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THE LOGIC OF SEQUENCES

A b s t r a c t. The notion of "sequences" is fundamental to practical reasoning in computer science, because it can appropriately represent "data (information) sequences", "program (execution) sequences", "action sequences", "time sequences", "trees", "orders" etc. The aim of this paper is thus to provide a basic logic for reasoning with sequences. A propositional modal logic LS of sequences is introduced as a Gentzen-type sequent calculus by extending Gentzen's LK for classical propositional logic. The completeness theorem with respect to a sequence-indexed semantics for LS is proved, and the cut-elimination theorem for LS is shown. Moreover, a first-order modal logic FLS of sequences, which is a first-order extension of LS, is introduced. The completeness theorem with respect to a first-order sequence-indexed semantics for FLS is proved, and the cut-elimination theorem for FLS is shown. LS and the monadic fragment of FLS are shown to be decidable.

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1. Introduction

The aim of this paper is to provide a basic logic for reasoning with sequences. The notion of "sequences" is fundamental to practical reasoning in computer science, because it can appropriately represent "data (information) sequences", "program (execution) sequences", "action sequences", "time sequences", "word (character or alphabet) sequences", "DNA sequences" etc. The notion of sequences is thus useful to represent the notions of "information", "computation", "trees", "orders", "preferences", "strings", "vectors" and "ontologies". In a view of reasoning with sequences, *dynamic logics* [6] are logics dealing with program (or action) sequences, *temporal logics* [5] are logics dealing with time sequences, and *Lambek calculus* [11] is a logic dealing with word sequences. Representing "information" by sequences is particularly suitable and important, since a sequence structure gives a *monoid* $\langle M, ;, \emptyset \rangle$ with an *informational interpretation* [16]:

1. M is a set of pieces of (ordered or prioritized) information (i.e., a set of sequences),
2. ; is a binary operator (on M) which combines two pieces of information (i.e., the concatenation operator on sequences),
3. $\emptyset$ is the empty piece of information (i.e., the empty sequence).

The informational interpretation for a monoid based semantics for *substructural logics* including Lambek calculus was proposed by Wansing [16] extending and generalizing Urquhart's interpretation [14] for semilattice (i.e., idempotent commutative monoid) semantics for *relevant logics*.

Handling the notion of sequences in a logic has recently been studied by several researchers. *Sequence logic* (SL), which is a parameterized logic where the formulas are *sequences of formulas*, was proposed and studied by Walicki et al. [15, 1]. In [1], the completeness and decidability theorems w.r.t. the class of dense linear orderings and the class of linear orderings are proved for SL. A predicate logic with *sequence variables* and *sequence function symbols* was introduced by Kutsita and Buchberger [10]. In [10], a Gentzen-type sequent calculus $\mathrm{G}^{\approx}$ for this logic was introduced, and the completeness theorem for $\mathrm{G}^{\approx}$ was proved. The three approaches based on

SL, $G^{\approx}$ and our logic are completely different from each other. For example, the differences between SL and our logic are shown as follows. SL uses sequences of formulas, but our logic uses a *sequence modal operator*. As mentioned in [1], SL can be viewed as a subsystem of linear-time temporal logic where the temporal aspects are completely separated from other logical aspects. Our logic may be viewed as a modified and combined subsystem of both propositional dynamic logic [6] and Prior's next-time temporal logic [12]. Our logic is considerably simple and natural, and it has some theoretically beneficial properties: strong completeness, cut-elimination, decidability and embeddability.

The contents of this paper are then summarized as follows.

In Section 2, the propositional case is discussed. Firstly, a propositional modal logic LS of sequences is introduced as a Gentzen-type sequent calculus by extending Gentzen's LK for classical propositional logic. In order to represent reasoning with information sequences, a sequence modal operator $[b]$ where b is a sequence is subsumed in LS. Then, a formula of the form $[b_1 ; b_2 ; \cdots ; b_n]\alpha$ intuitively means "α is true based on the sequence $b_1 ; b_2 ; \cdots ; b_n$ of (ordered or prioritized) information pieces", and a formula of the form $[\emptyset]\alpha$ intuitively means "α is true without any information (i.e., it is an eternal truth in the sense of classical logic)". The embedding theorem of LS into LK is proved, and the cut-elimination and decidability theorems for LS are obtained from the embedding theorem. Secondly, a simple and intuitive semantics, called a *sequence-indexed semantics*, is introduced for LS. This semantics is regarded as a natural extension of the standard two-valued semantics of classical logic. In this semantics, the valuations, denoted as $v_{\hat{d}}$, are indexed by a sequence $\hat{d}$, and a valuation $v_{\emptyset}$, which is indexed by the empty sequence $\emptyset$, just corresponds to a classical two-valued valuation. Then, $v_{\hat{d}}(\alpha) = t$ means "α is true based on a sequence $\hat{d}$ of information pieces" and $v_{\emptyset}(\alpha) = t$ means "α is eternally true without any information". The completeness theorem w.r.t. the sequence-indexed semantics for LS is proved based on an extension of Maehara's decomposition method for (propositional) LK. An alternative semantical proof of the cut-elimination theorem for LS is also obtained from this completeness proof.

In Section 3, the first-order case is discussed similarly. A first-order modal logic FLS of sequences, which is a first-order extension of LS, is introduced, and the completeness theorem w.r.t. a first-order sequence-indexed semantics for FLS is proved based on an extension of Schütte's method for (first-order) LK. The cut-elimination theorem for FLS is obtained in both the semantical way via the completeness theorem and the syntactical way via the embedding theorem of FLS into (first-order) LK. By a consequence of the cut-elimination theorem, FLS can be viewed as a conservative extension of LS. The monadic fragment of FLS is shown to be decidable. In Section 4, some remarks are presented. Firstly, some possible applications are presented. Secondly, it is remarked that extending and modifying the completeness results for LS and FLS by adding the standard non-deterministic choice operator $\cup$ used in dynamic logics are technically difficult in the present setting. Finally, a relationship between LS and a semilattice relevant logic is explained.

2. Propositional case

2.1 Sequent calculus

Formulas are constructed from countably many propositional variables, $\to$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation) and $[b]$ (sequence modal operator) where b is a sequence of characters. *Sequences* are constructed from countably many atomic sequences (i.e., characters), $\emptyset$ (empty sequence), ; (sequence composition or concatenation) and $\cdot^{-}$ (converse). Lower-case letters $b, c, ...$ are used for sequences, Greek lower-case letters $\alpha, \beta, ...$ are used for formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used for finite (possibly empty) sets of formulas. The symbol SE is used to denote the set of sequences. We write $A \equiv B$ to indicate the syntactical identity between A and B. The symbol ω is used to represent the set of natural numbers. An expression $[\emptyset]\alpha$ means α, and expressions $[\emptyset\ ;\ b]\alpha$ and $[b\ ;\ \emptyset]\alpha$ mean $[b]\alpha$. An expression of the form $\Gamma \Rightarrow \Delta$ is called a *sequent*. The terminological conventions regarding sequent calculus (e.g., antecedent, succedent etc.) are used. If a sequent S is provable in a sequent calculus L, then such a fact is denoted as $L \vdash S$ or $\vdash S$.

