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ON SOME PROPERTIES OF QUASI-MV ALGEBRAS AND $\sqrt{'}$ QUASI-MV ALGEBRAS. PART IV

A b s t r a c t. In the present paper, which is a sequel to [20, 4, 12], we investigate further the structure theory of quasi-MV algebras and $\sqrt{'}$quasi-MV algebras. In particular: we provide a new representation of arbitrary $\sqrt{'}$qMV algebras in terms of $\sqrt{'}$qMV algebras arising out of their MV* term subreducts of regular elements; we investigate in greater detail the structure of the lattice of $\sqrt{'}$qMV varieties, proving that it is uncountable, providing equational bases for some of its members, as well as analysing a number of slices of special interest; we show that the variety of $\sqrt{'}$qMV algebras has the amalgamation property; we provide an axiomatisation of the 1-assertional logic of $\sqrt{'}$qMV algebras; lastly, we reconsider the correspondence between Cartesian $\sqrt{'}$qMV algebras and a category of Abelian lattice-ordered groups with operators first addressed in [10].

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1. Introduction

Quasi-MV algebras are generalisations of MV algebras that have been introduced in [16] and investigated over the past few years. The original motivation for their study arises in connection with quantum computation; more precisely, as a result of the attempt to provide a convenient abstraction of the algebra over the set of all density operators of the Hilbert space $\mathbb{C}^2$, endowed with a suitable stock of quantum logical gates. Quite independently of this aspect, however, qMV algebras present several, purely algebraic, motives of interest within the frameworks of quasi-subtractive varieties [15] and of the subdirect decomposition theory for varieties [13]. $\sqrt{\prime}$*quasi-MV algebras* (for short, $\sqrt{\prime}$qMV algebras) were introduced as term expansions of qMV algebras by an operation of square root of the negation [9]. The above referenced papers contain the basics of the structure theory for these varieties, including appropriate standard completeness theorems w.r.t. the algebras over the complex numbers which constituted the starting point of the whole research project. In the subsequent papers [20, 4, 10, 12, 14] the algebraic properties of qMV algebras and $\sqrt{\prime}$qMV algebras were investigated in greater detail.

The present paper continues the series initiated with [20, 4, 12] by gathering some more results of the same kind. Actually, the main focus of the present article is on $\sqrt{\prime}$qMV algebras alone, but we preferred to keep the same title as in the previous members of the series to underscore the resemblance of the underlying approaches and themes. In particular, after a quick recap in § 2, aimed at making this paper as self-contained as possible, in § 3 we provide a new representation of arbitrary $\sqrt{\prime}$qMV algebras in terms of $\sqrt{\prime}$qMV algebras arising out of their MV* term subreducts of regular elements. In § 4 we investigate in greater detail the structure of the lattice of $\sqrt{\prime}$qMV varieties, explicitly proving for the first time that it is uncountable, providing equational bases for some of its members, as well as analysing a number of slices of special interest. § 5 amounts to a short note to the effect that the whole variety of $\sqrt{\prime}$qMV algebras has the amalgamation property. § 6 gives an axiomatisation of the 1-assertional logic of $\sqrt{\prime}$qMV algebras. Finally, in § 7 we reconsider the correspondence between Cartesian $\sqrt{\prime}$qMV algebras and a category of Abelian lattice-ordered groups with operators first addressed in [10], establishing a few additional results on that score.

The terminology and notation used in the paper is duly explained in the Preliminaries section. As to the rest, except where indicated otherwise, we keep to the conventions typically adopted in universal algebra and abstract algebraic logic.

2. Preliminaries

The concept of quasi-MV algebra is introduced first.

Definition 1. A *quasi-MV algebra* (for short, qMV algebra) is an algebra $\mathbf{A} = \langle A, \oplus, ', 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ satisfying the following equations:

A1. $x \oplus (y \oplus z) \approx (x \oplus z) \oplus y$
A2. $x'' \approx x$
A3. $x \oplus 1 \approx 1$
A4. $(x' \oplus y)' \oplus y \approx (y' \oplus x)' \oplus x$
A5. $(x \oplus 0)' \approx x' \oplus 0$
A6. $(x \oplus y) \oplus 0 \approx x \oplus y$
A7. $0' \approx 1$

We can think of a qMV algebra as identical to an MV algebra, except for the fact that 0 need not be a neutral element for the truncated sum $\oplus$. Of course, a qMV algebra is an MV algebra iff it satisfies the additional equation $x \oplus 0 \approx x$. An immediate consequence of Definition 1 is the fact that the class of qMV algebras is a *variety* in its signature. Henceforth, such a variety will be denoted by $q\mathbb{MV}$. The subvariety of MV algebras will be denoted by $\mathbb{MV}$.

Definition 2. We introduce the following abbreviations:

$$x \otimes y = (x' \oplus y')';$$
$$x \Cup y = x \oplus (x' \otimes y);$$
$$x \Cap y = x \otimes (x' \oplus y).$$

The relation $x \leq y$ defined by $x \Cap y = x \oplus 0$ is a preorder on any qMV algebra $\mathbf{A}$, which is however a partial order if and only if $\mathbf{A}$ is an MV algebra. Examples of "pure" qMV algebras, i.e. qMV algebras that are *not* MV algebras, are given by the next two structures over the complex numbers, $\mathbf{S}$ (for *square*) and $\mathbf{D}$ (for *disc*).

Example 3. (standard quasi-MV algebras). We introduce two *standard quasi-MV algebras.* $\mathbf{S}$ is the algebra $\left\langle [0,1] \times [0,1], \oplus^{\mathbf{S}}, {}'^{\mathbf{S}}, 0^{\mathbf{S}}, 1^{\mathbf{S}} \right\rangle$, where:

- $\langle a, b \rangle \oplus^{\mathbf{S}} \langle c, d \rangle = \left\langle \min(1, a+c), \frac{1}{2} \right\rangle$;
- $\langle a, b \rangle'^{\mathbf{S}} = \langle 1-a, 1-b \rangle$;
- $0^{\mathbf{S}} = \left\langle 0, \frac{1}{2} \right\rangle$;
- $1^{\mathbf{S}} = \left\langle 1, \frac{1}{2} \right\rangle$.

Note that $\langle a, b \rangle \oplus^{\mathbf{S}} \left\langle 0, \frac{1}{2} \right\rangle \neq \langle a, b \rangle$ whenever $b \neq \frac{1}{2}$.
$\mathbf{D}$ is the subalgebra of $\mathbf{S}$ whose universe is the set

$$\{\langle a, b \rangle : a, b \in R \text{ and } (1-2a)^2 + (1-2b)^2 \leq 1\}.$$

Next, we expand qMV algebras by an additional unary operation of *square root of the inverse* and by a constant k, which realises in the standard algebras the element $\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$.

Definition 4. A $\sqrt{'}$ *quasi-MV algebra* (for short, $\sqrt{'}$*qMV algebra)* is an algebra $\mathbf{A} = \left\langle A, \oplus, \sqrt{'}, 0, 1, k \right\rangle$ of type $\langle 2, 1, 0, 0, 0 \rangle$ such that, upon defining $a' = \sqrt{'}\sqrt{'}a$ for all $a \in A$, the following conditions are satisfied:

SQ1. $\langle A, \oplus, ', 0, 1 \rangle$ is a quasi-MV algebra;
SQ2. $k = \sqrt{'}k$;
SQ3. $\sqrt{'}(a \oplus b) \oplus 0 = k$ for all $a, b \in A$.

$\sqrt{'}$qMV algebras form a variety in their own similarity type, hereafter named $\sqrt{'}q\mathbb{MV}$. We remark in passing that it is impossible to add a square root of the inverse to a nontrivial MV algebra. Examples of $\sqrt{'}$qMV algebras are the following expansions of the standard qMV algebras over the complex numbers:

Example 5. (standard $\sqrt{'}$qMV algebras). We introduce two *standard* $\sqrt{'}$*qMV algebras.* $\mathbf{S}_r$ is the algebra $\left\langle [0,1] \times [0,1], \oplus^{\mathbf{S}_r}, \sqrt{'}^{\mathbf{S}_r}, 0^{\mathbf{S}_r}, 1^{\mathbf{S}_r}, k^{\mathbf{S}_r} \right\rangle$, where:

- $\left\langle [0,1] \times [0,1], \oplus^{\mathbf{S}_r}, {}'^{\mathbf{S}_r}, 0^{\mathbf{S}_r}, 1^{\mathbf{S}_r} \right\rangle$ is the qMV algebra $\mathbf{S}$ of Example 3;
- $\sqrt{'}^{\mathbf{S}_r} \langle a, b \rangle = \langle b, 1-a \rangle$;

- $k^{\mathbf{S}_r} = \left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$.

$\mathbf{D}_r$ *is the subalgebra of* $\mathbf{S}_r$ *whose universe is the set*

$$\{\langle a, b\rangle : a, b \in R \text{ and } (1-2a)^2 + (1-2b)^2 \leq 1\}.$$

An element a of a $\sqrt{'}$qMV algebra $\mathbf{A}$ is said to be:

- *regular*, if $a \oplus 0 = a$;
- *coregular*, if $\sqrt{'}a \oplus 0 = \sqrt{'}a$;
- *irregular*, if it is neither regular nor coregular.

A term (formula) $t(\overrightarrow{x})$ of the same type as $\sqrt{'}q\mathbb{MV}$ is called *regular* iff for all $\sqrt{'}$qMV algebras $\mathbf{A}$ and all $\overrightarrow{a} \in A$, $t^{\mathbf{A}}(\overrightarrow{a})$ is a regular element.

Definition 6. Let $\mathbf{A}$ be a $\sqrt{'}$qMV algebra and let $a, b \in A$. We set:

$$a\lambda b \text{ iff } a \oplus 0 = b \oplus 0 \text{ and } \sqrt{'}a \oplus 0 = \sqrt{'}b \oplus 0$$

or, equivalently,

$$a\lambda b \text{ iff } a \leq b, b \leq a, \sqrt{'}a \leq \sqrt{'}b \text{ and } \sqrt{'}b \leq \sqrt{'}a.$$

Moreover,

$$a\mu b \text{ iff either } a = b \text{ or neither } a \text{ nor } b \text{ is irregular.}$$

λ and μ are congruences on every $\sqrt{'}$qMV algebra whose intersection is the identity. These congruences allow us to introduce two special classes of $\sqrt{'}$qMV algebras: *Cartesian* algebras, where λ is the identity relation Δ, and *flat* algebras, where λ is the universal relation ∇.

Definition 7. A $\sqrt{'}$qMV algebra $\mathbf{A}$ is called *Cartesian* iff $\lambda = \Delta$, i.e. iff it satisfies the quasiequation

$$x \oplus 0 \approx y \oplus 0 \wedge \sqrt{'}x \oplus 0 \approx \sqrt{'}y \oplus 0 \Rightarrow x \approx y$$

A $\sqrt{'}$qMV algebra $\mathbf{A}$ is called *flat* iff $\lambda = \nabla$. We denote by $\mathbb{F}$ the class of flat $\sqrt{'}$qMV algebras, and by $\mathbb{C}$ the class of Cartesian $\sqrt{'}$qMV algebras.

$\mathbb{F}$ is a subvariety of $\sqrt{\prime}q\mathbb{MV}$, axiomatised relative to it by the equation $x \oplus 0 \approx 0$; $\mathbb{C}$, on the other hand, is a proper subquasivariety of $\sqrt{\prime}q\mathbb{MV}$. By extension, a congruence θ of a $\sqrt{\prime}$qMV algebra $\mathbf{A}$ is called *Cartesian (flat)* iff $\mathbf{A}/\theta$ is Cartesian (flat). Algebras with no irregular elements are called *strongly Cartesian*; strongly Cartesian $\sqrt{\prime}$qMV algebras are Cartesian, but not always conversely. For example, the algebras of Example 5 are Cartesian but not strongly Cartesian. The algebras in the next example, on the other hand, are flat.

Example 8. $\mathbf{F}_{100}$ is the algebra whose universe is the 2-element set $\{0, b\}$, s.t. all truncated sums equal 0, while $\sqrt{\prime}0 = 0$ and $\sqrt{\prime}b = b$. $\mathbf{F}_{020}$ is the algebra whose universe is the 3-element set $\{0, a, b\}$, whose semigroup reduct is again the constant 0-valued semigroup and whose table for $\sqrt{\prime}$ is given by

$\sqrt{\prime}$	
0	0
a	b
b	a

Both $\mathbf{F}_{100}$ and $\mathbf{F}_{020}$ are flat; moreover, $\mathbf{F}_{100}$ is simple, while $\mathbf{F}_{020}$ is a nonsimple subdirectly irreducible algebra having three congruences: $\Delta, \lambda = \nabla$ and the monolith θ whose cosets are $\{a, b\}$ and $\{0\}$.

