothesis). Therefore, u is w3-consistent (Proposition 4.5). Consequently, we have non-empty, w3-consistent theories z, u such that $R^T azu$, $A \in z$ and $B \notin u$. Now, by Lemma 5.3, u is extended to some $y \in K^C$ such that $u \subseteq y$ and $B \notin y$. Obviously, $R^C azy$. Next, by Lemma 5.5, z is extended to some $x \in K^C$ such that $z \subseteq x$ and $R^C axy$. Clearly, $A \in x$. Therefore, we have prime non-empty w3-consistent theories x, y such that $A \in x$, $B \notin y$ and $R^C axy$, as it was required.

3. 3 is easy to prove with the assistance of Lemma 5.6: P1 and P2 are trivial and P3 is easy by Lemma 4.4; Then, P4 and P5 follow by Lemma 5.1(2) and Lemma 5.1(3), respectively.

□

Then, the completeness of $\mathrm{B_{K2}}$ is immediate by Corollary 5.4 and Proposition 5.7:

Theorem 5.8 (Completeness of $\mathrm{B_{K2}}$). *If* $\vDash_{B_{K2}} A$, *then* $\vdash_{B_{K2}} A$.

6. Extensions of $\mathbf{B_{K2}}$

Consider the following axioms:

A10. $(B \to C) \to [(A \to B) \to (A \to C)]$
A11. $(A \to B) \to [(B \to C) \to (A \to C)]$
A12. $[A \to (A \to B)] \to (A \to B)$
A13. $[(A \to A) \to B] \to B$
A14. $(A \to B) \to [C \to (A \to B)]$
A15. $(A \to B) \to (\neg B \to \neg A)$
A16. $A \vee \neg A$

Axioms A10-A13 are, respectively, the *prefixing*, *suffixing*, *contraction* and *specialized assertion* axioms; A14 is a restricted (S4) version of axiom K; A15 is (one of the forms of) the weak contraposition axiom, and, finally, A16 is the principle of excluded middle.

Consider now the following definition and semantical postulates:

d3. $R^2abcd =_{\mathrm{df}} (\exists x \in K)(Rabx \ \ \& \ \ Rxcd)$

PA10. $R^2abcd \Rightarrow (\exists x \in K)(Rbcx \ \ \& \ \ Raxd)$

PA11. $R^2abcd \Rightarrow (\exists x \in K)(Racx \ \ \& \ \ Rbxd)$

PA12. $Raaa$

PA13. $(\exists x \in K)Raxa$

PA14. $R^2abcd \Rightarrow Racd$

PA15. $Rabc \Rightarrow Rac * b*$

PA16. $a \in O \Rightarrow a* \leq a$

We prove the following:

Proposition 6.1. *Given the logic B_{K2} and B_{K2}-semantics, PA10-PA16 are the postulates that correspond (c.p) to A10-A16, respectively.*

That is to say, given $\mathrm{B_{K2}}$ semantics, A10-A16 are proved valid with PA10-PA16, respectively; and, given the logic $\mathrm{B_{K2}}$, PA10-PA16 are proved canonically valid with A10-A16, respectively. (We shall use the abbreviature "cp" for "corresponding postulate", i.e., "postulate that corresponds to (a certain axiom)").

Proof. That this is so is proved in (or can easily be derived from) [12], with the exception perhaps of the cases regarding the correspondence between A12 and PA12 and A13 and PA13, which is proved in [10], Proposition 8. □

Remark 6.2. Notice that PA13 is not the c.p to A13 in relevant logics. The c.p to A13 is a bit more complicated in those logics (cf., e.g., [12]). On the other hand, note that in relevant logics, PA12 is the c.p to the modus ponens axiom $[(A \to B) \wedge A] \to B$.