Definition 2.1. Formulas and sequences are defined by the following grammar, assuming p and e represent propositional variables and atomic sequences, respectively:

$$\alpha ::= p \mid \alpha{\rightarrow}\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \neg\alpha \mid [b]\alpha$$
$$b ::= e \mid \emptyset \mid b\ ;\ b \mid b^{-}$$

A sequence of the form e or e^{-} where e is an atomic sequence is called a *literal sequence.*

An expression $[\hat{d}]$ is used to represent $[d_0][d_1]\cdots[d_i]$ with $i \in \omega$ and $d_0 \equiv \emptyset$, and an expression $\hat{d}$ is used to represent $d_0\ ;\ d_1\ ;\ \cdots ; d_i$ with $i \in \omega$ and $d_0 \equiv \emptyset$. Expressions $\emptyset\ ;\ b$ and $b\ ;\ \emptyset$ mean b. Remark that $[\hat{d}]$ and $\hat{d}$ can be empty and $\emptyset$, respectively.

Definition 2.2 (LS)**.** The initial sequents of LS are of the form: for any propositional variable p,

$$[\hat{d}]p \Rightarrow [\hat{d}]p.$$

The structural inference rules of LS are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{cut})$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (\text{we-left}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (\text{we-right}).$$

The logical inference rules of LS are of the form:

$$\frac{\Gamma \Rightarrow \Delta, [\hat{d}]\alpha \quad [\hat{d}]\beta, \Sigma \Rightarrow \Pi}{[\hat{d}](\alpha{\rightarrow}\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({\rightarrow}\text{left}) \qquad \frac{[\hat{d}]\alpha, \Gamma \Rightarrow \Delta, [\hat{d}]\beta}{\Gamma \Rightarrow \Delta, [\hat{d}](\alpha{\rightarrow}\beta)}\ ({\rightarrow}\text{right})$$

$$\frac{[\hat{d}]\alpha, \Gamma \Rightarrow \Delta}{[\hat{d}](\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left1}) \qquad \frac{[\hat{d}]\beta, \Gamma \Rightarrow \Delta}{[\hat{d}](\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left2})$$

$$\frac{\Gamma \Rightarrow \Delta, [\hat{d}]\alpha \quad \Gamma \Rightarrow \Delta, [\hat{d}]\beta}{\Gamma \Rightarrow \Delta, [\hat{d}](\alpha \wedge \beta)}\ (\wedge\text{right}) \qquad \frac{[\hat{d}]\alpha, \Gamma \Rightarrow \Delta \quad [\hat{d}]\beta, \Gamma \Rightarrow \Delta}{[\hat{d}](\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (\vee\text{left})$$

$$\frac{\Gamma \Rightarrow \Delta, [\hat{d}]\alpha}{\Gamma \Rightarrow \Delta, [\hat{d}](\alpha \vee \beta)}\ (\vee\text{right1}) \qquad \frac{\Gamma \Rightarrow \Delta, [\hat{d}]\beta}{\Gamma \Rightarrow \Delta, [\hat{d}](\alpha \vee \beta)}\ (\vee\text{right2})$$

$$\frac{\Gamma \Rightarrow \Delta, [\hat{d}]\alpha}{[\hat{d}](\neg\alpha), \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \qquad \frac{[\hat{d}]\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, [\hat{d}](\neg\alpha)}\ (\neg\text{right}).$$

The sequence inference rules of LS are of the form:

$$\frac{[\hat{d}][b][c]\alpha, \Gamma \Rightarrow \Delta}{[\hat{d}][b \; ; \; c]\alpha, \Gamma \Rightarrow \Delta} \text{ (;left)} \qquad \frac{\Gamma \Rightarrow \Delta, [\hat{d}][b][c]\alpha}{\Gamma \Rightarrow \Delta, [\hat{d}][b \; ; \; c]\alpha} \text{ (;right)}$$

$$\frac{[\hat{d}][b]\alpha, \Gamma \Rightarrow \Delta}{[\hat{d}][b^{--}]\alpha, \Gamma \Rightarrow \Delta} \text{ (}-\text{left)} \qquad \frac{\Gamma \Rightarrow \Delta, [\hat{d}][b]\alpha}{\Gamma \Rightarrow \Delta, [\hat{d}][b^{--}]\alpha} \text{ (}-\text{right)}$$

$$\frac{[\hat{d}][c^{-} \; ; \; b^{-}]\alpha, \Gamma \Rightarrow \Delta}{[\hat{d}][(b \; ; \; c)^{-}]\alpha, \Gamma \Rightarrow \Delta} \text{ (}-;\text{left)} \qquad \frac{\Gamma \Rightarrow \Delta, [\hat{d}][c^{-} \; ; \; b^{-}]\alpha}{\Gamma \Rightarrow \Delta, [\hat{d}][(b \; ; \; c)^{-}]\alpha} \text{ (}-;\text{right).}$$

Note that LS includes Gentzen's LK as a special case. Remark that a sequent calculus for Prior's next-time temporal logic [12] can be obtained from LS by deleting the sequence inference rules and replacing $[\hat{d}]$ by $\overbrace{\mathrm{X} \cdots \mathrm{X}}^{i}$ with $i \in \omega$ where X is the next-time operator.

Definition 2.3 (LK)**.** LK for propositional classical logic is obtained from LS by deleting the sequence inference rules and $[\hat{d}]$ appearing in the initial sequents and the logical inference rules. The names of the logical inference rules of LK are denoted by labeling "*LK*" in superscript position, e.g., ($\rightarrow$left^{LK}).

Proposition 2.4. *The rule of the form: for any non-empty sequence b,*

$$\frac{\Gamma \Rightarrow \Delta}{[b]\Gamma \Rightarrow [b]\Delta} \text{ (regu)}$$

is admissible in cut-free LS*.*

Proof. By induction on the cut-free proof P of $\Gamma \Rightarrow \Delta$ in LS. We distinguish the cases according to the last inference of P. We show only the following case.

Case (;left): The last inference of P is of the form:

$$\frac{[\hat{d}][c_1][c_2]\alpha, \Sigma \Rightarrow \Delta}{[\hat{d}][c_1 \; ; \; c_2]\alpha, \Sigma \Rightarrow \Delta} \text{ (;left).}$$

By induction hypothesis, we have LS $-$ (cut) $\vdash [b][\hat{d}][c_1][c_2]\alpha, [b]\Sigma \Rightarrow [b]\Delta$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ [b][\hat{d}][c_1][c_2]\alpha, [b]\Sigma \Rightarrow [b]\Delta\end{array}}{[b][\hat{d}][c_1 \; ; \; c_2]\alpha, [b]\Sigma \Rightarrow [b]\Delta} \text{ (;left).}$$

□

Proposition 2.5. *For any formula α, and any non-empty sequence b,* $\mathrm{LS} - (\mathrm{cut}) \vdash [b]\alpha \Rightarrow [b]\alpha$.

Proof. By induction on α. We show only the following case.

Case ($\alpha \equiv [c]\beta$ where c is a non-empty sequence): By induction hypothesis, we have $\mathrm{LS} - (\mathrm{cut}) \vdash [c]\beta \Rightarrow [c]\beta$. Then, we obtain the required fact $\mathrm{LS} - (\mathrm{cut}) \vdash [b][c]\beta \Rightarrow [b][c]\beta$ by applying the rule (regu), which is admissible in cut-free LS by Proposition 2.4. □

Proposition 2.6. *For any formula α,* $\mathrm{LS} - (\mathrm{cut}) \vdash \alpha \Rightarrow \alpha$.