More generally, we denote by $\mathbf{F}_{nmp}$ the finite flat algebra which contains n fixpoints for $\sqrt{\prime}$ beside 0, m fixpoints for the inverse which are not fixpoints for $\sqrt{\prime}$, and p elements which are not fixpoints under either operation.

By "MV* algebras" we mean expansions of MV algebras by an additional constant k, satisfying the axiom $k \approx k'$. This variety has been investigated by Lewin and his colleagues [17], who proved that: i) the category of such algebras is equivalent to the category of MV algebras; ii) the variety itself is generated as a quasivariety by the standard algebra over the $[0, 1]$ interval. Although e.g. all nontrivial Boolean algebras are ruled out by this definition, in virtue of the above-mentioned results the two concepts can be considered, for many purposes, interchangeable. Two important constructions yield $\sqrt{\prime}$qMV algebras out of an MV* algebra:

Definition 9. Let $\mathbf{A} = \langle A, \oplus^{\mathbf{A}}, {}^{\prime\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}}, k^{\mathbf{A}} \rangle$ be an MV* algebra. If $f(A)$ is a bijective copy of $A - \{k\}$ disjoint from A, the rotation of $\mathbf{A}$ is the algebra

$$\mathrm{Rt}\,(\mathbf{A}) = \left\langle A \cup f(A), \oplus^{\mathrm{Rt}(\mathbf{A})}, \sqrt{\prime}^{\mathrm{Rt}(\mathbf{A})}, 0^{\mathbf{A}}, 1^{\mathbf{A}}, k^{\mathbf{A}} \right\rangle$$

uniquely determined by the following stipulations:

$$a \oplus^{\mathrm{Rt}(\mathbf{A})} b = \begin{cases} a \oplus^{\mathbf{A}} b \text{ if } a, b \in A, \\ a \oplus^{\mathbf{A}} k^{\mathbf{A}} \text{ otherwise.} \end{cases}$$

$$\sqrt{\prime}^{\mathrm{Rt}(\mathbf{A})}(a) = \begin{cases} f(a) \text{ if } a \in A, \\ \left(f^{-1}(a)\right)^{\prime\mathbf{A}} \text{ otherwise.} \end{cases}$$

The *pair algebra* over $\mathbf{A}$ is the algebra

$$\mathcal{P}(\mathbf{A}) = \left\langle A^2, \oplus^{\mathcal{P}(\mathbf{A})}, \sqrt{\prime}^{\mathcal{P}(\mathbf{A})}, 0^{\mathcal{P}(\mathbf{A})}, 1^{\mathcal{P}(\mathbf{A})}, k^{\mathcal{P}(\mathbf{A})} \right\rangle$$

where:

- $\langle a, b\rangle \oplus^{\mathcal{P}(\mathbf{A})} \langle c, d\rangle = \left\langle a \oplus^{\mathbf{A}} c, k^{\mathbf{A}} \right\rangle$;
- $\sqrt{\prime}^{\mathcal{P}(\mathbf{A})} \langle a, b\rangle = \left\langle b, a^{\prime\mathbf{A}} \right\rangle$;
- $0^{\mathcal{P}(\mathbf{A})} = \left\langle 0^{\mathbf{A}}, k^{\mathbf{A}} \right\rangle$;
- $1^{\mathcal{P}(\mathbf{A})} = \left\langle 1^{\mathbf{A}}, k^{\mathbf{A}} \right\rangle$;
- $k^{\mathcal{P}(\mathbf{A})} = \left\langle k^{\mathbf{A}}, k^{\mathbf{A}} \right\rangle$.

As an illustration of the rotation construction, $\mathtt{Rt}\,(\mathbf{Ł}_5)$ is depicted in Fig. 2.2.

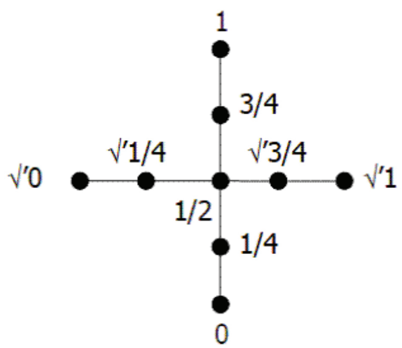


Fig. 2.2. $\mathtt{Rt}\,(\mathbf{Ł}_5)$.

$\mathcal{P}(\mathbf{A})$ is always Cartesian, while $\mathrm{Rt}(\mathbf{A})$ is even strongly Cartesian. Conversely, every Cartesian $\sqrt{'}$qMV algebra is embeddable into a pair algebra:

Theorem 10. *[9, Theorem 36] Every Cartesian $\sqrt{'}$qMV algebra $\mathbf{A}$ is embeddable into the pair algebra $\mathcal{P}(\mathbf{R_A})$ over its MV* term subreduct $\mathbf{R_A}$ of regular elements.*

Generic $\sqrt{'}$qMV algebras are not amenable to such a representation. However, we have the following result.

Theorem 11. *[9, Theorem 37] For every $\sqrt{'}$qMV algebra $\mathbf{Q}$, there exist a Cartesian algebra $\mathbf{C}$ and a flat algebra $\mathbf{F}$ such that $\mathbf{Q}$ can be subdirectly embedded into $\mathbf{C} \times \mathbf{F}$.*

A standard completeness theorem holds for $\sqrt{'}q\mathbb{MV}$:

Theorem 12. *[9, Corollary 53] $\mathbf{S}_r$ generates $\sqrt{'}$q$\mathbb{MV}$ as a variety.*

On the other hand, the subvariety of $\sqrt{'}q\mathbb{MV}$ generated by $\mathbf{D}_r$ is not finitely based, as shown in [14, Theorem 40].

We also make a note once and for all of the following result (a sort of restricted Jónsson's Lemma for $\sqrt{'}q\mathbb{MV}$ since these algebras are not congruence distributive), which will be repeatedly used in the sequel without special mention:

Lemma 13. *[12, Corollary 32] Let $\mathbb{K}$ be a class of $\sqrt{'}$qMV algebras. If $\mathbf{A} \in V(\mathbb{K})$ is a subdirectly irreducible Cartesian algebra, then $\mathbf{A} \in HSP_U(\mathbb{K})$.*

3. A representation theorem for $\sqrt{'}$qMV algebras

The two representation results for $\sqrt{'}$qMV algebras contained in Theorems 10 and 11 are flawed by a common shortcoming: the representation mappings are embeddings, rather than isomorphisms. It would be desirable to amend this defect and characterise $\sqrt{'}$qMV algebras along the lines of the analogous theorem for qMV algebras to be found in [4, § 2], where a generic qMV algebra is proved isomorphic to a qMV algebra arising out of an MV algebra with additional labels. This much will be accomplished in the present section.

Definition 14. Let $\mathbf{A}$ be an MV* algebra. A *numbered MV* algebra* over $\mathbf{A}$ is an ordered quintuple $\mathcal{A} = \langle \mathbf{A}, \gamma, \kappa_1, \kappa_2, \kappa_3 \rangle$, where γ is a cardinal function with domain A^2 and $\kappa_1, \kappa_2, \kappa_3$ are cardinals s.t.: 1) $\kappa_1 + \kappa_2 + \kappa_3 = \gamma\left(k^{\mathbf{A}}, k^{\mathbf{A}}\right)$; 2) if κ_2 is a natural number, then it is even; 3) if κ_3 is a natural number, then it is a multiple of 4.

If one thinks of a $\sqrt{'}$qMV algebra as a subalgebra of a pair algebra $\mathcal{P}(\mathbf{A})$ over an MV* algebra (possibly) along with an additional number of elements corresponding to non-singleton λ-cosets, then, intuitively, the function γ assigns to every member $\langle a, b \rangle$ the cardinality of $\langle a, b \rangle / \lambda$, while κ_1, κ_2 and κ_3 respectively express the number of fixpoints for $\sqrt{'}$, of fixpoints for $'$ that are not themselves fixpoints for $\sqrt{'}$, and of non-fixpoints for $'$ to be found in $\langle k, k \rangle / \lambda$. Bearing this interpretation in mind, we are ready to define *label* $\sqrt{'}$*qMV algebras.*

Definition 15. Let $\mathcal{A} = \langle \mathbf{A}, \gamma, \kappa_1, \kappa_2, \kappa_3 \rangle$ be a numbered MV* algebra. Let moreover

$$\begin{aligned} K_1 &= \{\delta + 1 : \delta < \kappa_1\}; \\ K_2 &= \{1 + \kappa_1 + \delta : \delta < \kappa_2\}; \\ K_3 &= \{1 + \kappa_1 + \kappa_2 + \delta : \delta < \kappa_3\}, \end{aligned}$$

and let g, h be, respectively, an involution on K_2 and a function of period 4 on K_3. A *label* $\sqrt{'}$*qMV algebra* on $\mathcal{A}$ is an algebra

$$\mathbf{B} = \left\langle \mathbf{B}, \oplus^{\mathbf{B}}, \sqrt{'}^{\mathbf{B}}, 0^{\mathbf{B}}, 1^{\mathbf{B}}, k^{\mathbf{B}} \right\rangle$$

of type $\langle 2, 1, 0, 0, 0 \rangle$ s.t.:

- $B = \bigcup_{a,b \in A} (\{\langle a, b \rangle\} \times \gamma(a, b))$;
- $\langle a_1, b_1, l_1 \rangle \oplus^{\mathbf{B}} \langle a_2, b_2, l_2 \rangle = \left\langle a_1 \oplus^{\mathbf{A}} a_2, k^{\mathbf{A}}, 0 \right\rangle$;
- $\sqrt{'}^{\mathbf{B}} \langle a, b, l \rangle = \begin{cases} \left\langle b, a'^{\mathbf{A}}, l \right\rangle, \text{ if } a \neq k \text{ or } b \neq k \\ \quad \text{or } (a = b = k \text{ and } l \in K_1) \\ \left\langle b, a'^{\mathbf{A}}, g(l) \right\rangle, \text{ if } a = b = k \text{ and } l \in K_2 \\ \left\langle b, a'^{\mathbf{A}}, h(l) \right\rangle, \text{ if } a = b = k \text{ and } l \in K_3 \end{cases}$
- $0^{\mathbf{B}} = \left\langle 0^{\mathbf{A}}, k^{\mathbf{A}}, 0 \right\rangle$;

- $1^{\mathbf{B}} = \left\langle 1^{\mathbf{A}}, k^{\mathbf{A}}, 0 \right\rangle$;
- $k^{\mathbf{B}} = \left\langle k^{\mathbf{A}}, k^{\mathbf{A}}, 0 \right\rangle$.

Observe that we omitted some angle brackets and parentheses for the sake of notational irredundancy; accordingly, we sometimes refer to elements of B as "triples", with a slight linguistic abuse. Keeping in mind our previous intuitive description of a $\sqrt{\prime}$qMV algebra $\mathbf{Q}$ as a subalgebra of the pair algebra $\mathcal{P}(\mathbf{R_Q})$ over the MV* algebra $\mathbf{R_Q}$ (possibly) along with an additional number of elements corresponding to non-singleton λ-cosets, every member $a \in \mathbf{Q}$ appears in B as the triple consisting of its projections $a \oplus 0$ and $\sqrt{\prime}a \oplus 0$ and a label uniquely characterising a within a/λ. We remark that B is defined in such a way as to exclude triples whose first projection a and second projection b are such that $\gamma(a,b) = 0$. Intuitively, this corresponds to the fact that, in general, not all elements of $\mathcal{P}(\mathbf{R_Q})$ belong to the subalgebra $\mathbf{Q}$.

We now show that the name "label $\sqrt{\prime}$qMV algebra" is not a misnomer.