Now, let $\mathrm{SB_{K2}}$ be any extension of $\mathrm{B_{K2}}$ with any selection of A10-A16. (As above, it is supposed that the rules of $\mathrm{B_{K2}}$ can now be applied to the theorems of $\mathrm{SB_{K2}}$.) And let $\mathrm{SB_{K2}}$-models be defined, similarly, as

B_{K2}-models except for the addition of the c.p to the new axioms added. Furthermore, define SB_{K2}-validity in a similar way to which B_{K2}-validity was defined. (Cf. Definitions 3.1 and 3.2.) Finally, provide a definition of the SB_{K2} canonical model similar to that of the B_{K2} canonical model (cf. Definitions 4.1, 4.2). It is clear that from Proposition 6.1 and Theorems 3.3, 5.8, we have:

Theorem 6.3 (Soundness and completeness of SB_{K2})**.** $\vDash_{SB_{K2}} A$ *iff* $\vdash_{SB_{K2}} A$.

We shall not study here the different logics definable from B_{K2} by adding some (or all) of A10-A16 and the relations that they maintain to each other. Nevertheless, let us remark that A1, A11, A12 and A14 together with MP axiomatize the implicative fragment of Lewis' S4 (cf. [5]), whence it is easy to see that B_+ plus these axioms is a logic deductively equivalent to the positive fragment of Lewis' S4, $S4_+$ (cf. [3]). Therefore, B_{K2} plus A11, A12, A14, A15 and A16 can intuitively be described as $S4_+$ supplemented with the double negations axioms (A7, A8), all forms of the De Morgan laws (T4, T5), all forms of the contraposition axioms (they are easily derived from A15 with A7 and A8) and the principle of excluded middle A16 (equivalently, the principle of non-contradiction $\neg(A \wedge \neg A)$). This logic can provisionally be labelled $S4_{K2}$. Notice that $S4_{K2}$ plus axiom ECQ ("E contradictione quodlibet")

$$\text{ECQ. } (A \wedge \neg A) \rightarrow B$$

is easily shown to be deductively equivalent to Lewis' S4 (cf. [3]). The logic $S4_{K2}$ also lacks all forms of the reductio axioms. Actually, we can state the following:

Proposition 6.4.

1. *$S4_{K2}$ plus any of the formulas and rules that follow (all of them deriv-*

able in S4) is a logic deductively equivalent to S4:

$$(a).\ (A \wedge \neg A) \to B$$
$$(b).\ (A \to \neg A) \to \neg A$$
$$(c).\ (\neg A \to A) \to A$$
$$(d).\ \vdash A \to B \Rightarrow \vdash (A \to \neg B) \to \neg A$$
$$(e).\ \vdash \neg A \to B \Rightarrow \vdash (\neg A \to \neg B) \to A$$
$$(f).\ \vdash A \to B \Rightarrow \vdash (\neg A \to B) \to B$$
$$(g).\ (A \to B) \to \neg(A \wedge \neg B)$$
$$(h).\ (A \to \neg B) \to \neg(A \wedge B)$$
$$(i).\ [B \to (A \wedge \neg A)] \to \neg B$$
$$(j).\ [(A \to B) \wedge (A \to \neg B)] \to \neg A$$

2. *None of the formulas (a)-(j) is derivable in $S4_{K2}$.*

Proof.

1. By $S4_{K2}$ and any of (a)-(j), we have:

$$(1).\ (A \wedge \neg A) \to \neg(A \to A)$$

   By A9, Con and A8,

$$(2).\ \neg(A \to A) \to B$$

   Then, axiom ECQ is immediate by (1) and (2).

2. Cf. Appendix.

□

Therefore, from an intuitive point of view, $S4_{K2}$ could equivalently be described as Lewis' S4 without the reductio axioms.

We end this section with the following remark:

Remark 6.5. The reductio rules in the following forms are derivable in $S4_{K2}$:

$$(\vdash A \to B \ \& \ \vdash A \to \neg B) \Rightarrow \vdash \neg A$$
$$(\vdash \neg A \to B \ \& \ \vdash \neg A \to \neg B) \Rightarrow \vdash A$$
$$(\vdash A \to B \ \& \ \vdash \neg A \to B) \Rightarrow \vdash B$$

7. SB_{K2} logics are sc-paraconsistent

In this section SB_{K2} logics are shown to be sc-paraconsistent.

Consider the following relation of semantical consequence.