Proof. By induction on α. We use Proposition 2.5 for the case $\alpha \equiv [b]\beta$. □

An expression $\alpha \Leftrightarrow \beta$ is an abbreviation for the pair of sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$.

Proposition 2.7. *The following sequents are provable in cut-free* LS*: for any formulas α, β, and any sequences b, c,*

1. $[b](\alpha \sharp \beta) \Leftrightarrow [b]\alpha \sharp [b]\beta$ *where* $\sharp \in \{\rightarrow, \wedge, \vee\}$,
2. $[b](\neg\alpha) \Leftrightarrow \neg([b]\alpha)$,
3. $[b\ ;\ c]\alpha \Leftrightarrow [b][c]\alpha$,
4. $[b^{--}]\alpha \Leftrightarrow [b]\alpha$,
5. $[(b\ ;\ c)^{-}] \Leftrightarrow [c^{-}\ ;\ b^{-}]\alpha$.

Note that by Propositions 2.4 and 2.7, LS is stronger than the basic normal modal logic K. Remark also that LS is not a fragment of propositional dynamic logic (PDL), since PDL has no axiom: $([b]\alpha \rightarrow [b]\beta) \rightarrow [b](\alpha \rightarrow \beta)$.

2.2 Embedding, cut-elimination and decidability

Definition 2.8. Let $\Psi := \{p, q, r, ...\}$ be a fixed countable non-empty set of propositional variables. Then, we define the sets $\Psi_{\hat{d}} := \{p_{\hat{d}} \mid p \in \Psi\}$ ($\hat{d} \in \mathrm{SE}$) of propositional variables where $p_{\emptyset} := p$, i.e., $\Psi_{\emptyset} := \Psi$. The language (or the set of formulas) $\mathcal{L}_{\rm LS}$ of LS is obtained from Ψ, $\rightarrow, \wedge, \vee, \neg$ and $[b]$. The language (or the set of formulas) $\mathcal{L}_{\rm LK}$ of LK is obtained from $\bigcup_{\hat{d}\in {\rm SE}} \Psi_{\hat{d}}$, $\rightarrow, \wedge, \vee$ and $\neg$.

A mapping f from $\mathcal{L}_{\rm LS}$ to $\mathcal{L}_{\rm LK}$ is defined by:

1. $f([\hat{d}]p) := p_{\hat{d}} \in \Psi_{\hat{d}}$ for every $p \in \Psi$,
2. $f([\hat{d}](\alpha \sharp \beta)) := f([\hat{d}]\alpha) \sharp f([\hat{d}]\beta)$ where $\sharp \in \{\rightarrow, \wedge, \vee\}$,
3. $f([\hat{d}]\neg\alpha) := \neg f([\hat{d}]\alpha)$,
4. $f([\hat{d}][b\ ;\ c]\alpha) := f([\hat{d}][b][c]\alpha)$,
5. $f([\hat{d}][b^{--}]\alpha) := f([\hat{d}][b]\alpha)$,
6. $f([\hat{d}][(b\ ;\ c)^{-}]\alpha)) := f([\hat{d}][c^{-}\ ;\ b^{-}]\alpha)$.

Let Γ be a set of formulas in $\mathcal{L}_{\rm LS}$. Then, an expression $f(\Gamma)$ means the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$.

Theorem 2.9 (Embedding)**.** *Let Γ and Δ be sets of formulas in $\mathcal{L}_{\rm LS}$, and f be the mapping defined in Definition 2.8. Then:*

1. LS $\vdash \Gamma \Rightarrow \Delta$ *iff* LK $\vdash f(\Gamma) \Rightarrow f(\Delta)$.

2. LS $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$ *iff* LK $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$.

Proof. We show only (1), since (2) can be obtained as the subproof of (1).

• ($\Longrightarrow$): By induction on the proof P of $\Gamma \Rightarrow \Delta$ in LS. We distinguish the cases according to the last inference of P. We show some cases.

Case ($[\hat{d}]p \Rightarrow [\hat{d}]p$): The last inference of P is of the form: $[\hat{d}]p \Rightarrow [\hat{d}]p$. Since $f([\hat{d}]p)$ coincides with the propositional variable $p_{\hat{d}}$ by the definition of f, we obtain the required fact: LK $\vdash f([\hat{d}]p) \Rightarrow f([\hat{d}]p)$.

Case ($\vee$right1): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, [\hat{d}]\alpha}{\Gamma \Rightarrow \Delta, [\hat{d}](\alpha \vee \beta)}\ (\vee\text{right1}).$$

By induction hypothesis, we have LK $\vdash f(\Gamma) \Rightarrow f(\Delta), f([\hat{d}]\alpha)$. Then, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f([\hat{d}]\alpha)\end{array}}{f(\Gamma) \Rightarrow f(\Delta), f([\hat{d}]\alpha) \vee f([\hat{d}]\beta)}\ (\vee\text{right}1^{LK})$$

where $f([\hat{d}]\alpha) \vee f([\hat{d}]\beta)$ coincides with $f([\hat{d}](\alpha \vee \beta))$ by the definition of f.

Case (;left): The last inference of P is of the form:

$$\dfrac{[\hat{d}][b][c]\alpha, \Gamma \Rightarrow \Delta}{[\hat{d}][b\ ;\ c]\alpha, \Gamma \Rightarrow \Delta}\ (;\text{left}).$$

By induction hypothesis, we have LK $\vdash f([\hat{d}][b][c]\alpha), f(\Gamma) \Rightarrow f(\Delta)$. Then, we obtain the required fact, since $f([\hat{d}][b][c]\alpha)$ coincides with $f([\hat{d}][b\ ;\ c]\alpha)$ by the definition of f.

• ($\Longleftarrow$): By induction on the proof Q of $f(\Gamma) \Rightarrow f(\Delta)$ in LK. We distinguish the cases according to the last inference of Q. We show only the following case.

Case ($\to$left^{LK}): The last inference of Q is of the form:

$$\dfrac{f(\Gamma) \Rightarrow f(\Pi), f([\hat{d}]\alpha) \quad f([\hat{d}]\beta), f(\Sigma) \Rightarrow f(\Delta)}{f([\hat{d}]\alpha) \to f([\hat{d}]\beta), f(\Gamma), f(\Sigma) \Rightarrow f(\Pi), f(\Delta)}\ (\to\text{left}^{LK}).$$

where $f([\hat{d}]\alpha) \to f([\hat{d}]\beta)$ coincides with $f([\hat{d}](\alpha \to \beta))$ by the definition of f. By induction hypothesis, we have LS $\vdash \Gamma \Rightarrow \Pi, [\hat{d}]\alpha$ and LS $\vdash [\hat{d}]\beta, \Sigma \Rightarrow \Delta$. Then, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Pi, [\hat{d}]\alpha\end{array} \quad \begin{array}{c}\vdots\\ [\hat{d}]\beta, \Sigma \Rightarrow \Delta\end{array}}{[\hat{d}](\alpha \to \beta), \Gamma, \Sigma \Rightarrow \Pi, \Delta}\ (\to\text{left}).$$

□

Theorem 2.10 (Cut-elimination)**.** *The rule* (cut) *is admissible in cut-free* LS*.*