Lemma 16. *Every label $\sqrt{\prime}$qMV algebra is a $\sqrt{\prime}$qMV algebra.*

Proof. We check only a few representative axioms, leaving the remainder of this task to the reader and omitting all unnecessary subscripts and superscripts.

$$\begin{aligned}\sqrt{\prime}\sqrt{\prime}\langle a,b,l\rangle \oplus \langle 0,k,0\rangle &= \langle a',b',l^*\rangle \oplus \langle 0,k,0\rangle \\ &= \langle a',k,0\rangle \\ &= \sqrt{\prime}\sqrt{\prime}\langle a,k,0\rangle \\ &= \sqrt{\prime}\sqrt{\prime}\left(\langle a,b,l\rangle \oplus \langle 0,k,0\rangle\right).\end{aligned}$$

That $\sqrt{\prime}\sqrt{\prime}k = k$ is clear enough, while

$$\begin{aligned}\sqrt{\prime}\left(\langle a_1,b_1,l_1\rangle \oplus \langle a_2,b_2,l_2\rangle\right) \oplus \langle 0,k,0\rangle &= \sqrt{\prime}\left(\langle a_1 \oplus a_2, k, 0\rangle\right) \oplus \langle 0,k,0\rangle \\ &= \left\langle k, (a_1 \oplus a_2)', 0\right\rangle \oplus \langle 0,k,0\rangle \\ &= \langle k,k,0\rangle.\end{aligned}$$

□

Before going on to show that every $\sqrt{\prime}$qMV algebra is isomorphic to a label $\sqrt{\prime}$qMV algebra, we establish a useful auxiliary lemma.

Lemma 17. *If $\mathbf{A}$ is a $\sqrt{\prime}$qMV algebra and $a \in \mathbf{A}$, then the function $f(x) = \sqrt{\prime}x$ is a bijection between a/λ and $\sqrt{\prime}a/\lambda$.*

Proof. Injectivity is clear: if $\sqrt{\prime}b = \sqrt{\prime}c$, then $b = \sqrt{\prime}\sqrt{\prime}\sqrt{\prime}\sqrt{\prime}b = \sqrt{\prime}\sqrt{\prime}\sqrt{\prime}\sqrt{\prime}c = c$.

As regards surjectivity, suppose $b \in \sqrt{\prime}a/\lambda$, i.e. $b \oplus 0 = \sqrt{\prime}a \oplus 0$ and $\sqrt{\prime}b \oplus 0 = a' \oplus 0$. Then $\sqrt{\prime}b' \oplus 0 = \left(\sqrt{\prime}b \oplus 0\right)' = (a' \oplus 0)' = a \oplus 0$, while $b'' \oplus 0 = b \oplus 0 = \sqrt{\prime}a \oplus 0$, whence $\sqrt{\prime}b' \in a/\lambda$ and, clearly, $f(\sqrt{\prime}b') = b$. $\square$

We now have to define the target structure of our representation. If $\mathbf{Q}$ is an arbitrary $\sqrt{\prime}$qMV algebra, then the term subreduct $\mathbf{R_Q}$ of regular elements is an MV* algebra, whence

$$\mathcal{R}_{\mathcal{Q}}= \langle \mathbf{R_Q},\ \gamma, \kappa_1, \kappa_2, \kappa_3 \rangle$$

where:

- $\gamma\,(a,b) = \left|\left\{c \in Q : c \oplus 0 = a \text{ and } \sqrt{\prime}c \oplus 0 = b\right\}\right|$;
- $\kappa_1 = \left|\left\{c \in Q : c \oplus 0 = \sqrt{\prime}c \oplus 0 = k \text{ and } \sqrt{\prime}c = c\right\}\right|$;
- $\kappa_2 = \left|\left\{c \in Q : c \oplus 0 = \sqrt{\prime}c \oplus 0 = k \text{ and } \sqrt{\prime}c \neq c \text{ and } c = c'\right\}\right|$;
- $\kappa_3 = \left|\left\{c \in Q : c \oplus 0 = \sqrt{\prime}c \oplus 0 = k \text{ and } c \neq c'\right\}\right|$,

is a numbered MV* algebra. The fact that κ_2 (κ_3) is the union of two (four) disjoint equipotent subsets via the bijection induced by $\sqrt{\prime}$ automatically determines an obvious involution g on K_2 and a corresponding function h of period 4 on K_3, and this, in turn, according to Definition 15, univocally specifies a label $\sqrt{\prime}$qMV algebra on $\mathcal{R}_{\mathcal{Q}}$, which we call $\mathbf{B}^{g,h}_{\mathbf{Q}}$. We now prove that:

Theorem 18. *Every $\sqrt{\prime}$qMV algebra $\mathbf{Q}$ is isomorphic to a label $\sqrt{\prime}$qMV algebra $\mathbf{B}^{g,h}_{\mathbf{Q}}$ on the numbered MV* algebra $\mathcal{R}_{\mathcal{Q}}$ over its own term subreduct $\mathbf{R_Q}$ of regular elements.*

Proof. For $a \in Q$, let $a/\lambda = \left\{c_j : j < \gamma\left(a \oplus 0, \sqrt{\prime}a \oplus 0\right)\right\}$, where $b = c_0$ in case $b = b \oplus 0$. If $a = c_i$, we define $\varphi\,(a) = \left\langle a \oplus 0, \sqrt{\prime}a \oplus 0, i\right\rangle$. We first have to check that φ is one-one. However, if $\varphi\,(a) = \varphi\,(b)$, we have in particular that $\left\langle a \oplus 0, \sqrt{\prime}a \oplus 0\right\rangle = \left\langle b \oplus 0, \sqrt{\prime}b \oplus 0\right\rangle$, whence $a/\lambda = b/\lambda$. Since $i = j$, we get that $a = c_i = c_j = b$. Also, φ is onto $B^{g,h}_{\mathbf{Q}}$ because a generic element of $B_{\mathbf{Q}}^{g,h}$ has the form $\langle a,b,i\rangle$, whence $\gamma(a,b)\neq 0$ and so there exists a $c\in Q$ s.t. $c=c_i$ in $\left\{d\in Q : d\oplus 0=a \text{ and } \sqrt{\prime}d\oplus 0=b\right\}$; clearly, $\varphi(c)=\langle a,b,i\rangle$.

It remains to check that φ is a homomorphism. However, applying the appropriate $\sqrt{\prime}$qMV axioms and our stipulation that $q=c_0$ in case $q=q\oplus 0$,

$$\begin{aligned}\varphi\left(a\oplus^{\mathbf{Q}} b\right) &= \left\langle a\oplus^{\mathbf{Q}} b,k,0\right\rangle\\ &= \left\langle a\oplus^{\mathbf{Q}} 0,\sqrt{\prime}a\oplus^{\mathbf{Q}} 0,i\right\rangle \oplus^{\mathbf{B}_{\mathbf{Q}}^{g,h}} \left\langle b\oplus^{\mathbf{Q}} 0,\sqrt{\prime}b\oplus^{\mathbf{Q}} 0,j\right\rangle\\ &= \varphi(a)\oplus^{\mathbf{B}_{\mathbf{Q}}^{g,h}}\varphi(b).\end{aligned}$$

In a similar fashion, we can prove that the constants are all preserved. As regards the square root of the negation, we have to go through a case-splitting argument. If $a\notin k/\lambda$, we observe that by Lemma 17 the equivalence classes a/λ and $\sqrt{\prime}a/\lambda$ can be enumerated in such a way that a and $\sqrt{\prime}a$ are assigned the same label i. Then

$$\begin{aligned}\varphi\left(\sqrt{\prime}^{\mathbf{Q}}a\right) &= \left\langle \sqrt{\prime}^{\mathbf{Q}}a\oplus^{\mathbf{Q}} 0, a'^{\mathbf{Q}}\oplus^{\mathbf{Q}} 0,i\right\rangle\\ &= \sqrt{\prime}^{\mathbf{B}_{\mathbf{Q}}^{g,h}}\left\langle a\oplus^{\mathbf{Q}} 0,\sqrt{\prime}^{\mathbf{Q}}a\oplus^{\mathbf{Q}} 0,i\right\rangle\\ &= \sqrt{\prime}^{\mathbf{B}_{\mathbf{Q}}^{g,h}}\varphi(a).\end{aligned}$$

In the remaining cases, we only have to make sure that the application of φ gets the third component of $\varphi\left(\sqrt{\prime}a\right)$ right, because the definition of $\sqrt{\prime}$ in label $\sqrt{\prime}$qMV algebras is identical in all cases relatively to the first two components. Indeed, if $a\in k/\lambda$ and $a=\sqrt{\prime}a=c_i$, then $\pi_3\left(\varphi\left(\sqrt{\prime}^{\mathbf{Q}}a\right)\right)= i=\pi_3\left(\sqrt{\prime}^{\mathbf{B}_{\mathbf{Q}}^{g,h}}\varphi(a)\right)$ because a is a fixpoint for $\sqrt{\prime}$, while if $a\in k/\lambda$, $a\neq\sqrt{\prime}a$ and $a=a'=c_i$, then $\pi_3\left(\varphi\left(\sqrt{\prime}^{\mathbf{Q}}a\right)\right)=g(i)=\pi_3\left(\sqrt{\prime}^{\mathbf{B}_{\mathbf{Q}}^{g,h}}\varphi(a)\right)$. The remaining fourth case is handled similarly, using the function h. □

4. The lattice of subvarieties of $\sqrt{\prime}q\mathbb{MV}$

Recall that a finite Łukasiewicz chain is of the form

$$\text{Ł}_{n+1}=\left\langle\{0,\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n},1\},\oplus,',0,1\right\rangle$$

for $n > 0$ where $x \oplus y = \min(1, x+y)$ and $x' = 1-x$. Alternatively $\mathbf{Ł}_{n+1} = \langle\{0, 1, \ldots, n\}, \oplus, ', 0, n\rangle$ where $x \oplus y = \min(n, x+y)$ and $x' = n-x$. Let $C = \mathbb{Z} \times \mathbb{Z}$ be ordered lexicographically by $\langle a, b\rangle < \langle c, d\rangle$ if and only if $a < b$ or ($a = b$ and $c < d$). The countable *Łukasiewicz chains with infinitesimals* are defined by $\mathbf{Ł}_{n+1,\varepsilon} = \langle\{x \in C : \langle 0, 0\rangle \leq x \leq \langle n, 0\rangle\}, \oplus, ', \langle 0, 0\rangle, \langle n, 0\rangle\rangle$, where $\langle a, b\rangle \oplus \langle c, d\rangle = \min(\langle n, 0\rangle, \langle a+c, b+d\rangle)$ and $\langle a, b\rangle' = \langle n, 0\rangle - \langle a, b\rangle$. The elements $\langle i, 0\rangle$ are the *standard elements* and the remaining elements are the *infinitesimals*, with $\langle 0, 1\rangle$ denoted by ε. The join-irreducible MV varieties are generated by either $Ł_n$ or $Ł_{n,\varepsilon}$ or the standard MV-algebra $\mathbf{Ł}_{[0,1]} = \langle[0, 1], \oplus, ', 0, 1\rangle$, and all other varieties are generated by finite collections of these algebras, hence there are only countably many MV varieties [11]. The same result holds for quasi MV-algebras [4, § 3], though the classification of subvarieties is somewhat more involved.

Although the lattice of $\sqrt{'}$qMV varieties was investigated in detail in [12] and in [14], several problems concerning its structure were left open. In particular, it was conjectured that, although there are only countably many subvarieties of $q\mathbb{MV}$, the number of $\sqrt{'}$qMV varieties is uncountable — however, the above-referenced papers did not settle the issue either way. After dispatching a mandatory recap of known results in the next subsection, we go on to fill some gaps concerning the structure of some slices and to provide equational bases for some interesting varieties.

4.1 Structure of the lattice

The lattice $\mathcal{L}^V(\sqrt{'}q\mathbb{MV})$ of subvarieties of $\sqrt{'}q\mathbb{MV}$ can be depicted as in Fig. 4.1.1: the whole lattice sits upon the chain consisting of the four varieties which contain only flat algebras: the trivial variety, its unique cover $V(\mathbf{F}_{100})$ (axiomatised relative to $\sqrt{'}q\mathbb{MV}$ by the single equation $x \approx \sqrt{'}x$), $V(\mathbf{F}_{020})$ (axiomatised by $x \approx x'$) and the variety of all flat algebras, $\mathbb{F} = V(\mathbf{F}_{004})$ (axiomatised by $x \oplus 0 \approx 0$).

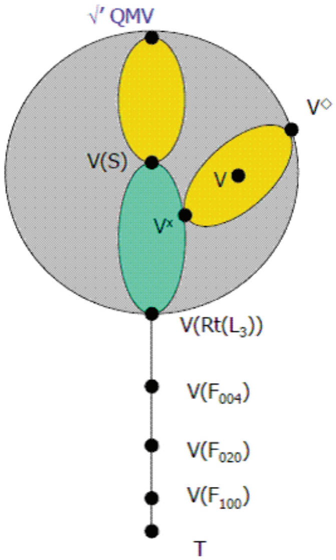


Fig. 4.1.1. The lattice of subvarieties of $\sqrt{'}q\mathbb{MV}$. $\mathbb{V}^{\times}$ and $\mathbb{V}^{\Diamond}$ are shorthands for, respectively, $V\left(\{\mathtt{Rt}(\mathbf{A})\colon \mathbf{A}\in\mathbb{V}_{SI}\}\right)$ and $V\left(\{\wp(\mathbf{A})\colon \mathbf{A}\in\mathbb{V}_{SI}\}\right)$.