Definition 7.1. Let Γ be a set of wffs and A a wff. Then, $\Gamma \vDash A$ iff for all $a \in K$ in all SB_{K2}-models, if $a \vDash \Gamma$, then $a \vDash A$ ($a \vDash \Gamma$ iff $a \vDash B$ for every $B \in \Gamma$).

As in standard relevant logics, we can prove the following lemmas:

Lemma 7.2. $(a \leq b \ \& \ a \vDash A) \Rightarrow b \vDash B$

Proof. Induction on the length of A (cf., e.g., [12]). □

Lemma 7.3. $\vDash_{SB_{K2}} A \to B$ *iff for all* $a \in K$ *in all* SB_{K2}*-models,* $a \vDash A \Rightarrow a \vDash B$

Proof. By using Lemma 7.2, P1 and d1 (cf. §3). □

Then, the meaning of the consequence relation in Definition 7.1 is clear. Let $\Gamma = \{B_1, ..., B_m\}$. By Lemma 7.3, $\Gamma \vDash A$ iff $\vDash_{SB_{K2}} (B_1 \wedge ... \wedge B_m) \to A$. So, the following can be proved without difficulty.

Proposition 7.4. *Each* SB_{K2} *is sc-paraconsistent.*

Proof. We prove that rule ECQ (cf. Definition 1.6)

$$A \wedge \neg A \vDash B$$

is not SB_{K2}-valid. Axiom ECQ

$$(A \wedge \neg A) \to B$$

is not provable in $S4_{K2}$ (Proposition 6.4), so, neither is it in any of the SB_{K2} logics. By the completeness theorem (Theorem 6.3), $(A \wedge \neg A) \to B$ is not valid. By Lemma 7.3, there is some $x \in K$ in some SB_{K2} model such that $x \vDash A \wedge \neg A$ and $x \nvDash B$. Therefore, $A \wedge \neg A \nvDash B$ by Definition 7.1. □

Nevertheless, we can show that rule ECQ fails according to a stricter, more adequate (to SB_{K2} logics) consequence relation. Consider the following definition.

Definition 7.5. Let Γ be a set of wffs. Then, $\Gamma \vDash A$ iff for all $a \in O$ in all $\mathrm{SB_{K2}}$ models, if $a \vDash \Gamma$, then $a \vDash A$ ($a \vDash \Gamma$ iff $a \vDash B$ for every $B \in \Gamma$).

Then, it is proved:

Proposition 7.6. *Let M be the canonical model. Then, for some $x \in O^C$ and wff A, B, $x \vDash A \wedge \neg A$ and $x \nvDash B$.*

Proof. Let p_i be the i-th propositional variable. Consider now the following set of formulas $y = \{B \mid \vdash_{\mathrm{SB_{K2}}} A \;\&\; \vdash_{\mathrm{SB_{K2}}} [A \wedge (p_i \wedge \neg p_i)] \to B\}$. It is easy to prove that y is a $\mathrm{SB_{K2}}$-theory (it is closed under adjunction and $\mathrm{SB_{K2}}$-entailment). Moreover, it is regular (it contains all the theorems of $\mathrm{SB_{K2}}$ by A2), but is sc-inconsistent: $(p_i \wedge \neg p_i) \in y$. Nevertheless, we show that y is a-consistent. The rule $\vdash_{\mathrm{SB_{K2}}} A \Rightarrow \vdash_{\mathrm{SB_{K2}}} [A \wedge (p_i \wedge \neg p_i)] \to C$ is not provable in $\mathrm{S4_{K2}}$ (cf. Appendix). So, there is a wff B and a theorem A such that $\nvdash_{\mathrm{SB_{K2}}} [A \wedge (p_i \wedge \neg p_i)] \to B$. By definition of y, $B \notin y$ (in fact, it would be easy to select a particular wff B and theorem A. Cf. Appendix). Therefore, y is a -consistent. Now, by using Lemma 5.3, y is extended to a (regular) prime, a-consistent theory x such that $(p_i \wedge \neg p_i) \in x$ but $B \notin x$. By definition of the canonical models (cf. Definition 4.2), $x \vDash (p_i \wedge \neg p_i)$, $x \nvDash B$, whence $p_i \wedge \neg p_i \nvDash B$ by Definition 7.5. □

Consequently, it is proved:

Proposition 7.7. *All SB_{K2} logics —B_{K2} included— are sc-paraconsistent.*

Proof. By Proposition 7.6 and Definition 1.6. □

However, in Proposition 7.13 below, it is proved that none of the $\mathrm{SB_{K2}}$ is w3-paraconsistent.