Proof. Suppose LS $\vdash \Gamma \Rightarrow \Delta$. Then, we have LK $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 2.9 (1). We obtain LK $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by the well-known cut-elimination theorem for LK. By Theorem 2.9 (2), we obtain the required fact LS $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$. □

Theorem 2.11 (Decidability)**.** LS *is decidable.*

Proof. By the decidability of LK, for each α, it is possible to decide if $f(\alpha)$ is LK-provable. Then, by Theorem 2.9, LS is decidable. □

2.3 Sequence-indexed semantics

We have the following fact: for any formulas $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n$, the sequent $\alpha_1, ..., \alpha_m \Rightarrow \beta_1, ..., \beta_n$ is provable in LS if and only if so is $\alpha_1 \wedge \cdots \wedge \alpha_m \Rightarrow \beta_1 \vee \cdots \vee \beta_n$. Let Γ be a set $\{\alpha_1, ..., \alpha_m\}$ $(m \geq 0)$. Then, Γ^* means $\alpha_1 \vee \cdots \vee \alpha_m$ if $m \geq 1$, and otherwise $\neg(p \rightarrow p)$ where p is a fixed propositional variable. Also Γ_* means $\alpha_1 \wedge \cdots \wedge \alpha_m$ if $m \geq 1$, and otherwise $p \rightarrow p$ where p is a fixed propositional variable.

Definition 2.12. *Valuations* $v_{\hat{d}}$ $(\hat{d} \in \mathrm{SE})$, also called *sequence-indexed valuations*, are mappings from the set of all propositional variables to the set $\{t, f\}$ of truth values, such that for each $\hat{d}$, the truth value of any propositional letter at the sequence $\hat{d}$ is assigned. Each valuation $v_{\hat{d}}$ is extended to a mapping from the set of all formulas to $\{t, f\}$ by the following prescriptions:

1. $v_{\hat{d}}(\alpha \wedge \beta) = t$ iff $v_{\hat{d}}(\alpha) = v_{\hat{d}}(\beta) = t$,
2. $v_{\hat{d}}(\alpha \vee \beta) = t$ iff $v_{\hat{d}}(\alpha) = t$ or $v_{\hat{d}}(\beta) = t$,
3. $v_{\hat{d}}(\alpha \rightarrow \beta) = t$ iff $v_{\hat{d}}(\alpha) = f$ or $v_{\hat{d}}(\beta) = t$,
4. $v_{\hat{d}}(\neg\alpha) = t$ iff $v_{\hat{d}}(\alpha) = f$,
5. for each literal sequence e, $v_{\hat{d}}([e]\alpha) = t$ iff $v_{\hat{d}\,;\,e}(\alpha) = t$,
6. $v_{\hat{d}}([b\,;\,c]\alpha) = t$ iff $v_{\hat{d}}([b][c]\alpha) = t$,
7. $v_{\hat{d}}([b^{--}]\alpha) = t$ iff $v_{\hat{d}}([b]\alpha) = t$,
8. $v_{\hat{d}\,;\,b^{--}}(\alpha) = t$ iff $v_{\hat{d}\,;\,b}(\alpha) = t$,

9. $v_{\hat{d}}([(b\ ;\ c)^-]\alpha) = t$ iff $v_{\hat{d}}([c^-\ ;\ b^-]\alpha) = t$,

10. $v_{\hat{d}\ ;\ (b\ ;\ c)^-}(\alpha) = t$ iff $v_{\hat{d}\ ;\ c^-\ ;\ b^-}(\alpha) = t$.

A formula α is called a *tautology* if $v_{\emptyset}(\alpha) = t$ holds for any valuations $v_{\hat{d}}$ ($\hat{d} \in$ SE). A sequent of the form $\Gamma \Rightarrow \Delta$ is called a tautology if so is the formula $\Gamma_* \to \Delta^*$.

Proposition 2.13. *The following hold: for any valuations $v_{\hat{d}}$, any formula α, any non-empty sequence c, and any sequences b_1, b_2,*

1. $v_{\hat{d}}([c]\alpha) = t$ *iff* $v_{\hat{d}\ ;\ c}(\alpha) = t$,

2. $v_{\emptyset}([\hat{d}]\alpha) = t$ *iff* $v_{\hat{d}}(\alpha) = t$,

3. $v_{\hat{d}}([(b_1\ ;\ b_2)^-]\alpha) = t$ *iff* $v_{\hat{d}}([b_2^-][b_1^-]\alpha) = t$.

Proof. Since (3) is obvious and (2) is derived using (1), we only show (1) by induction on c.

Case ($c \equiv e$ where e is a literal sequence): By the definition of the valuations.

Case ($c \equiv b_1\ ;\ b_2$): $v_{\hat{d}}([b_1\ ;\ b_2]\alpha) = t$ iff $v_{\hat{d}}([b_1][b_2]\alpha) = t$ iff $v_{\hat{d}\ ;\ b_1}([b_2]\alpha) = t$ (by induction hypothesis) iff $v_{\hat{d}\ ;\ b_1\ ;\ b_2}(\alpha) = t$ (by induction hypothesis).

Case ($c \equiv b^{--}$): $v_{\hat{d}}([b^{--}]\alpha) = t$ iff $v_{\hat{d}}([b]\alpha) = t$ iff $v_{\hat{d}\ ;\ b}(\alpha) = t$ (by induction hypothesis) iff $v_{\hat{d}\ ;\ b^{--}}(\alpha) = t$.

Case ($c \equiv (b_1\ ;\ b_2)^-$): $v_{\hat{d}}([(b_1\ ;\ b_2)^-]\alpha) = t$ iff $v_{\hat{d}}([b_2^-\ ;\ b_1^-]\alpha) = t$ iff $v_{\hat{d}\ ;\ b_2^-}([b_1^-]\alpha) = t$ (by induction hypothesis) iff $v_{\hat{d}\ ;\ b_2^-\ ;\ b_1^-}(\alpha) = t$ (by induction hypothesis) iff $v_{\hat{d}\ ;\ (b_1\ ;\ b_2)^-}(\alpha) = t$. □

Theorem 2.14 (Soundness). *For any sequent S, if* LS $\vdash S$, *then S is a tautology.*

Proof. By induction on the proof P of S in LS. We distinguish the cases according to the last inference of P. Since the proof is similar to Lemma 2.17, we show only the case ($\wedge$right).