On top of this chain, the dark grey area represents the sublattice $\mathcal{L}^{V}(\mathbb{S})$ of varieties generated by strongly Cartesian algebras. The bottom of this sublattice is $V(\mathtt{Rt}\,(\text{Ł}_3))$, the variety generated by the smallest nontrivial (5-element) Cartesian algebra, while its top is the variety $V(\mathbb{S})$ generated by all strongly Cartesian algebras. The main results we proved concerning $\mathcal{L}^{V}(\mathbb{S})$ are listed below.

Theorem 19. *$V(\mathbb{S})$ is axiomatised relative to $\sqrt{'}q\mathbb{MV}$ by the single equation*

$$x \Cup \sqrt{'}x \geq k.$$

Interpreted over Cartesian algebras whose regular elements are linearly ordered, such an equation says that any element a is either greater than or equal to k or such that its square root of the negation is greater than or equal to k. Because of the properties of $\sqrt{'}$, this is equivalent (over Cartesian algebras with linearly ordered regular elements) to every element being either regular or coregular.

If we define $\mathtt{Rt}\,(\mathbb{V})$ as $V\left(\{\mathtt{Rt}(\mathbf{A}) : \mathbf{A} \in\mathbb{V}\}\right)$ for a variety $\mathbb{V}$ of MV* algebras, it is possible to prove that:

Theorem 20. *The lattice $\mathcal{L}^V(\mathbb{MV}^*)$ of all nontrivial MV* varieties is isomorphic to $\mathcal{L}^V(\mathbb{S})$ via the mapping $\varphi(\mathbb{V}) = \mathtt{Rt}(\mathbb{V})$.*

The light grey areas represent what we (in [12]) called "slices", i.e. intervals in $\mathcal{L}^V(\sqrt{'}q\mathbb{MV})$ whose bottom elements are members of $\mathcal{L}^V(\mathbb{S})$. By a *non-flat* variety of $\sqrt{'}$qMV algebras we mean a variety which contains at least an algebra not in $\mathbb{F}$ (equivalently, as we have seen, a variety above or equal to $V(\mathtt{Rt}(\mathbf{Ł}_3))$). We have that:

Lemma 21. *A non-flat $\sqrt{'}qMV$ algebra $\mathbf{A}$ is subdirectly irreducible iff $\mathtt{Rt}(\mathbf{R_A})$ is subdirectly irreducible iff $\mathcal{P}(\mathbf{R_A})$ is subdirectly irreducible. If $\mathbb{V}$ is a non-flat variety, the varieties $\mathbb{V}$, $V(\{\mathtt{Rt}(\mathbf{R_A})\colon \mathbf{A} \in \mathbb{V}\})$, and $V(\{\mathcal{P}(\mathbf{R_A})\colon \mathbf{A} \in \mathbb{V}\})$ have the same strongly Cartesian and flat subdirectly irreducible members.*

Slices are precisely intervals of $\mathcal{L}^V(\sqrt{'}q\mathbb{MV})$ of the form

$$[V(\{\mathtt{Rt}(\mathbf{A})\colon \mathbf{A} \in \mathbb{V}_{SI}\}), V(\{\mathcal{P}(\mathbf{A})\colon \mathbf{A} \in \mathbb{V}_{SI}\})],$$

for some variety $\mathbb{V}$ of MV* algebras. Every non-flat variety is contained in some slice:

Lemma 22. *Every non-flat variety $\mathbb{V}$ belongs to the interval*

$$[V(\{\mathtt{Rt}(\mathbf{R_A})\colon \mathbf{A} \in \mathbb{V}\}), V(\{\mathcal{P}(\mathbf{R_A})\colon \mathbf{A} \in \mathbb{V}\})].$$

The preceding results have a noteworthy consequence: by our description of flat varieties, as well as by Theorem 20 and Lemma 22, $V(\mathbf{F}_{100})$ is the single atom of $\mathcal{L}^V(\sqrt{'}q\mathbb{MV})$. However, the class of congruence lattices of algebras in $V(\mathbf{F}_{100})$ coincides with the class of all equivalence lattices over some set, whence no nontrivial variety $\mathbb{V}$ in $\mathcal{L}^V(\sqrt{'}q\mathbb{MV})$ satisfies any nontrivial congruence identity.

The simplest slices have the form $\mathcal{S}_n = [V(\mathtt{Rt}(\mathbf{Ł}_{2n+1})), V(\mathcal{P}(\mathbf{Ł}_{2n+1}))]$, for some $n \in \mathbb{N}$. If $\mathbf{A} \leq \mathcal{P}(\mathbf{Ł}_{2n+1})$, then $V(\mathbf{A})$ is join-irreducible, and, conversely, every join-irreducible member of $\mathcal{S}_n$ is of the above form. Moreover, since $\mathcal{P}(\mathbf{Ł}_{2n+1})$ is finite, by Lemma 13 all subdirectly irreducible Cartesian algebras in $V(\mathcal{P}(\mathbf{Ł}_{2n+1}))$ belong to $HS(\mathcal{P}(\mathbf{Ł}_{2n+1}))$. Further, $\mathcal{P}(\mathbf{Ł}_{2n+1})$ has no nontrivial Cartesian congruences, and thus, by the relative congruence extension property for Cartesian algebras [20, Lemma 45], the same holds for its subalgebras. It follows that HS above can be replaced by S. The next theorem yields a fairly complete description of the slices $\mathcal{S}_n$:

Theorem 23. *The lattice $\mathcal{S}_n$ contains a subposet order-isomorphic to the interval $[\mathtt{Rt}(\mathbf{Ł}_{2n+1}), \mathcal{P}(\mathbf{Ł}_{2n+1})]$ in the lattice of subalgebras of $\mathcal{P}(\mathbf{Ł}_{2n+1})$, and is itself isomorphic to the lattice of order ideals of the poset $\mathcal{P}^+(n^2)$ of all nonempty subsets of a set with n^2 elements.*

4.2 There are uncountably many subvarieties of $\sqrt{'}q\mathbb{MV}$

In this subsection we first show that the top slice of the lattice of subvarieties of $\sqrt{'}q\mathbb{MV}$, whose bottom element is $V\left(\mathtt{Rt}\left(\mathbf{MV}_{[0,1]}\right)\right)$ and whose top element is the whole of $\sqrt{'}q\mathbb{MV}$, contains uncountably many elements. Subsequently, we prove that we do not have to wait until we reach the top slice in order to find an uncountable one: there are uncountably many varieties of $\sqrt{'}$qMV algebras, even if we restrict ourselves to varieties generated by algebras obtained from Łukasiewicz chains with infinitesimals.

Recall that in [14, § 2.2] appropriate $\sqrt{'}$qMV terms $\chi_i^{\langle a,b\rangle}(x)$ $(1 \leq i \leq 4)$ were used with the property that, if $\langle a,b\rangle$ and $\langle c,d\rangle$ are elements of S_r,

Lemma 24. *1. $\chi_1^{\langle a,b\rangle}(\langle c,d\rangle) \neq 1$ iff $c < a$ and $d < b$,*

2. $\chi_2^{\langle a,b\rangle}(\langle c,d\rangle) \neq 1$ iff $c < a$ and $d > b$,

3. $\chi_3^{\langle a,b\rangle}(\langle c,d\rangle) \neq 1$ iff $c > a$ and $d > b$,

4. $\chi_4^{\langle a,b\rangle}(\langle c,d\rangle) \neq 1$ iff $c > a$ and $d < b$.

In particular, if $a, b, c, d \in [0,1]$, the $\chi_i^{\langle a,b\rangle}$'s have the following form, for some MV terms[1] $\lambda_a, \lambda_b, \rho_a, \rho_b$:

- $\chi_1^{\langle a,b\rangle}(x) = \lambda_a(x) \Cup \lambda_b(\sqrt{'}x)$
- $\chi_2^{\langle a,b\rangle}(x) = \lambda_a(x) \Cup \rho_b(\sqrt{'}x)$
- $\chi_3^{\langle a,b\rangle}(x) = \rho_a(x) \Cup \rho_b(\sqrt{'}x)$
- $\chi_4^{\langle a,b\rangle}(x) = \rho_a(x) \Cup \lambda_b(\sqrt{'}x)$

[1] Actually, unbeknownst to us, the terms $\lambda_a, \lambda_b, \rho_a, \rho_b$ had been defined, although in a different notation, by Aguzzoli [1], to whom it is fair to credit their introduction.

A rather obvious geometric intuition for visualising the terms $\chi_i^{\langle a,b\rangle}(x)$ is that each of these defines its own *rejection rectangle*, consisting of all points $u \in S_r$ that falsify $\chi_i^{\langle a,b\rangle}(u) = 1$ (Fig. 4.2.1). More precisely, these rectangles are as follows:

- for $\chi_1^{\langle a,b\rangle}(u)$, the lower left-hand corner is $\langle 0,0\rangle$ and the upper right-hand corner is $\langle a,b\rangle$,
- for $\chi_2^{\langle a,b\rangle}(u)$, the upper left-hand corner is $\langle 0,1\rangle$ and the lower right-hand corner is $\langle a,b\rangle$,
- for $\chi_3^{\langle a,b\rangle}(u)$, the upper right-hand corner is $\langle 1,1\rangle$ and the lower left-hand corner is $\langle a,b\rangle$,
- for $\chi_4^{\langle a,b\rangle}(u)$, the lower right-hand corner is $\langle 1,0\rangle$ and the upper left-hand corner is $\langle a,b\rangle$.

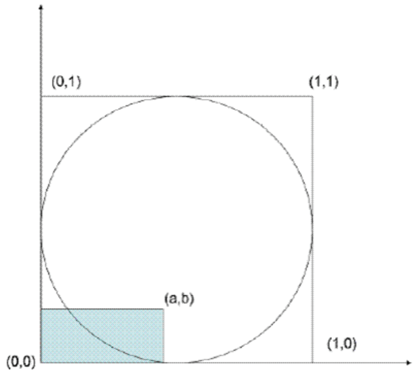


Fig. 4.2.1. Rejection rectangle for $\chi_1^{\langle a,b\rangle}(x)$. $\langle c,d\rangle$ is in the rectangle iff $\chi_1^{\langle a,b\rangle}(\langle c,d\rangle) \neq \left\langle 1,\frac{1}{2}\right\rangle$.

Using these terms, we can show that:

Theorem 25. *The top slice in* $\mathcal{L}^V(\sqrt{'}q\mathbb{MV})$ *contains uncountably many varieties.*

Proof. Consider the line segment with endpoints $\left\langle 0,\frac{1}{2}\right\rangle, \left\langle \frac{1}{2},0\right\rangle$ in S_r, and let $\langle a_0,...,a_k,...\rangle$ be any countable sequence of points in the segment converging to $\left\langle \frac{1}{2},0\right\rangle$. For $X \subseteq N$, let $\mathbf{A}_X$ be the smallest subalgebra of $\mathbf{S}_r$ which includes $\mathtt{Rt}\left(\mathbf{MV}_{[0,1]}\right)$ and contains $\{a_k : k \in X\}$. It will suffice to show that, if $X \neq Y$, then $\mathbf{A}_X$ and $\mathbf{A}_Y$ generate different varieties. In fact, if $X \neq Y$, then w.l.g. there will be an $a_j \in \mathbf{A}_X$ which does not belong to $\mathbf{A}_Y$. Since the sequence $\langle a_0, ..., a_k, ...\rangle$ is countable, there will be some neighbourhood N of a_j (in the standard Euclidean topology of the plane) and some $b \in N$ such that b is point-wise greater than a_j and has the property that the rejection rectangle associated with the term $\chi_1^b(x)$ includes a_j but no other a_k, for $k \neq j$. Therefore, $\mathbf{A}_Y \vDash \chi_1^b(x) \approx 1$, but $\mathbf{A}_X \nvDash \chi_1^b(x) \approx 1$, for $\chi_1^b(a_j) \neq 1$. □

We now show that uncountability is not restricted to the top slice. Let

$$t_n(x) = (((n+1)x)' \oplus \sqrt{'}x) \Cup (nx \oplus (\sqrt{'}x)') \Cup 2x \Cup 2\sqrt{'}x,$$

where the notation nx is defined by $0x = 0$ and $nx = x \oplus (n-1)x$ for $n > 0$. For each set S of positive integers we define a subalgebra of $\mathcal{P}(\mathbf{Ł}_{3\varepsilon})$ by

$$\begin{aligned} \mathbf{A}_S \quad = \quad & \mathrm{Rt}(\mathbf{Ł}_{3\varepsilon}) \cup \{\langle 2\varepsilon, j\varepsilon\rangle, \left\langle j\varepsilon, (2\varepsilon)'\right\rangle, \left\langle (2\varepsilon)', (j\varepsilon)'\right\rangle, \left\langle (j\varepsilon)', 2\varepsilon\right\rangle : \\ & j = 2i+1 \text{ for } i \in S\} \end{aligned}$$

Theorem 26. *Let S, T be two distinct sets of positive integers.*

1. $\mathbf{A}_S \models t_n(x) \approx 1$ *if and only if* $n \notin S$*;*
2. $V(\mathbf{A}_S) \neq V(\mathbf{A}_T)$.