Let us now investigate to which proof-theoretical consequence relation the semantical relation introduced in Definition 7.5 corresponds. Consider the following definition.

Definition 7.8. Let Γ be a set of wffs and A a wff. Then, $\Gamma \vdash_{\mathrm{SB_{K2}}} A$ iff there is a finite sequence of wffs $B_1, ...B_m$ such that B_m is A and for each i ($1 \leq i \leq m$), one of the following is the case: (1) $B_i \in \Gamma$ (2) B_i is a theorem (3) B_i is by Adj or by MP.

On the other hand, the set of consequences of a set of wff Γ (CnΓ) is defined in a classical way as follows:

Definition 7.9. $\text{Cn}\Gamma = \{A \mid \Gamma \vdash_{\text{SB}_{\text{K2}}} A\}$

Then, we immediately have:

Proposition 7.10. *For any set of wffs* Γ, *Cn*Γ *is a* SB_{K2} *theory.*

Proof. It is clear that $\text{Cn}\Gamma$ is closed under adjunction and SB_{K2}-entailment. □

And, moreover, we have:

Theorem 7.11. *If* $\Gamma \vdash_{SB_{K2}} A$*, then* $\Gamma \vDash A$*.*

Proof. Induction on the proof of A from Γ, $\Gamma \vdash_{\text{SB}_{\text{K2}}} A$. The cases in which $A \in \Gamma$ and A is by Adj are trivial. So, suppose that A is by MP. Then, for some wff C, $\Gamma \vdash_{\text{SB}_{\text{K2}}} C \to A$, $\Gamma \vdash_{\text{SB}_{\text{K2}}} C$. By hypothesis, $\Gamma \vDash C \to A$, $\Gamma \vDash C$. Now, let $a \vDash \Gamma$ for some $a \in O$ in some SB_{K2}-model. Then, $a \vDash C$, $a \vDash C \to A$. By clause (iv) (Definition 3.1), for all $x, y \in K$, $(Raxy \ \& \ x \vDash C) \Rightarrow y \vDash A$. As $a \in O$, by d1, $(x \leq y \ \& \ x \vDash C) \Rightarrow y \vDash A$. By P1, $a \leq a$. So, $a \vDash A$, as it was to be proved. Finally, if A is a theorem of SB_{K2}, the case follows by Theorem 6.3 (soundness of SB_{K2}). □

Theorem 7.12. *If* $\Gamma \vDash A$ *, then* $\Gamma \vdash_{SB_{K2}} A$*.*

Proof. Suppose $\Gamma \nvdash_{\text{SB}_{\text{K2}}} A$. If Γ is empty, then the proof is immediate by Theorem 6.3. So, suppose that Γ is a non-empty set. Obviously, $A \notin \text{Cn}\Gamma$. Then, by Lemma 5.3, there is some (regular) x in K^C (i.e., some $x \in O^C$) such that $\text{Cn}\Gamma \subseteq x$ and $A \notin x$. As $\Gamma \subseteq \text{Cn}\Gamma$, $\Gamma \subseteq x$. Now, by definitions of the canonical model (cf. Definition 4.2), $x \vDash^C \Gamma$ (i.e., $x \vDash^C B$ for every $B \in x$) and $x \nvDash^C A$, i.e., $\Gamma \nvDash A$, as was to be proved. □

Finally, it is proved:

Proposition 7.13. *None of the* SB_{K2} *logics —*B_{K2} *included— is w3-paraconsistent.*

Proof. Suppose $A \to B$ is a theorem of SB_{K2}. It is clear that $\neg(A \to B) \vdash_{\text{SB}_{\text{K2}}} C$, by T8 (cf. Definition 7.8). Then, SB_{K2} is not w3-paraconsistent by Definition 1.10. □

Let us end the paper with some remarks on the consequence relations just introduced, on consistency, and on paraconsistency.