Case ($\wedge$right): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, [\hat{d}]\alpha \quad \Gamma \Rightarrow \Delta, [\hat{d}]\beta}{\Gamma \Rightarrow \Delta, [\hat{d}](\alpha \wedge \beta)}\ (\wedge\text{right}).$$

We show that $\Gamma_*{\rightarrow}\Delta^* \vee [\hat{d}](\alpha \wedge \beta)$ is a tautology. Suppose that (1) $v_\emptyset(\Gamma_*) = t$. Then, we show that $v_\emptyset(\Delta^* \vee [\hat{d}](\alpha \wedge \beta)) = t$. If $v_\emptyset([\hat{d}](\alpha \wedge \beta)) = t$, then $v_\emptyset(\Delta^* \vee [\hat{d}](\alpha \wedge \beta)) = t$. Thus, suppose that $v_\emptyset([\hat{d}](\alpha \wedge \beta)) = v_{\hat{d}}(\alpha \wedge \beta) = v_{\hat{d}}(\alpha) = v_{\hat{d}}(\beta) = f$. Then, we have (2) $v_\emptyset([\hat{d}]\alpha) = v_{\hat{d}}(\alpha) = f$ and (3) $v_\emptyset([\hat{d}]\beta) = v_{\hat{d}}(\beta) = f$. On the other hand, by induction hypothesis, we have that $\Gamma_*{\rightarrow}\Delta^* \vee [\hat{d}]\alpha$ and $\Gamma_*{\rightarrow}\Delta^* \vee [\hat{d}]\beta$ are tautologies. Then, we have (4) $v_\emptyset(\Gamma_*{\rightarrow}\Delta^* \vee [\hat{d}]\alpha) = t$ and (5) $v_\emptyset(\Gamma_*{\rightarrow}\Delta^* \vee [\hat{d}]\beta) = t$. By (1–5), we obtain $v_\emptyset(\Delta^*) = t$. Therefore we have $v_\emptyset(\Delta^* \vee [\hat{d}](\alpha \wedge \beta)) = t$. □

2.4 Completeness

Definition 2.15. A *decomposition* of a sequent S is defined as of the form S_0 or $S_0; S_1$ by

1a. $\Gamma \Rightarrow \Delta, [\hat{d}]\alpha$; $\Gamma \Rightarrow \Delta, [\hat{d}]\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}](\alpha \wedge \beta)$,

1b. $[\hat{d}]\alpha, [\hat{d}]\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}](\alpha \wedge \beta), \Gamma \Rightarrow \Delta$,

2a. $\Gamma \Rightarrow \Delta, [\hat{d}]\alpha, [\hat{d}]\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}](\alpha \vee \beta)$,

2b. $[\hat{d}]\alpha, \Gamma \Rightarrow \Delta$; $[\hat{d}]\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}](\alpha \vee \beta), \Gamma \Rightarrow \Delta$,

3a. $[\hat{d}]\alpha, \Gamma \Rightarrow \Delta, [\hat{d}]\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}](\alpha{\rightarrow}\beta)$,

3b. $\Gamma \Rightarrow \Delta, [\hat{d}]\alpha$; $[\hat{d}]\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}](\alpha{\rightarrow}\beta), \Gamma \Rightarrow \Delta$,

4a. $[\hat{d}]\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}]\neg\alpha$,

4b. $\Gamma \Rightarrow \Delta, [\hat{d}]\alpha$ is a decomposition of $[\hat{d}]\neg\alpha, \Gamma \Rightarrow \Delta$,

5a. $\Gamma \Rightarrow \Delta, [\hat{d}][b][c]\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}][b \,;\, c]\alpha$,

5b. $[\hat{d}][b][c]\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}][b \,;\, c]\alpha, \Gamma \Rightarrow \Delta$,

6a. $\Gamma \Rightarrow \Delta, [\hat{d}][b]\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}][b^{--}]\alpha$,

6b. $[\hat{d}][b]\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}][b^{--}]\alpha, \Gamma \Rightarrow \Delta$,

7a. $\Gamma \Rightarrow \Delta, [\hat{d}][c^- \,;\, b^-]\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}][(b \,;\, c)^-]\alpha$,

7b. $[\hat{d}][c^- \,;\, b^-]\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}][(b \,;\, c)^-]\alpha, \Gamma \Rightarrow \Delta$.

Definition 2.16. A *decomposition tree of* S is a tree produced by a decomposition procedure starting from S. A *complete decomposition tree of* S is such that the formulas occurring in the leaves have the form: p or $[e_1][e_2]\cdots[e_n]p$, with p: propositional variable and e_i: literal sequence.

Note that, by construction, the decomposition procedure terminates. Remark that a decomposition tree corresponds to a bottom-up proof search tree of LS.

Lemma 2.17. *Let* S_0 *or* $S_0; S_1$ *be a decomposition of* S. *If* S *is a tautology, then so are* S_0 *and* S_1.

Proof. We prove that if $v_\emptyset(S) = t$, then $v_\emptyset(S_0) = t$ and $v_\emptyset(S_1) = t$. We show some cases.

(3b): Suppose that $[\hat{d}](\alpha\to\beta) \wedge \Gamma_*\to\Delta^*$ is a tautology. First, we show that $\Gamma_*\to\Delta^* \vee [\hat{d}]\alpha$ is a tautology. Suppose that (1) $v_\emptyset(\Gamma_*) = t$. We show $v_\emptyset(\Delta^* \vee [\hat{d}]\alpha) = t$. If $v_\emptyset([\hat{d}]\alpha) = t$, then $v_\emptyset(\Delta^* \vee [\hat{d}]\alpha) = t$. Thus, suppose that $v_\emptyset([\hat{d}]\alpha) = v_{\hat{d}}(\alpha) = f$. Then, $v_{\hat{d}}(\alpha\to\beta) = t$, and hence (2) $v_\emptyset([\hat{d}](\alpha\to\beta)) = v_{\hat{d}}(\alpha\to\beta) = t$. On the other hand, we have (3) $v_\emptyset([\hat{d}](\alpha\to\beta) \wedge \Gamma_*\to\Delta^*) = t$ by the hypothesis. Thus, we obtain $v_\emptyset(\Delta^*) = t$ by (1), (2) and (3). Therefore $v_\emptyset(\Delta^* \vee [\hat{d}]\alpha) = t$. Second, we show that $[\hat{d}]\beta \wedge \Gamma_*\to\Delta^*$ is a tautology. Suppose that (4) $v_\emptyset([\hat{d}]\beta \wedge \Gamma_*) = t$. Then, (5) $v_\emptyset([\hat{d}]\beta) = v_{\hat{d}}(\beta) = t$ and (6) $v_\emptyset(\Gamma_*) = t$. By (5), we have $v_{\hat{d}}(\alpha\to\beta) = t$, and hence (7) $v_\emptyset([\hat{d}](\alpha\to\beta)) = v_{\hat{d}}(\alpha\to\beta) = t$. By (3), (6) and (7), we obtain $v_\emptyset(\Delta^*) = t$.

(5b): We show that if $[\hat{d}][b\ ;\ c]\alpha, \Gamma \Rightarrow \Delta$ is a tautology, then so is $[\hat{d}][b][c]\alpha, \Gamma \Rightarrow \Delta$, i.e., $v_\emptyset([\hat{d}][b\ ;\ c]\alpha \wedge \Gamma_*\to\Delta^*) = t$ implies $v_\emptyset([\hat{d}][b][c]\alpha \wedge \Gamma_*\to\Delta^*) = t$. It is sufficient to show that $v_\emptyset([\hat{d}][b\ ;\ c]\alpha) = t$ implies $v_\emptyset([\hat{d}][b][c]\alpha) = t$. This is shown by: $v_\emptyset([\hat{d}][b\ ;\ c]\alpha) = t$ iff $v_{\hat{d}}([b\ ;\ c]\alpha) = t$ iff $v_\emptyset([\hat{d}][b][c]\alpha) = t$. □

Lemma 2.18. *The following hold:*

1. *Suppose that each* α_j *or* β_k *in* $\{\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n\}$ *is a formula of the form* p *or* $[e_1][e_2]\cdots[e_n]p$ *where* p *is a propositional variable and* e_i *is a literal sequence. Then, the sequent* $\alpha_1, ..., \alpha_m \Rightarrow \beta_1, ..., \beta_n$ *is a tautology if and only if there are* α_j $(j \leq m)$ *and* β_k $(k \leq n)$ *such that* $\alpha_j \equiv \beta_k$.