Proof. (1) Note that $\mathbf{A}_S \not\models t_n(x) \approx 1$ is equivalent to $2c \neq 1$, $2\sqrt{'}c \neq 1$, $((n+1)c)' \oplus \sqrt{'}c \neq 1$ and $nc \oplus (\sqrt{'}c)' \neq 1$ for some $c \in A_S$. The first two inequations ensure that $x^{\mathbf{A}_S} = \langle a, b\rangle$ for some $a, b < k = \frac{1}{2}$, hence $a = 2\varepsilon$ and $b = j\varepsilon$ for some $j = 2i+1$ where $i \in S$.
So,

$$\begin{aligned} ((n+1)\langle a, b\rangle)' \oplus \sqrt{'}\langle a, b\rangle \quad &= \quad \left\langle 1-(n+1)a, \frac{1}{2}\right\rangle \\ &= \quad \left\langle \min(1, 1-(n+1)a+b, \frac{1}{2}\right\rangle \neq 1 \end{aligned}$$

if and only if $1-(n+1)a+b < 1$, which is equivalent to $b < (n+1)a$, i.e., $j\varepsilon < 2(n+1)\varepsilon$, so $2i+1 \leq 2n+1$, hence $i \leq n$.

Similarly $n(a, b) \oplus (\sqrt{'}\langle a, b\rangle)' = \left\langle na, \frac{1}{2}\right\rangle \oplus \langle 1-b, a\rangle \neq 1$ if and only if $na + 1 - b < 1$, or equivalently $2n\varepsilon < (2i+1)\varepsilon$, hence $n \leq i$. It follows that the identity $t_n(x) \approx 1$ fails in $\mathbf{A}_S$ precisely when $n = i$ for some $i \in S$.

(2) is an immediate consequence of (1), since either $n \in S \backslash T$ or $n \in T \backslash S$, so the identity $t_n(x) \approx 1$ distinguishes the two varieties. □

The proof given above can be adapted to subalgebras of $\mathcal{P}(\mathbf{Ł}_{2m+1,\varepsilon})$.

Corollary 27. *For $m > 0$ the lattice of subvarieties of $V(\mathcal{P}(\mathbf{Ł}_{2m+1,\varepsilon}))$ is uncountable.*

Although the $Ł_{3\varepsilon}$-slice contains uncountably many varieties, it is possible to describe parts of the poset of join-irreducible varieties near the bottom of the slice. For a finite set $S \subseteq N$, let

$$\mathbf{B}_S = \mathtt{Rt}(\mathbf{Ł}_{3\varepsilon}) \cup \{\langle i\varepsilon, 0\rangle, \langle 0, (i\varepsilon)'\rangle, \langle (i\varepsilon)', 1\rangle, \langle 1, i\varepsilon\rangle : i \in S\}$$

Theorem 28. *Let S, T be finite subsets of N. Then $V(\mathbf{B}_S) \subseteq V(\mathbf{B}_T)$ if and only if there is a positive integer m such that $\{mn : n \in S\} \subseteq T$.*

Proof. For the forward implication, let $y_0, y_1, \ldots$ be a sequence of distinct variables, let $M = \max(T)$, assume $V(\mathbf{B}_S) \subseteq V(\mathbf{B}_T)$ and consider the equation

$$e_S : \bigvee_{n \in S} [(((nx \leftrightsquigarrow y_n)^M)' \oplus y_n') \Cup 2y_n \Cup 2\sqrt{\prime} y_n] \approx 1,$$

where $x \leftrightsquigarrow y = (x' \oplus y) \Cap (y' \oplus x)$ and $\bigvee$ generalises $\Cup$ to finitely but otherwise arbitrarily many arguments. Note that e_S fails in $\mathbf{B}_S$ since if we let $x^{\mathbf{B}_S} = \langle \varepsilon, 0\rangle$ and $y_n^{\mathbf{B}_S} = \langle n\varepsilon, 0\rangle$ then each of the terms in the join gives a value strictly less than 1. Therefore e_S also fails in $\mathbf{B}_T$ for some assignment to the variables. From $2y_n^{\mathbf{B}_T} < 1$ and $2\sqrt{\prime} y_n^{\mathbf{B}_T} < 1$ we deduce that the y_n are assigned irregular elements, hence for all $n \in S$, $y_n^{\mathbf{B}_T} = \langle q_n\varepsilon, 0\rangle$ for some $q_n \in T$. Moreover, $x^{\mathbf{B}_T} = \langle m\varepsilon, 0\rangle$ or $x^{\mathbf{B}_T} = \langle m\varepsilon, \frac{1}{2}\rangle$ for $m > 0$, since in all other cases the term $((nx \leftrightsquigarrow y_n)^M)' \oplus y_n'$ evaluates to 1. In addition $((nx^{\mathbf{B}_T} \leftrightsquigarrow y_n^{\mathbf{B}_T})^M)' \oplus (y_n^{\mathbf{B}_T})' < 1$ implies $(nx^{\mathbf{B}_T} \leftrightsquigarrow y_n^{\mathbf{B}_T})^M \not\leq (y_n^{\mathbf{B}_T})' \oplus 0$. If $nx^{\mathbf{B}_T} \leftrightsquigarrow y_n^{\mathbf{B}_T} < 1$ then $nx^{\mathbf{B}_T} \leftrightsquigarrow y_n^{\mathbf{B}_T} \leq \langle \varepsilon, \frac{1}{2}\rangle'$, hence $(nx^{\mathbf{B}_T} \leftrightsquigarrow y_n^{\mathbf{B}_T})^M \leq \langle (M\varepsilon)', \frac{1}{2}\rangle \leq y_n^{\mathbf{B}_T\prime} \oplus 0$, a contradiction. Therefore $nx^{\mathbf{B}_T} \leftrightsquigarrow y_n^{\mathbf{B}_T} = 1$, whence $nm\varepsilon = q_n\varepsilon$. Since $q_n \in T$ for all $n \in S$, we conclude that $\{mn : n \in S\} \subseteq T$.

For the reverse implication, suppose $\{mn : n \in S\} \subseteq T$ for some $m > 0$. Define the map $h : B_S \to B_T$ by $h(\langle i\varepsilon, j\varepsilon\rangle) = \langle mi\varepsilon, mj\varepsilon\rangle$, and extend it homomorphically to all of B_S. This map is always an embedding on the regular and coregular elements of $\mathbf{B}_S$, and by assumption $\langle mi, 0\rangle \in B_T$ for all $i \in S$, whence the map is also an embedding on the irregular elements. Therefore $\mathbf{B}_S \in V(\mathbf{B}_T)$, as required. □

Note that the above result implies that $V(\mathbf{B}_S)$ and $V(\mathbf{B}_T)$ are distinct if $S \neq T$, but this property does not hold for infinite sets S, T in general. For example if $S = N \setminus \{0\}$ and $T = N$ then $\mathbf{B}_S$ is a subalgebra of $\mathbf{B}_T$, and $\mathbf{B}_T$ is a homomorphic image of any nonprincipal ultrapower of B_S, hence $V(\mathbf{B}_S) = V(\mathbf{B}_T)$. Similarly the top variety of the $\text{Ł}_{3\varepsilon}$-slice, which is generated by the pair algebra $\mathcal{P}(\text{Ł}_{3\varepsilon})$, is also generated by the subalgebra obtained by removing the 4 "corners" $\langle 0,0\rangle, \langle 1,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle$, or indeed, by removing any finite set of irregular points that is invariant under $\sqrt{'}$.

4.3 Equational bases for some subvarieties

In [12, 14] the lattice $\mathcal{L}^V(\sqrt{'}q\mathbb{MV})$ was described to some extent, but — differently from what had been done for $\mathcal{L}^V(q\mathbb{MV})$ in [4] — no equational bases were given for individual subvarieties. Here, we provide such bases at least for some reasonably simple cases. We start with an easy task: axiomatising the varieties generated by strongly Cartesian algebras. By Theorem 20 every such variety is the rotation of some variety of MV* algebras.

Lemma 29. *Let $\mathbb{V}$ be a variety of MV* algebras whose equational basis w.r.t. $\mathbb{MV}^*$ is $\mathcal{E}$. Then $\mathtt{Rt}(\mathbb{V})$ is axiomatised relative to $\sqrt{'}q\mathbb{MV}$ by $\mathcal{E}$ and the strongly Cartesian equation*

$$\left(x \Cup \sqrt{'}x\right) \oplus k \approx 1.$$

Proof. From left to right, $\mathtt{Rt}(\mathbf{A}) \in \mathtt{Rt}(\mathbb{V})$ is a $\sqrt{'}$qMV algebra which satisfies $\left(x \Cup \sqrt{'}x\right) \oplus k \approx 1$ by Theorems 19 and 20. Moreover, since $\mathcal{E}$ can be taken to be a set of *normal* MV* equations[2] by results in [7, Chapter 8], $\mathbf{A}$ will satisfy $\mathcal{E}$ as a qMV algebra, whence it will satisfy these equations altogether. Conversely, let $\mathbf{A}$ be a s.i. $\sqrt{'}$qMV algebra which satisfies both $\mathcal{E}$ and $\left(x \Cup \sqrt{'}x\right) \oplus k \approx 1$. Being subdirectly irreducible, it is either Cartesian

[2]Recall that an equation $t \approx s$ (of a given type) is said to be normal iff either t and s are the same variable or else neither t nor s is a variable [6].

or flat. If the latter, then $\mathbf{A} \in \mathtt{Rt}(\mathbb{V})$ because flat algebras are contained in every variety generated by strongly Cartesian algebras. If the former, then its MV* term subreduct $\mathbf{R_A}$ is also subdirectly irreducible and, therefore, linearly ordered. As a consequence, the axiom $\left(x \Cup \sqrt{'}x\right) \oplus k \approx 1$ expresses the fact that any element is either above k or such that its own square root of the negation is above k. It follows that $\mathbf{A} = \mathtt{Rt}(\mathbf{B})$ for some MV* algebra $\mathbf{B}$. Since $\mathbf{A}$ satisfies $\mathcal{E}$, however, $\mathbf{B}$ (having fewer elements) also satisfies it and thus $\mathbf{A} \in \mathtt{Rt}(\mathbb{V})$. □

By Theorem 23, each slice whose bottom element is the variety generated by the rotation $\mathtt{Rt}(\text{Ł}_{2n+1})$ of a single finite Łukasiewicz chain Ł_{2n+1}, and whose top element is the variety generated by the full pair algebra $\mathcal{P}(\text{Ł}_{2n+1})$, has exactly 2^{n^2} join irreducible elements, one for each set of irregular elements in any one "quadrant" of $\mathcal{P}(\text{Ł}_{2n+1})$. We are now going to give explicit equational bases for all of them. For this purpose, it will be expedient to identify their generating algebras with subalgebras of $\mathbf{S}_r$. If we do so, each meet and join irreducible variety in any such slice can be identified with the variety generated by the algebra $\mathbf{A}_p$, obtained by removing from $\mathcal{P}(\text{Ł}_{2n+1})$ exactly the point $p = \left\langle \frac{m_1}{2n}, \frac{m_2}{2n} \right\rangle$, together with $\sqrt{'}p, p', \sqrt{'}p'$. With no loss of generality, of course, p can be taken to reside in the first quadrant, i.e. $m_1, m_2 \in \{0, \ldots, n-1\}$.