Remark 7.14. On Definition 7.8: Suppose that the following two clauses are added to clauses (1)-(3) in Definition 7.8 (cf. §2): (4) B_i is by Suf. (5) B_i is by Pref. Then, Theorem 7.11 would not be provable because not for any $a \in O$ in any $\mathrm{SB_{K2}}$ model, $a \vDash B \to C \Rightarrow a \vDash (A \to B) \to (A \to C)$, $a \vDash A \to B \Rightarrow a \vDash (B \to C) \to (A \to C)$. On the other hand, suppose that clause (2) in this definition is replaced by (2′): B_i is an axiom. Then, not all the theorems would necessarily be derivable from a given set of wff Γ in logics without the prefixing axiom A10 or the suffixing axiom A11. These are the reasons why clauses (4), (5) and (2′) do not appear in Definition 7.8. Nevertheless, it is obvious that clause (2) can be replaced by clause (2′) in the case of any $\mathrm{SB_{K2}}$ logic in which A10 and A11 are theorems. In this case, a classically defined concept of deducibilty would result.

Remark 7.15. On the proof-theoretical consequence relation corresponding to the semantical relation in Definition 7.1: consider the following consequence relation:

Definition 7.16. Let Γ be a set of wffs. Then, $\Gamma \vdash A$ iff there is a finite sequence $B_1, ..., B_m$ where B_m is A and for each i $(1 \leq i \leq m)$, one of the following is the case: (1) $B_i \in \Gamma$ (2) B_i is by Adj (3) B_i is by $\mathrm{SB_{K2}}$ -entailment (B_i is by $\mathrm{S_{K2}}$ -entailment iff for some wff C, $\Gamma \vdash C$, $C \to B_i$ being a theorem of $\mathrm{SB_{K2}}$).

Now, let $\vDash$ be the consequence relation introduced in Definition 7.1. The proof of the following theorem is left to reader.

Theorem 7.17. $\Gamma \vdash A$ *iff* $\Gamma \vDash A$.

Remark 7.18. Strong soundness and completeness: it is clear that Theorems 7.11 and 7.12 are a proof of the strong (i.e., in respect of deducibilty) soundness and completeness of each $\mathrm{SB_{K2}}$ logic in which A10 and A11 hold.

Remark 7.19. Adequacy to w1-consistency: in Remark 4.8 it was noted that $\mathrm{SB_{K2}}$ logics are not adequate to sc-consistency. But, are they adequate to w1-consistency? In order to prove that they actually are (cf. Lemma 4.7 and Remark 4.8), the following rules would be needed:

r1. $(\vdash A \ \& \ \vdash B) \Rightarrow \vdash \neg A \to (B \to C)$

r2. $(\vdash A \ \& \ \vdash B) \Rightarrow \vdash \neg A \to \neg B$

These rules are not provable in any of the SB_{K2} logics (cf. Appendix). Furthermore, they are not even admissible in any SB_{K2} logics in which $\neg(A \wedge \neg A)$ holds. Otherwise,

$$(A \wedge \neg A) \to \neg(A \to A)$$

is immediate by r2, and then, axiom ECQ

$$(A \wedge \neg A) \to B$$

is also immediate (cf. Proposition 6.4). Consequently, SB_{K2} logics in which A16 $(A \vee \neg A)$ holds are not adequate to w1-consistency. Regarding the logics without A16 (equivalently, without $\neg(A \wedge \neg A)$), the question of their adequacy is left open: are r1 and r2 admissible? If they are, logics without A16 are in principle equivalently adequate to w3-consistency and w1-consistency.

Remark 7.20. w1-paraconsistency: let S be any logic in which $A \to \neg\neg A$ and $\neg(A \wedge \neg A)$ hold. It is clear that if S is sc-paraconsistent, then S is w1-consistent. Therefore, all SB_{K2} logics in which A16 is a theorem are w1-paraconsistent in addition to being sc-paraconsistent. Concerning SB_{K2} logics without A16, the question of their w1-paraconsistency is left open.