2. LS $-$ (cut) $\vdash [\hat{d}]\alpha, \Gamma \Rightarrow \Delta, [\hat{d}]\alpha$.

Lemma 2.19. *Let S_0 or $S_0; S_1$ be a decomposition of S. If S_0 (S_0 and S_1) is (are) provable in cut-free* LS*, then so is S.*

Theorem 2.20 (Strong completeness)**.** *For any sequent S, if S is a tautology, then* LS $-$ (cut) $\vdash S$.

Proof. Suppose that a sequent S is a tautology. Then, all the leaves of a complete decomposition tree of S are tautologies by using Lemma 2.17 repeatedly. Then, these leaves are provable in cut-free LS by Lemma 2.18. By using Lemma 2.19 repeatedly for the complete decomposition tree of S, all the sequents in the tree are provable in cut-free LS. Therefore, in particular, S is provable in cut-free LS. □

An alternative semantical proof of Theorem 2.10 (Cut-elimination) is obtained as follows. Suppose that LS $\vdash S$ for any sequent S. Then, S is a tautology by Theorem 2.14. We then obtain LS $-$ (cut) $\vdash S$ by Theorem 2.20.

3. First-order case

3.1 Sequent calculus, embedding and cut-elimination

The notations used in the previous section are also adopted in this section. For the sake of simplicity, a first-order language L without individual constants and function symbols is adopted. *Formulas* of FLS are constructed from countably many predicate symbols $p, q, ...$, countably many individual variables $x, y, ...$, and the logical connectives $\rightarrow$, $\wedge$, $\vee$, $\neg$, $[b]$, $\forall$ (universal quantifier) and $\exists$ (existencial quantifier). *Sequences* of FLS are the same as that of LS. The term "propositional variable" used in the previous section is replaced by "atomic formula" in the following discussion. Lower-case letters $p, q, ...$ are also used to denote atomic formulas. We adopt the notation $\alpha[y/x]$ as the formula which is obtained from a formula α by replacing all free occurrences of an individual variable x in α by an arbitrary individual variable y, but avoiding the clash of variables.

Definition 3.1 (FLS)**.** FLS is obtained from LS by adding the inference rules of the form:

$$\frac{[\hat{d}]\alpha[y/x], \Gamma \Rightarrow \Delta}{[\hat{d}]\forall x\alpha, \Gamma \Rightarrow \Delta}\ (\forall left) \qquad \frac{\Gamma \Rightarrow \Delta, [\hat{d}]\alpha[z/x]}{\Gamma \Rightarrow \Delta, [\hat{d}]\forall x\alpha}\ (\forall right)$$

$$\frac{[\hat{d}]\alpha[z/x], \Gamma \Rightarrow \Delta}{[\hat{d}]\exists x\alpha, \Gamma \Rightarrow \Delta}\ (\exists left) \qquad \frac{\Gamma \Rightarrow \Delta, [\hat{d}]\alpha[y/x]}{\Gamma \Rightarrow \Delta, [\hat{d}]\exists x\alpha}\ (\exists right)$$

where y in (∀left) and (∃right) is an arbitrary individual variable, and z in (∀right) and (∃left) is an individual variable which has the eigenvariable condition, i.e., z does not occur as a free individual variable in the lower sequent of the rule.

Definition 3.2 (FLK)**.** FLK for first-order (predicate) classical logic is obtained from LK by adding the $[\hat{d}]$-free versions of the quantifier rules (∀left), (∀right), (∃left) and (∃right).

Proposition 3.3. *The rule* (regu) *is admissible in cut-free* FLS*.*

Proposition 3.4. *For any formula α, and any non-empty sequence b,* FLS − (cut) $\vdash [b]\alpha \Rightarrow [b]\alpha$*.*

Proof. By using Proposition 3.3. □

Proposition 3.5. *For any formula α,* FLS − (cut) $\vdash \alpha \Rightarrow \alpha$*.*

Proof. By using Proposition 3.4. □

Proposition 3.6. *For any formula α, any sequence b, and any $Q \in \{\forall x, \exists x\}$,* FLS − (cut) $\vdash [b](Q\alpha) \Leftrightarrow Q([b]\alpha)$*.*

Definition 3.7. Let $\Phi := \{p, q, r, ...\}$ be a fixed countable non-empty set of atomic formulas. Then, we define the sets $\Phi_{\hat{d}} := \{p_{\hat{d}} \mid p \in \Phi\}$ ($\hat{d} \in$ SE) of atomic formulas where $p_{\emptyset} := p$, i.e., $\Phi_{\emptyset} := \Phi$. The language (or the set of formulas) $\mathcal{L}_{\text{FLS}}$ of FLS is obtained from $\Phi, \rightarrow, \wedge, \vee, \neg, \forall, \exists$ and $[b]$. The language (or the set of formulas) $\mathcal{L}_{\text{FLK}}$ of FLK is obtained from $\bigcup_{\hat{d} \in \text{SE}} \Phi_{\hat{d}}$, $\rightarrow, \wedge, \vee, \neg, \forall$ and $\exists$.

A mapping f from $\mathcal{L}_{\text{FLS}}$ to $\mathcal{L}_{\text{FLK}}$ is defined by the conditions presented in Definition 2.8 and the following conditions: $f([\hat{d}](Q\alpha)) := Qf([\hat{d}]\alpha)$ where $Q \in \{\forall x, \exists x\}$.

Theorem 3.8 (Embedding)**.** *Let* Γ *and* Δ *be sets of formulas in* $\mathcal{L}_{\text{FLS}}$*, and* f *be the mapping defined in Definition 3.7. Then:*

1. FLS $\vdash \Gamma \Rightarrow \Delta$ *iff* FLK $\vdash f(\Gamma) \Rightarrow f(\Delta)$.

2. FLS $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$ *iff* FLK $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$.

Using Theorem 3.8 and the cut-elimination theorem for FLK, we can obtain the following theorem.

Theorem 3.9 (Cut-elimination)**.** *The rule* (cut) *is admissible in cut-free* FLS.

The *monadic fragment* of FLK, i.e., all predicate symbols take only one arity and there are no function symbols, is known as decidable. Since the provability of the monadic fragment of FLS can be transformed into that of the monadic fragment of FLK by (a slightly modified version of) Theorem 3.8, the following theorem can also be obtained.

Theorem 3.10 (Decidability)**.** *The monadic fragment of* FLS *is decidable.*

3.2 Sequence-indexed semantics

Definition 3.11. $A := \langle U, \{I^{\hat{d}}\}_{\hat{d}\in \text{SE}}\rangle$ is called a *model* (or *sequence-indexed model*) if the following conditions hold:

1. U is a non-empty set,
2. $I^{\hat{d}}$ ($\hat{d} \in$ SE) are mappings such that $p^{I^{\hat{d}}} \subseteq U^n$ (i.e., $p^{I^{\hat{d}}}$ are n-ary relations on U) for each n-ary predicate symbol p.