Theorem 30. *If $\mathcal{E}$ axiomatises $V(\text{Ł}_{2n+1})$ relative to $\mathbb{MV}^*$, then $V(\mathbf{A}_p)$ is axiomatised relative to $\sqrt{'}q\mathbb{MV}$ by $\mathcal{E}$ as well as $t_p(x) \approx 1$, where*

$$t_p(x) = \chi_1^{\left\langle \frac{m_1+1}{2n}, \frac{m_2+1}{2n} \right\rangle}(x) \Cup \chi_3^{\left\langle \frac{m_1-1}{2n}, \frac{m_2-1}{2n} \right\rangle}(x).$$

Proof. After observing that the term $t_p(x)$ can be further unwound as

$$\lambda_{\frac{m_1+1}{2n}}(x) \Cup \rho_{\frac{m_1-1}{2n}}(x) \Cup \lambda_{\frac{m_2+1}{2n}}(\sqrt{'}x) \Cup \rho_{\frac{m_2-1}{2n}}(\sqrt{'}x),$$

our proof goes through a number of claims.

Claim 31. *In the standard MV* algebra $\mathbf{MV}_{[0,1]}$, $\lambda_{\frac{m_1+1}{2n}}(a) \Cup \rho_{\frac{m_1-1}{2n}}(a) < 1$ iff $a \in \left(\frac{m_1-1}{2n}, \frac{m_1+1}{2n}\right)$.*

In fact, by Lemma 15 in [14], $\lambda_{\frac{m_1+1}{2n}}(a) = 1$ iff $a > \frac{m_1+1}{2n}$, while $\rho_{\frac{m_1-1}{2n}}(a) = 1$ iff $a < \frac{m_1-1}{2n}$. Therefore, the indicated join is 1 exactly for the points that lie outside of the open interval $\left(\frac{m_1-1}{2n}, \frac{m_1+1}{2n}\right)$. Now the following claims are immediate consequences of Claim 31:

Claim 32. *In* $\mathbf{S}_r$, $\lambda_{\frac{m_1+1}{2n}}(a) \Cup \rho_{\frac{m_1-1}{2n}}(a) < 1$ *iff* $a \in \left(\frac{m_1-1}{2n}, \frac{m_1+1}{2n}\right)$.

Claim 33. *In* $\mathbf{S}_r$, $t_p(a) < 1$ *iff a belongs to the open square with centre* p *and radius* $\frac{1}{2n}$.

Having established these claims, it follows that $\mathbf{A}_p$ satisfies $t_p(x) \approx 1$, while any subdirectly irreducible Cartesian algebra in the slice satisfying $t_p(x) \approx 1$ must be a subalgebra of $\mathcal{P}(\mathbf{Ł}_{2n+1})$ in the light of the remarks preceding Theorem 23 and at the same time exclude the point p, i.e. be a subalgebra of $\mathbf{A}_p$. □

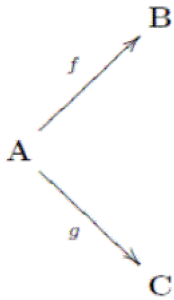
Corollary 34. *An arbitrary join irreducible variety* $V(\mathbf{A})$ *in the slice whose bottom element is* $V(\mathtt{Rt}(Ł_{2n+1}))$ *is axiomatised relative to* $\sqrt{'}q\mathbb{MV}$ *by* $\mathcal{E}$ *as well as* $\{t_p(x) \approx 1 : p \notin A\}$, *where* $p = \left\langle \frac{m_1}{2n}, \frac{m_2}{2n} \right\rangle$ *for* $m_1, m_2 \in \{0, \ldots, n-1\}$.

5. $\sqrt{'}q\mathbb{MV}$ has the amalgamation property

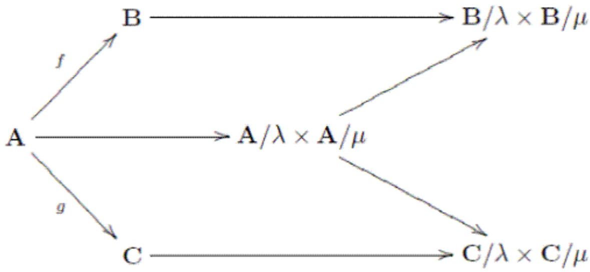
An amalgam is a tuple $\langle \mathbf{A}, f, \mathbf{B}, g, \mathbf{C} \rangle$ such that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are structures of the same signature, and $f : \mathbf{A} \to \mathbf{B}, g : \mathbf{A} \to \mathbf{C}$ are embeddings (injective morphisms). A class $\mathbb{K}$ of structures is said to have the *amalgamation property* if for every amalgam with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{K}$ and $A \neq \emptyset$ there exists a structure $\mathbf{D} \in \mathbb{K}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}, g' : \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$. A couple of decades ago, Mundici proved that MV algebras have the amalgamation property [19], and his result was extended to the variety $q\mathbb{MV}$ in [4, § 6.1]. In the same paper (§ 6.2) it was proved that both Cartesian and flat $\sqrt{'}$qMV algebras amalgamate, but the property was not established for the entire variety of $\sqrt{'}$qMV algebras, although it was to be expected that it would hold. Since taking this further step is not completely trivial, we answer the question in the affirmative in this subsection.

Theorem 35. *The variety of* $\sqrt{'}qMV$ *algebras enjoys the amalgamation property.*

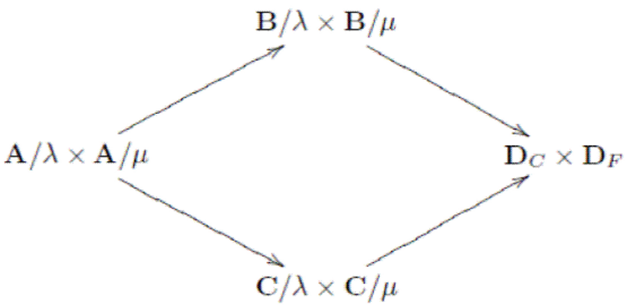
Proof. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be $\sqrt{'}$qMV algebras such that:



where f, g are embeddings. By the Third isomorphism theorem and the representation theorem for $\sqrt{\prime}$qMV algebras the following diagram commutes:



But Cartesian and flat $\sqrt{\prime}$qMV algebras possess the amalgamation property. Therefore there exist a Cartesian algebra $\mathbf{D}_C$, and a flat algebra $\mathbf{D}_F$ such that the following is commutative:



Thus, combining the previous two diagrams, we see that $\mathbf{D}_C \times \mathbf{D}_F$ amalgamates $\langle \mathbf{A}, f, \mathbf{B}, g, \mathbf{C} \rangle$. □

6. The 1-assertional logic of $\sqrt{'}q\mathbb{MV}$

Recall that the 1-assertional logic [3] of a class $\mathbb{K}$ of similar algebras of type ν (containing at least one constant 1) is the logic whose language is ν and whose consequence relation $\vdash_{\mathbb{K}}$ is defined for all $\Gamma \cup \{\alpha\} \subseteq \mathrm{For}(\nu)$ as follows:

$$\Gamma \vdash_{\mathbb{K}} \alpha \text{ if and only if } \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathbb{K}} \alpha \approx 1,$$

where $\vDash_{\mathbb{K}}$ is the equational consequence relation of the class $\mathbb{K}$. Although this consequence relation need not, in general, be finitary [8], it can be forced to be such by changing its definition into

$$\Gamma \vdash_{\mathbb{K}} \alpha \text{ iff there is a finite } \Gamma' \subseteq \Gamma \text{ s.t. } \left\{\gamma \approx 1 : \gamma \in \Gamma'\right\} \vDash_{\mathbb{K}} \alpha \approx 1.$$

Hereafter, we will adopt the latter definition of 1-assertional logic. Since we will deal with logics on the same language, we will also identify logics with their associated consequence relation, with a slight linguistic abuse.

Among the several abstract logics related to $\sqrt{'}q\mathbb{MV}$ that were introduced and motivated in [21], there were the 1-assertional logics $\vdash_{\sqrt{'}q\mathbb{W}}$ of the variety $\sqrt{'}q\mathbb{W}$ (a term equivalent variant of $\sqrt{'}q\mathbb{MV}$ in the language $\left\{\to, \sqrt{'}, 0, 1\right\}$, where $x \to y = x' \oplus y$) and $\vdash_{\mathbb{CW}}$ of the quasivariety $\mathbb{CW}$ of Cartesian algebras (also formulated in the same language; $\mathbb{W}$ stands for Wajsberg algebras). Such logics differ profoundly from each other as regards their abstract algebraic logical properties. For example, while the latter is a regularly algebraisable logic whose equivalent algebraic semantics is $\mathbb{CW}$, the former is not even protoalgebraic. The above-referenced paper provides an axiomatisation of $\vdash_{\mathbb{CW}}$ that streamlines the algorithmic axiomatisation obtained from the standard axiomatic presentation of the relatively point regular quasivariety $\mathbb{CW}$ by the Blok-Pigozzi method [2], as well as a characterisation of its deductive filters. For the non-protoalgebraic logic $\vdash_{\sqrt{'}q\mathbb{W}}$, the axiomatisation problem is not trivial and cannot be tackled by standard methods, since we cannot construct anything like the Lindenbaum algebra of the logic. The aim of the present section is giving an answer to this problem.

For a start, since $\mathbb{CW}$ is a subquasivariety of $\sqrt{'}q\mathbb{W}$, we observe that:

Lemma 36. *If* $\alpha_1, ..., \alpha_n \vdash_{\sqrt{'}q\mathbb{W}} \alpha$, *then* $\alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} \alpha$.

We also recall the following lemma, first proved in [21, Lemma 19]. Here and in the sequel, $\sqrt{\prime}^{(n)}\alpha$ is inductively defined by $\sqrt{\prime}^{(0)}\alpha = \alpha$ and $\sqrt{\prime}^{(m+1)}\alpha = \sqrt{\prime}\left(\sqrt{\prime}^{(m)}\alpha\right)$.

Lemma 37. $\alpha_1, ..., \alpha_n \vdash_{\sqrt{\prime}q\mathbb{W}} \sqrt{\prime}^{(m)}p$ *iff at least one of the following conditions hold:*

1. *For some integer* $k \equiv m \pmod 4$ $\sqrt{\prime}^{(k)}p \in \{\alpha_1, ..., \alpha_n\}$*;*
2. *For some integer* $k \not\equiv m \pmod 4$ $\sqrt{\prime}^{(k)}p \in \{\alpha_1, ..., \alpha_n\}$ *and* $\alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} 0$.

The next result shows that although the converse of Lemma 36 need not be true in general, we can nonetheless infer some information from its premiss.

Lemma 38. $\alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} \alpha$ *iff* $\alpha_1, ..., \alpha_n \vdash_{\sqrt{\prime}q\mathbb{W}} \alpha \leftrightarrow 1$*, where*

$$\alpha \leftrightarrow \beta = (\alpha \to \beta) \otimes (\beta \to \alpha) \otimes \left(\sqrt{\prime}\alpha \to \sqrt{\prime}\beta\right) \otimes \left(\sqrt{\prime}\beta \to \sqrt{\prime}\alpha\right).$$

Proof. Left to right. Suppose $\alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} \alpha$, and let $\mathbf{A}$ be a $\sqrt{\prime}$ qW algebra. Suppose further that $\overrightarrow{a} \in A^i$, where i is the number of variables in the indicated formulas, and that $\alpha_1^{\mathbf{A}}(\overrightarrow{a}) = ... = \alpha_n^{\mathbf{A}}(\overrightarrow{a}) = 1$. Now, the quotient $\mathbf{A}/\lambda$ is a Cartesian algebra, whence our hypothesis that $\alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} \alpha$ implies $\alpha^{\mathbf{A}/\lambda}(\overrightarrow{a}/\lambda) = 1^{\mathbf{A}/\lambda}$, i.e. $\alpha^{\mathbf{A}}(\overrightarrow{a})\,\lambda 1$. Unwinding this statement, we get that

$$\alpha^{\mathbf{A}}(\overrightarrow{a}) \to 1 = 1 \to \alpha^{\mathbf{A}}(\overrightarrow{a}) = \sqrt{\prime}\alpha^{\mathbf{A}}(\overrightarrow{a}) \to \sqrt{\prime}1 = \sqrt{\prime}1 \to \sqrt{\prime}\alpha^{\mathbf{A}}(\overrightarrow{a}) = 1,$$

and so $\alpha^{\mathbf{A}}(\overrightarrow{a}) \leftrightarrow 1 = 1$.