Finally, a note on rules K and assertion.

Remark 7.21. Rule K

$$\text{K. } \vdash A \Rightarrow \vdash B \to A$$

is not derivable in any of the logics in this paper (cf. Appendix). But in SB_{K2} logics with A16, it is not even admissible. Otherwise, axiom ECQ would be immediate: by K and $\neg(A \wedge \neg A)$, $\vdash \neg B \to \neg(A \wedge \neg A)$. Then, $\vdash (A \wedge \neg A) \to B$ by T3.

Rule assertion

$$\text{Asser. } \vdash A \Rightarrow \vdash (A \to B) \to B$$

is derivable in none of the logics in this paper (cf. Appendix). But in SB_{K2} logics with A16, it is not even admissible because rule K is immediate by Asser and A9.

8. Appendix

Consider the following set of matrices where the designated values are starred:

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	3	3	2
*2	0	0	3	3	1
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

This set satisfies the axioms and rules of $S4_{K2}$, but falsifies the following rules and theses (cf. Remark 4.8, Remark 7.19, Remark 7.21 and Proposition 7.6):

1. $(A \wedge \neg A) \to B$ $(A = 1,\ B = 0)$
2. $(A \to \neg A) \to \neg A$ $(A = 1)$
3. $(\neg A \to A) \to A$ $(A = 2)$
4. $\vdash A \to B \Rightarrow\ \vdash (A \to \neg B) \to \neg A$ $(A = B = 1)$
5. $\vdash \neg A \to B \Rightarrow\ \vdash (\neg A \to \neg B) \to A$ $(A = 2,\ B = 1)$
6. $\vdash A \to B \Rightarrow\ \vdash (\neg A \to B) \to B$ $(A = 1,\ B = 2)$
7. $(A \to B) \to \neg(A \wedge \neg B)$ $(A = B = 1)$
8. $(A \to \neg B) \to \neg(A \wedge B)$ $(A = B = 1)$
9. $[B \to (A \wedge \neg A)] \to \neg B$ $(A = B = 1)$
10. $[(A \to B) \wedge (A \to \neg B)] \to \neg A$ $(A = B = 1)$
11. $\vdash A \Rightarrow\ \vdash B \to A$ $(A = 2,\ B = 3)$
12. $\vdash A \Rightarrow\ \vdash (A \to B) \to B$ $(A = B = 2)$
13. $\vdash A \Rightarrow\ \vdash [A \wedge (p_i \wedge \neg p_i)] \to C$ $(A = 2,\ p_i = 1,\ C = 0)$
14. $(A \wedge \neg A) \to (B \to C)$ $(A = B = 1,\ C = 0)$
15. $(A \wedge \neg A) \to [B \to (A \wedge \neg A)]$ $(A = 1,\ B = 2)$

16. $(\vdash A\ \&\ \vdash B) \Rightarrow \vdash \neg A \rightarrow (B \rightarrow C)$ $(A = B = 2,\ C = 0)$

17. $(\vdash A\ \&\ \vdash B) \Rightarrow \vdash \neg A \rightarrow \neg B$ $(A = 2,\ B = 3)$

It is proved with this set of matrices that 1-3, 7-10 and 14, 15 are not *theorems* of $S4_{K2}$; and that 4-6, 11-13 and 16, 17 are not *derivable rules* of $S4_{K2}$. But, in addition, it can easily be shown that the rules are not even admissible: (1) By Proposition 6.4, rules 4, 5 and 6 are not admissible (2) Let p_i be a fixed propositional variable and change A in 11-13, 16 and 17 for $\neg(p_i \wedge \neg p_i)$ and B in 16 and 17 for $p_i \rightarrow p_i$. Then, it is obvious that these rules cannot be admissible in $S4_{K2}$.

Notes

- Work supported by research projects FFI2008-05859/FISO and FFI2011-28494 financed by the Spanish Ministry of Science and Innovation.
- G. Robles is supported by Program Ramón y Cajal of the Spanish Ministry of Science and Innovation.
- I thank a referee of the RML for his (her) comments and suggestions on a previous draft of this paper.

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