We introduce the notation $\underline{u}$ for the name of $u \in U$, and denotes $L[A]$ for the language obtained from L by adding the names of all the elements of U. A formula α is called a *closed formula* if α has no free individual variable. A formula of the form $\forall x_1 \cdots \forall x_m \alpha$ is called the *universal closure* of α if the free variables of α are $x_1, ..., x_m$. We write $cl(\alpha)$ for the universal closure of α.

Definition 3.12. Let $A := \langle U, \{I^{\hat{d}}\}_{\hat{d}\in \text{SE}}\rangle$ be a model. The consequence relations $A \models_{\hat{d}} \alpha$ ($\hat{d} \in$ SE) for any closed formula α of $L[A]$ are defined by:

1. $A \models_{\hat{d}} p(\underline{x}_1, ..., \underline{x}_n)$ iff $(x_1, ..., x_n) \in p^{I^{\hat{d}}}$ for each n-ary atomic formula $p(\underline{x}_1, ..., \underline{x}_n)$,
2. $A \models_{\hat{d}} \alpha \wedge \beta$ iff $A \models_{\hat{d}} \alpha$ and $A \models_{\hat{d}} \beta$,
3. $A \models_{\hat{d}} \alpha \vee \beta$ iff $A \models_{\hat{d}} \alpha$ or $A \models_{\hat{d}} \beta$,
4. $A \models_{\hat{d}} \alpha \rightarrow \beta$ iff not-($A \models_{\hat{d}} \alpha$) or $A \models_{\hat{d}} \beta$,
5. $A \models_{\hat{d}} \neg\alpha$ iff not-($A \models_{\hat{d}} \alpha$),
6. $A \models_{\hat{d}} \forall x\alpha$ iff $A \models_{\hat{d}} \alpha[\underline{u}/x]$ for all $u \in U$,
7. $A \models_{\hat{d}} \exists x\alpha$ iff $A \models_{\hat{d}} \alpha[\underline{u}/x]$ for some $u \in U$,
8. for each literal sequence e, $A \models_{\hat{d}} [e]\alpha$ iff $A \models_{\hat{d}\ ;\ e} \alpha$,
9. $A \models_{\hat{d}} [b\ ;\ c]\alpha$ iff $A \models_{\hat{d}} [b][c]\alpha$,
10. $A \models_{\hat{d}} [b^{--}]\alpha$ iff $A \models_{\hat{d}} [b]\alpha$,
11. $A \models_{\hat{d}\ ;\ b^{--}} \alpha$ iff $A \models_{\hat{d}\ ;\ b} \alpha$,
12. $A \models_{\hat{d}} [(b\ ;\ c)^{-}]\alpha$ iff $A \models_{\hat{d}} [c^{-}\ ;\ b^{-}]\alpha$,
13. $A \models_{\hat{d}\ ;\ (b\ ;\ c)^{-}} \alpha$ iff $A \models_{\hat{d}\ ;\ c^{-}\ ;\ b^{-}} \alpha$.

The consequence relations $A \models_{\hat{d}} \alpha$ ($\hat{d} \in$ SE) for any formula α of L are defined by ($A \models_{\hat{d}} \alpha$ iff $A \models_{\hat{d}} cl(\alpha)$). A formula α of L is called *valid* if $A \models_{\emptyset} \alpha$ holds for each model A. A sequent $\Gamma \Rightarrow \Delta$ of L is called valid if so is the formula $\Gamma_* \rightarrow \Delta^*$.

Proposition 3.13. *The following hold: for any consequence relations* $\models_{\hat{d}}$*, any formula* α*, any non-empty sequence* c*, and any sequences* b_1, b_2*,*

1. $A \models_{\hat{d}} [c]\alpha$ *iff* $A \models_{\hat{d}\ ;\ c} \alpha$*,*
2. $A \models_{\emptyset} [\hat{d}]\alpha$ *iff* $A \models_{\hat{d}} \alpha$*,*
3. $A \models_{\hat{d}} [(b_1\ ;\ b_2)^{-}]\alpha$ *iff* $A \models_{\hat{d}} [b_2^{-}][b_1^{-}]\alpha$*.*

Theorem 3.14 (Soundness)**.** *For any sequent* S*, if* FLS $\vdash S$*, then* S *is valid.*

Proof. By induction on the proof P of S. We distinguish the cases according to the last inference of P. We show only the following case.

Case (∀right): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, [\hat{d}]\alpha[z/x]}{\Gamma \Rightarrow \Delta, [\hat{d}]\forall x\alpha} \ (\forall\text{right}).$$

We show that "$\Gamma \Rightarrow \Delta, [\hat{d}]\alpha[z/x]$ is valid" implies "$\Gamma \Rightarrow \Delta, [\hat{d}]\forall x\alpha$ is valid". By the hypothesis, (i): $\forall z_1 \cdots \forall z_n \forall z(\Gamma_* \rightarrow (\Delta^* \vee ([\hat{d}]\alpha[z/x])))$ (where $z_1, ..., z_n$ are the free individual variables occurring in $\Gamma \Rightarrow \Delta, [\hat{d}]\forall x\alpha$) is valid. We show that $A \models_\emptyset \forall z_1 \cdots \forall z_n(\Gamma_* \rightarrow (\Delta^* \vee ([\hat{d}]\forall x\alpha)))$ for any model $A := \langle U, \omega_l, \{I^{\hat{d}}\}_{\hat{d}\in\mathrm{SE}}\rangle$, i.e., we show that for any $u_1, ..., u_n \in U$, $A \models_\emptyset \underline{\Gamma}_* \rightarrow (\underline{\Delta}^* \vee ([\hat{d}]\forall x\underline{\alpha}))$, where $\underline{\Gamma}_*, \underline{\Delta}^*$ and $\underline{\alpha}$ are respectively obtained from Γ_*, Δ^* and α by replacing $z_1, ..., z_n$ by $\underline{u}_1, ..., \underline{u}_n$ [1]. By (i), we have $A \models_\emptyset (\underline{\Gamma}_* \rightarrow (\underline{\Delta}^* \vee ([\hat{d}]\underline{\alpha}[z/x])))[\underline{w}/z]$ for any $w \in U$. By the eigenvariable condition, z is not occurring freely in $\underline{\Gamma}_*, \underline{\Delta}^*$ and $\underline{\alpha}$. Thus, $\underline{\Gamma}_*[\underline{w}/z]$ and $\underline{\Delta}^*[\underline{w}/z]$ are equivalent to $\underline{\Gamma}_*$and $\underline{\Delta}^*$ respectively, and $\underline{\alpha}[z/x][\underline{w}/z]$ is equivalent to $\underline{\alpha}[\underline{w}/z][\underline{w}/x]$, i.e., $\underline{\alpha}[\underline{w}/x]$. Therefore, for any $w \in U$, we have that (a): $A \models_\emptyset \underline{\Gamma}_* \rightarrow (\underline{\Delta}^* \vee [\hat{d}]\underline{\alpha}[\underline{w}/x])$. Suppose that (b): $[A \models_\emptyset \underline{\Gamma}_*$ and not-$(A \models_\emptyset \underline{\Delta}^*)]$. Then, by (a), we have that for any $w \in U$, $A \models_\emptyset [\hat{d}]\underline{\alpha}[\underline{w}/x]$, i.e., $A \models_{\hat{d}} \underline{\alpha}[\underline{w}/x]$. Therefore, we obtain (c): $A \models_{\hat{d}} \forall x\underline{\alpha}$, and hence $A \models_\emptyset [\hat{d}]\forall x\underline{\alpha}$. This means that (b) implies (c), i.e., $A \models_\emptyset \underline{\Gamma}_*$ implies $(A \models_\emptyset \underline{\Delta}^*$ or $A \models_\emptyset [\hat{d}]\forall x\underline{\alpha})$. Therefore, we have the required fact that $A \models_\emptyset \underline{\Gamma}_* \rightarrow (\underline{\Delta}^* \vee ([\hat{d}]\forall x\underline{\alpha}))$ for any $u_1, ...u_n \in U$.