Right to left. Suppose $\alpha_1, ..., \alpha_n \vdash_{\sqrt{\prime}q\mathbb{W}} \alpha \leftrightarrow 1$, and let $\mathbf{A}$ be a Cartesian algebra. Suppose further that $\overrightarrow{a} \in A^i$, and that $\alpha_1^{\mathbf{A}}(\overrightarrow{a}) = ... = \alpha_n^{\mathbf{A}}(\overrightarrow{a}) = 1$. Since $\mathbf{A}$ is in particular a $\sqrt{\prime}$ qW algebra, $\alpha^{\mathbf{A}}(\overrightarrow{a}) \leftrightarrow 1 = 1$ and, since the immediate subformulas of $\alpha \leftrightarrow 1$ are all regular,

$$\alpha^{\mathbf{A}}(\overrightarrow{a}) \to 1 = 1 \to \alpha^{\mathbf{A}}(\overrightarrow{a}) = \sqrt{\prime}\alpha^{\mathbf{A}}(\overrightarrow{a}) \to \sqrt{\prime}1 = \sqrt{\prime}1 \to \sqrt{\prime}\alpha^{\mathbf{A}}(\overrightarrow{a}) = 1.$$

This means $1 \to \alpha^{\mathbf{A}}(\overrightarrow{a}) = 1$ and $1 \to \sqrt{\prime}\alpha^{\mathbf{A}}(\overrightarrow{a}) = 1 \to \sqrt{\prime}1$; since $\mathbf{A}$ is Cartesian, $\alpha^{\mathbf{A}}(\overrightarrow{a}) = 1$. □

An immediate consequence of the above lemma is:

Corollary 39. $\alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} 0$ *iff* $\alpha_1, ..., \alpha_n \vdash_{\sqrt{\prime}q\mathbb{W}} 0$.

Lemma 40. *For* $m \geq 0$,

$$\alpha_1, ..., \alpha_n \vdash_{\sqrt{\prime}q\mathbb{W}} \sqrt{\prime}^{(m)} (\alpha \to \beta) \text{ iff } \alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} \sqrt{\prime}^{(m)} (\alpha \to \beta).$$

Proof. The left-to-right direction follows from Lemma 36. For the converse direction, suppose $\alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} \sqrt{\prime}^{(m)} (\alpha \to \beta)$ and let $\mathbf{A}$ be a $\sqrt{\prime}$ qW algebra. Suppose further that $\overrightarrow{a} \in A^i$, and that $\alpha_1^{\mathbf{A}} (\overrightarrow{a}) = ... = \alpha_n^{\mathbf{A}} (\overrightarrow{a}) = 1$. By Lemma 38, $\sqrt{\prime}^{(m)} (\alpha \to \beta) (\overrightarrow{a}) \leftrightarrow 1 = 1$; in full,

$$\left(1 \to \sqrt{\prime}^{(m)} (\alpha \to \beta) (\overrightarrow{a})\right) \otimes \left(\sqrt{\prime}^{(m)} (\alpha \to \beta) (\overrightarrow{a}) \to 1\right) \otimes$$
$$\left(\sqrt{\prime}1 \to \sqrt{\prime}^{(m+1)} (\alpha \to \beta) (\overrightarrow{a})\right) \otimes \left(\sqrt{\prime}^{(m+1)} (\alpha \to \beta) (\overrightarrow{a}) \to \sqrt{\prime}1\right) = 1,$$

and so the immediate subformulas of the preceding formula, being regular, all evaluate to 1. Now, if m is odd, from $1 \to \sqrt{\prime}^{(m)} (\alpha \to \beta) (\overrightarrow{a}) = 1$ we get $1 = k$. In other words $\mathbf{A}$ is flat, whence $\sqrt{\prime}^{(m)} (\alpha \to \beta) (\overrightarrow{a}) = 1$. If m is even, then either $1 \to (\alpha \to \beta) (\overrightarrow{a}) = 1$ or $1 \to (\alpha \to \beta)' (\overrightarrow{a}) = 1$, which respectively imply either $(\alpha \to \beta) (\overrightarrow{a}) = 1$ or $(\alpha \to \beta)' (\overrightarrow{a}) = 1$. $\square$

Corollary 41. $\vdash_{\sqrt{\prime}q\mathbb{W}}$ *and* $\vdash_{\mathbb{CW}}$ *have the same theorems.*

Proof. From Lemma 40, since all the theorems of $\vdash_{\mathbb{CW}}$ have the form $\sqrt{\prime}^{(m)} (\alpha \to \beta)$, for some $m \geq 0$. It is also a consequence of the fact that $\mathbb{CW}$ and $\sqrt{\prime}q\mathbb{W}$ satisfy the same equations [9]. $\square$

The next Theorem gives a complete characterisation of the valid entailments of $\vdash_{\sqrt{\prime}q\mathbb{W}}$.

Theorem 42. $\alpha_1, ..., \alpha_n \vdash_{\sqrt{\prime}q\mathbb{W}} \alpha$ *iff at least one of the following conditions hold:*

1. $\alpha = \sqrt{\prime}^{(m)} (\beta \to \gamma)$ (for some formulas β, γ and some $m \geq 0$) or $\alpha = 0$ or $\alpha = 1$, and $\alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} \alpha$;
2. $\alpha = \sqrt{\prime}^{(m)} p$ (for some $m \geq 0$) and for some integer $k \equiv m \pmod 4$ $\sqrt{\prime}^{(k)} p \in \{\alpha_1, ..., \alpha_n\}$;
3. $\alpha = \sqrt{\prime}^{(m)} p$ (for some $m \geq 0$) and for some integer $k \not\equiv m \pmod 4$ $\sqrt{\prime}^{(k)} p \in \{\alpha_1, ..., \alpha_n\}$ and $\alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} 0$.

Proof. From Lemmas 37 and 40. For the cases $\alpha = 0$ or $\alpha = 1$, use Corollaries 41 and 39. □

We are now going to define a Hilbert system whose syntactic derivability relation will prove to be equivalent to $\vdash_{\sqrt{'}q\mathbb{W}}$. This system is both an expansion and a rule extension of the Hilbert system qŁ for the logic of quasi-Wajsberg algebras introduced in [5], and the techniques used to prove completeness are heavily indebted to the tools adopted in the mentioned paper.

Definition 43. The deductive system $\vdash_{\sqrt{'}q\text{Ł}}$, formulated in the signature $\left\langle \rightarrow, \sqrt{'}, 1, 0 \right\rangle$, has the following postulates:

A1. $\alpha \rightarrow (\beta \rightarrow \alpha)$
A2. $(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma))$
A3. $((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow ((\beta \rightarrow \alpha) \rightarrow \alpha)$
A4. $(\alpha' \rightarrow \beta') \rightarrow (\beta \rightarrow \alpha)$
A5. 1
A6. $\sqrt{'}\alpha \rightarrow \sqrt{'}\beta$, for α, β regular form.
A7. $\left(1 \rightarrow \sqrt{'}\,(\alpha \rightarrow \beta)\right) \leftrightarrow \sqrt{'}\left(1 \rightarrow \sqrt{'}\,(\alpha \rightarrow \beta)\right)$
qMP. $1 \rightarrow \alpha, 1 \rightarrow (\alpha \rightarrow \beta) \vdash 1 \rightarrow \beta$
Areg1. $1 \rightarrow \sqrt{'}^{(m)}\,(\alpha \rightarrow \beta) \vdash \sqrt{'}^{(m)}\,(\alpha \rightarrow \beta)$ $(0 \leq m \leq 3)$
Areg2. $1 \rightarrow 0 \vdash 0$
Reg. $\alpha \vdash 1 \rightarrow \alpha$
Inv. $\alpha \dashv\vdash \alpha''$
Flat. $\alpha, 0 \vdash \sqrt{'}\alpha$
GR. $\alpha, \beta \vdash \sqrt{'}\alpha \rightarrow \sqrt{'}\beta$

Lemma 44. *The Cartesian logic* $\vdash_{\mathbb{CW}}$*, as axiomatised in [21, Definition 20], is the rule extension of* $\vdash_{\sqrt{'}q\text{Ł}}$ *by the rule*

$$\texttt{MP}^*.\ \alpha, \alpha \rightarrow \beta, \sqrt{'}\alpha \rightarrow \sqrt{'}\beta, \sqrt{'}\beta \rightarrow \sqrt{'}\alpha \vdash \beta.$$

Proof. For the sole missing axiom, observe that by (Flat) $\sqrt{'}\alpha, 0 \vdash \alpha'$ and $\alpha, 0 \vdash \sqrt{'}\alpha$, whence by (Cut) we have our conclusion. □

The next lemma will prove very useful in the sequel and will be mostly employed without special mention.

Lemma 45. *If* $\alpha_1, ..., \alpha_n \vdash_{\mathbb{W}} \alpha$ *and* $\alpha_1, ..., \alpha_n, \alpha$ *are regular formulas, then* $\alpha_1, ..., \alpha_n \vdash_{\sqrt{'}q\text{Ł}} \alpha$*.*

Proof. From the assumptions $\alpha_1, ..., \alpha_n$, by (Reg) we conclude $1 \to \alpha_1, ..., 1 \to \alpha_n$, whence there is a proof in $\vdash_{\sqrt{}'q\text{Ł}}$ of $1 \to \alpha$ using (qMP). Our claim follows then by (Areg1-2). □

We now need a syntactic analogue of one direction in Lemma 38.

Lemma 46. *If* $\alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} \alpha$ *then* $\alpha_1, ..., \alpha_n \vdash_{\sqrt{}'qŁ} \alpha \leftrightarrow 1$.

Proof. In consideration of Lemma 44, we proceed by induction on the derivation of α from $\alpha_1, ..., \alpha_n$, in the Hilbert system given in the same lemma.

If α is an axiom, then it is both a $\vdash_{\sqrt{}'q\text{Ł}}$ axiom and a regular formula, whence $\sqrt{}'\alpha \to \sqrt{}'1$ and $\sqrt{}'1 \to \sqrt{}'\alpha$ are both $\vdash_{\sqrt{}'q\text{Ł}}$-provable by (GR), while $1 \to \alpha$ is $\vdash_{\sqrt{}'q\text{Ł}}$-provable by (Reg). Since $\alpha \to 1$ is $\vdash_{\sqrt{}'q\text{Ł}}$-provable by the completeness theorem for the subsystem qŁ [5, Theorem 21], we conclude that the conjunction of regular formulas $\alpha \leftrightarrow 1$ is also such.

Now, let $\alpha = 1 \to \beta$ be obtained from $\alpha_1, ..., \alpha_{n-1}, \beta$ by the rule (Reg). We have to prove that $\alpha_1, ..., \alpha_{n-1}, \beta \vdash_{\sqrt{}'q\text{Ł}} (1 \to \beta) \leftrightarrow 1$. However, as already noticed $(1 \to \beta) \to 1$ is $\vdash_{\sqrt{}'q\text{Ł}}$-provable, while $1 \to (1 \to \beta)$ is obtained from β by two applications of (Reg). $\sqrt{}'(1 \to \beta) \to \sqrt{}'1$ and its converse are $\vdash_{\sqrt{}'q\text{Ł}}$-provable by (A6), whence we obtain our conclusion. The rules (Areg1-2), (qMP) and (GR) are dispatched similarly.

Let $\alpha = \sqrt{}'\beta$ be obtained from $\alpha_1, ..., \alpha_{n-1}, \beta, 0$ by the rule (Flat). We have to prove that $\alpha_1, ..., \alpha_{n-1}, \beta, 0 \vdash_{\sqrt{}'q\text{Ł}} \sqrt{}'\beta \leftrightarrow 1$, where, in full,

$$\sqrt{}'\beta \leftrightarrow 1 = \left(\sqrt{}'\beta \to 1\right) \otimes \left(1 \to \sqrt{}'\beta\right) \otimes \left(\beta' \to \sqrt{}'1\right) \otimes \left(\sqrt{}'1 \to \beta'\right).$$

However, (i) $\sqrt{}'\beta \to 1$ is $\vdash_{\sqrt{}'q\text{Ł}}$-provable by the completeness theorem for the subsystem qŁ; (ii) $1 \to \sqrt{}'\beta$ can be derived from $\beta, 0$ by (Flat) and (Reg); (iii) from $\beta, 0$ we get $\sqrt{}'\beta$ by (Flat) and then $\beta' \to \sqrt{}'1$ and $\sqrt{}'1 \to \beta'$ by (A5) and (GR). The rule (Inv) is dispatched similarly.

Finally, let $\alpha = \beta$ be obtained from $\alpha_1, ..., \alpha_{n-4}, \gamma, \gamma \to \beta, \sqrt{}'\gamma \to \sqrt{}'\beta, \sqrt{}'\beta \to \sqrt{}'\gamma$ by the rule (MP*). By induction hypothesis,

$$\alpha_1, ..., \alpha_{n-4} \vdash_{\sqrt{}'q\text{Ł}} \gamma \leftrightarrow 1, (\gamma \to \beta) \leftrightarrow 1,$$

$$\left(\sqrt{}'\gamma \to \sqrt{}'\beta\right) \leftrightarrow 1, \left(\sqrt{}'\beta \to \sqrt{}'\gamma\right) \leftrightarrow 1.$$

We must show that $\alpha_1, ..., \alpha_{n-4}, \gamma, \gamma \to \beta, \sqrt{'}\gamma \to \sqrt{'}\beta, \sqrt{'}\beta \to \sqrt{'}\gamma \vdash'_{\sqrt{'}q\text{Ł}}$ $\beta \leftrightarrow 1$, where, in full,

$$\beta \leftrightarrow 1 = (\beta \to 1) \otimes (1 \to \beta) \otimes \left(\sqrt{'}\beta \to \sqrt{'}1\right) \otimes \left(\sqrt{'}1 \to \sqrt{'}\beta\right).$$

However, (i) $\beta \to 1$ is $\vdash_{\sqrt{'}q\text{Ł}}$-provable by the completeness theorem for the subsystem qŁ; (ii) applying (Reg) to the premisses $\gamma, \gamma \to \beta$ we obtain $1 \to \gamma, 1 \to (\gamma \to \beta)$, whence $1 \to \beta$ follows by (qMP); (iii) our induction hypothesis[3] yields $1 \to \left(\sqrt{'}\beta \to \sqrt{'}\gamma\right)$, whence $\sqrt{'}\beta \to \sqrt{'}\gamma$ follows from (Areg1). By ind. hyp. again, we obtain $\sqrt{'}\gamma \to \sqrt{'}1$, whence by transitivity (legitimate by Lemma 45) we conclude $\sqrt{'}\beta \to \sqrt{'}1$. For $\sqrt{'}1 \to \sqrt{'}\beta$ we argue similarly. □

Lemma 47. $\sqrt{'}^{(m)}(\alpha \to \beta) \leftrightarrow 1 \vdash_{\sqrt{'}q\text{Ł}} \sqrt{'}^{(m)}(\alpha \to \beta)$ *for all* $m \geq 0$.

Proof. From our hypothesis we deduce $1 \to \sqrt{'}^{(m)}(\alpha \to \beta)$, whence our conclusion follows by (Areg1). □

Lemma 48. *If* $\alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} 0$ *then* $\alpha_1, ..., \alpha_n \vdash_{\sqrt{'}q\text{Ł}} 0$.

Proof. By Lemma 46, if $\alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} 0$ then $\alpha_1, ..., \alpha_n \vdash_{\sqrt{'}q\text{Ł}} 0 \leftrightarrow 1$, whence we deduce $1 \to 0$ and then 0 by (Areg2). □

We are now ready to establish the main result of this section.

Theorem 49. $\alpha_1, ..., \alpha_n \vdash_{\sqrt{'}q\text{Ł}} \alpha$ *iff* $\alpha_1, ..., \alpha_n \vdash_{\sqrt{'}q\mathbb{W}} \alpha$.

Proof. From left to right, we proceed through a customary inductive argument. Conversely, suppose that $\alpha_1, ..., \alpha_n \vdash_{\sqrt{'}q\mathbb{W}} \alpha$. Then, at least one of the conditions (1)-(3) in Theorem 42 obtains.

If (1) holds, then either $\alpha = \sqrt{'}^{(m)}(\beta \to \gamma)$ for some formulas β, γ and some $m \geq 0$, or $\alpha = 0$ or $\alpha = 1$; moreover, $\alpha_1, ..., \alpha_n \vdash_{\mathbb{CW}} \alpha$. If $\alpha = \sqrt{'}^{(m)}(\beta \to \gamma)$, by Lemma 46 $\alpha_1, ..., \alpha_n \vdash_{\sqrt{'}q\text{Ł}} \sqrt{'}^{(m)}(\beta \to \gamma) \leftrightarrow 1$, whence our conclusion follows applying Lemma 47. If $\alpha = 0$ we reach the same conclusion by Lemma 48, while if $\alpha = 1$ (A5) suffices.

[3] Observe that the (MP*) step is the only locus in our proof where the inductive hypothesis is actually used.

If (2) holds, we must show that $\alpha_1, ..., \alpha_{n-1}, \sqrt{'}^{(k)} p \vdash_{\sqrt{'}q\text{Ł}} \sqrt{'}^{(m)} p$. Since $k \equiv m \pmod 4$, either $k = m$ (and so there is nothing to prove) or our conclusion can be attained by (Inv).

Finally, if (3) holds, we can assume that $\alpha_1, ..., \alpha_{n-1}, \sqrt{'}^{(k)} p \vdash_{\mathbb{CW}} 0$. To show that $\alpha_1, ..., \alpha_{n-1}, \sqrt{'}^{(k)} p \vdash_{\sqrt{'}q\text{Ł}} \sqrt{'}^{(m)} p$, we apply Lemma 48 to get

$$\alpha_1, ..., \alpha_{n-1}, \sqrt{'}^{(k)} p \vdash_{\sqrt{'}q\text{Ł}} 0,$$

whence by (Flat) $\alpha_1, ..., \alpha_{n-1}, \sqrt{'}^{(k)} p \vdash_{\sqrt{'}q\text{Ł}} \sqrt{'}^{(k+1)} p$. From here, we proceed to our conclusion by as many applications of (Flat) and (Inv) as needed. □

7. Cartesian $\sqrt{'}$qMV algebras and Abelian PR-groups

Abelian PR-groups were defined in [10] as an expansion of Abelian ℓ-groups by two operations P, R that for $\mathbb{C}$ behave like a projection onto the first coordinate and a clockwise rotation by $\pi/2$ radians. It was proved that: a) every Cartesian $\sqrt{'}$ quasi-MV algebra is embeddable into an interval in a particular Abelian PR-group; b) the category of pair algebras is equivalent both to the category of such ℓ-groups (with strong order unit), and to the category of MV algebras. As a byproduct of these results a purely group-theoretical equivalence was obtained, namely between the mentioned category of Abelian PR-groups and the category of Abelian ℓ-groups (both with strong order unit).

Although these results shed some light on the geometrical structure of Cartesian $\sqrt{'}$qMV algebras, as well as on their relationships with better known classes of algebras, they suffer from a shortcoming. In fact, the classes of objects in the above-mentioned categories do not form varieties, whence the connection between these theorems and the general theory of categorical equivalence for varieties [18] remains to some extent unclear. In particular, the fact that pair algebras are generated by $\mathbf{S}_r$ does not translate automatically into the fact that the variety of Abelian PR-groups is generated by the standard PR-group over the complex numbers. Here we prove a categorical equivalence for a larger variety of negation groupoids with operators, which includes Abelian groups and Abelian ℓ-groups. This result restricts to an equivalence between Abelian ℓ-groups and Abelian PR-groups, whence we can derive that the complex numbers actually generate the latter variety.

Definition 50. An *operator* with respect to the signature $\langle +, 0\rangle$ is an n-ary operation f that satisfies the identities

$$f(x_1, \ldots, x_i + y_i, \ldots, x_n) \approx f(x_1, \ldots, x_i, \ldots, x_n) + f(x_1, \ldots, y_i, \ldots, x_n)$$

and $f(0, 0, \ldots, 0) \approx 0$.

Definition 51. A *negation groupoid with operators* is an algebra $\mathbf{A} = \langle A, +, 0, -, f_1, f_2, \ldots\rangle$ such that the identities $x+0 \approx 0+x \approx x$, $-(-x) \approx x$ are satisfied and $-, f_1, f_2, \ldots$ are operators. A *projection-rotation groupoid with operators*, or PR-groupoid for short, is a negation groupoid with operators $\langle A, +, 0, -, f_1, f_2, \ldots, P, R\rangle$ (so P, R are also operators) such that the following identities hold for all $x, x_1, \ldots, x_n \in A$ and $i = 1, 2, \ldots$:

1. $P(-x) = -P(x)$
2. $Pf_i(x_1, \ldots, x_n) = f_i(P(x_1), \ldots, P(x_n))$
3. $PP(x) = P(x)$
4. $RR(x) = -x$
5. $PR(f_i(x_1, \ldots, x_n)) = f_i(PR(x_1), \ldots, PR(x_n))$
6. $PRP(x) = 0$
7. $P(x) + -RPR(x) = x$

Every negation groupoid $\mathbf{A}$ with operators gives rise to a PR-groupoid $F(\mathbf{A}) = \langle A \times A, +, \langle 0, 0\rangle, -, f_1, f_2, \ldots, P, R\rangle$ where $+, -, f_i$ are defined pointwise, $P(\langle a, b\rangle) = \langle a, 0\rangle$ and $R(\langle a, b\rangle) = \langle b, -a\rangle$. The operator identities and (1)-(5) are clearly satisfied, and checking (6), (7) is simple: $PRP(\langle a, b\rangle) = P(\langle 0, -a\rangle) = \langle 0, 0\rangle$, while

$$P(\langle a, b\rangle) + -RPR(\langle a, b\rangle) = \langle a, 0\rangle + -R(\langle b, 0\rangle) = \langle a, 0\rangle + \langle 0, b\rangle = \langle a, b\rangle.$$

Theorem 52. *Given a PR-groupoid* $\mathbf{A} = \langle A, +, 0, -, f_1, f_2, \ldots, P, R\rangle$, *define* $G(\mathbf{A}) = \langle P(A), +, 0, -, f_1, f_2, \ldots\rangle$. *Then* $G(\mathbf{A})$ *is a negation groupoid with operators, and the maps* $e : \mathbf{A} \to FG(\mathbf{A})$ *given by* $e(x) = \langle P(x), PR(x)\rangle$ *and* $d : \mathbf{B} \to FG(\mathbf{B})$ *given by* $d(x) = \langle x, 0\rangle$ *are isomorphisms. Moreover* F, G *are functors that give a categorical equivalence between the algebraic categories of negation groupoids with operators and PR-groupoids.*

Proof. $e(x+y) = \langle P(x+y), PR(x+y)\rangle = e(x)+e(y)$ and $e(0) = \langle 0, 0\rangle$ since P, R are operators. Similarly $e(-x) = -e(x)$ and $e(f_i(x_1, \ldots, x_n)) = f_i(e(x_1), \ldots, e(x_n))$ follow from (1), (2), (5). The homomorphism property for P, R is computed by

$$\begin{aligned} e(P(x)) &= \langle PP(x), PRP(x)\rangle = \langle P(x), 0\rangle = P(\langle P(x), PR(x)\rangle) = P(e(x)) \\ e(R(x)) &= \langle PR(x), PRR(x)\rangle = \langle PR(x), -P(x)\rangle = R(\langle P(x), PR(x)\rangle) \\ &= R(e(x)). \end{aligned}$$

If $e(x) = e(y)$ then $P(x) = P(y)$ and $PR(x) = PR(y)$, so (7) implies $x = y$, whence e is injective. Given $\langle P(x), P(y)\rangle \in FG(\mathbf{A})$, let $z = P(x) + R(-P(y))$. Then

$$\begin{aligned} e(z) &= \langle PP(x) + PR(-P(y)), PRP(x) + PRR(-P(y))\rangle \\ &= \langle P(x) + -PRP(y), PP(y)\rangle \\ &= \langle P(x), P(y)\rangle \end{aligned}$$

hence e is surjective. Similarly, checking that d is an isomorphism of negation groupoids with operators is straightforward.

For a homomorphism h between negation groupoids with operators, we define a homomorphism between the corresponding PR-groupoids by $F(h)(\langle a, b\rangle) = \langle h(a), h(b)\rangle$. Likewise for a homomorphism h between PR-groupoids, let $G(h)$ be the restriction of h to the image of P, then $G(h)$ is a homomorphism of negation groupoids with operators. Moreover, it is easy to check that F, G are functors. □

Corollary 53. *The varieties of negation groupoids with operators and PR-groupoids are categorically equivalent. The equivalence restricts to Abelian* ℓ*-groups and Abelian PR-groups, whence the variety of Abelian PR-groups is generated by* $\langle \mathbb{C}, \wedge, \vee, +, -, 0, P, R\rangle$, *where* $\langle \mathbb{C}, \wedge, \vee, +, -, 0\rangle$ *is the*

ℓ-group of the complex numbers (considered as $\mathbb{R}^2$), and P, R are defined by:

$$P(\langle a,b\rangle) = \langle a,0\rangle\,;$$
$$R(\langle a,b\rangle) = \langle b,-a\rangle\,.$$

We note that this result does not apply (in the current form) to non-Abelian (ℓ-)groups since the assumption that $-$ is an operator in a group implies that $+$ is commutative[4].

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