□

3.3 Completeness

Definition 3.15. A sequent $\Gamma \Rightarrow \Delta$ is called *saturated* if

s1. $[\hat{d}](\alpha \wedge \beta) \in \Gamma$ implies $([\hat{d}]\alpha \in \Gamma$ and $[\hat{d}]\beta \in \Gamma)$,

s2. $[\hat{d}](\alpha \wedge \beta) \in \Delta$ implies $([\hat{d}]\alpha \in \Delta$ or $[\hat{d}]\beta \in \Delta)$,

s3. $[\hat{d}](\alpha \vee \beta) \in \Gamma$ implies $([\hat{d}]\alpha \in \Gamma$ or $[\hat{d}]\beta \in \Gamma)$,

s4. $[\hat{d}](\alpha \vee \beta) \in \Delta$ implies $([\hat{d}]\alpha \in \Delta$ and $[\hat{d}]\beta \in \Delta)$,

[1] Remark that $([\hat{d}]\forall x\alpha)[\underline{u}_1/z_1, ..., \underline{u}_n/z_n]$ (the simultaneous substitution) is equivalent to $[\hat{d}]\forall x(\alpha[\underline{u}_1/z_1, ..., \underline{u}_n/z_n])$, i.e., $[\hat{d}]\forall x\underline{\alpha}$.

s5. $[\hat{d}](\alpha \to \beta) \in \Gamma$ implies $([\hat{d}]\alpha \in \Delta$ or $[\hat{d}]\beta \in \Gamma)$,

s6. $[\hat{d}](\alpha \to \beta) \in \Delta$ implies $([\hat{d}]\alpha \in \Gamma$ and $[\hat{d}]\beta \in \Delta)$,

s7. $[\hat{d}]\neg\alpha \in \Gamma$ implies $[\hat{d}]\alpha \in \Delta$,

s8. $[\hat{d}]\neg\alpha \in \Delta$ implies $[\hat{d}]\alpha \in \Gamma$,

s9. $[\hat{d}]\forall x\alpha \in \Gamma$ implies $([\hat{d}]\alpha[y/x] \in \Gamma$ for any individual variable $y)$,

s10. $[\hat{d}]\forall x\alpha \in \Delta$ implies $([\hat{d}]\alpha[z/x] \in \Delta$ for some individual variable $z)$,

s11. $[\hat{d}]\exists x\alpha \in \Gamma$ implies $([\hat{d}]\alpha[z/x] \in \Gamma$ for some individual variable $z)$,

s12. $[\hat{d}]\exists x\alpha \in \Delta$ implies $([\hat{d}]\alpha[y/x] \in \Delta$ for any individual variable $y)$,

s13. $[\hat{d}][b\ ;\ c]\alpha \in \Gamma$ implies $[\hat{d}][b][c]\alpha \in \Gamma$,

s14. $[\hat{d}][b\ ;\ c]\alpha \in \Delta$ implies $[\hat{d}][b][c]\alpha \in \Delta$,

s15. $[\hat{d}][b^{--}]\alpha \in \Gamma$ implies $[\hat{d}][b]\alpha \in \Gamma$,

s16. $[\hat{d}][b^{--}]\alpha \in \Delta$ implies $[\hat{d}][b]\alpha \in \Delta$,

s17. $[\hat{d}][(b\ ;\ c)^{-}]\alpha \in \Gamma$ implies $[\hat{d}][c^{-}\ ;\ b^{-}]\alpha \in \Gamma$,

s18. $[\hat{d}][(b\ ;\ c)^{-}]\alpha \in \Delta$ implies $[\hat{d}][c^{-}\ ;\ b^{-}]\alpha \in \Delta$.

A sequent $\Gamma \Rightarrow \Delta$ is called an *infinite sequent* if Γ or Δ are infinite (countable) sets of formulas. An infinite sequent $\Gamma \Rightarrow \Delta$ is called *provable* if two finite subsets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ exist, such that $\mathrm{LS} \vdash \Gamma' \Rightarrow \Delta'$.

Definition 3.16. A *decomposition* of a sequent (or infinite sequent) S is defined as of the form S_0 or $S_0; S_1$ by

1a. $\Gamma \Rightarrow \Delta, [\hat{d}](\alpha \wedge \beta), [\hat{d}]\alpha\ ;\ \Gamma \Rightarrow \Delta, [\hat{d}](\alpha \wedge \beta), [\hat{d}]\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}](\alpha \wedge \beta)$,

1b. $[\hat{d}]\alpha, [\hat{d}]\beta, [\hat{d}](\alpha \wedge \beta), \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}](\alpha \wedge \beta), \Gamma \Rightarrow \Delta$,

2a. $\Gamma \Rightarrow \Delta, [\hat{d}](\alpha \vee \beta), [\hat{d}]\alpha, [\hat{d}]\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}](\alpha \vee \beta)$,

2b. $[\hat{d}]\alpha, [\hat{d}](\alpha \vee \beta), \Gamma \Rightarrow \Delta\ ;\ [\hat{d}]\beta, [\hat{d}](\alpha \vee \beta), \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}](\alpha \vee \beta), \Gamma \Rightarrow \Delta$, 3a. $[\hat{d}]\alpha, \Gamma \Rightarrow \Delta, [\hat{d}](\alpha{\rightarrow}\beta), [\hat{d}]\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}](\alpha{\rightarrow}\beta)$,

3b. $[\hat{d}](\alpha{\rightarrow}\beta), \Gamma \Rightarrow \Delta, [\hat{d}]\alpha$; $[\hat{d}]\beta, [\hat{d}](\alpha{\rightarrow}\beta), \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}](\alpha{\rightarrow}\beta), \Gamma \Rightarrow \Delta$,

4a. $[\hat{d}]\alpha, \Gamma \Rightarrow \Delta, [\hat{d}]\neg\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}]\neg\alpha$,

4b. $[\hat{d}]\neg\alpha, \Gamma \Rightarrow \Delta, [\hat{d}]\alpha$ is a decomposition of $[\hat{d}]\neg\alpha, \Gamma \Rightarrow \Delta$,

5a. $\Gamma \Rightarrow \Delta, [\hat{d}]\forall x\alpha, [\hat{d}]\alpha[z/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}]\forall x\alpha$ where z is a *fresh* free individual variable, i.e., z is not occurring in $\Gamma \Rightarrow \Delta, [\hat{d}]\forall x\alpha$,

5b. $[\hat{d}]\alpha[y_1/x], ..., [\hat{d}]\alpha[y_m/x], [\hat{d}]\forall x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}]\forall x\alpha, \Gamma \Rightarrow \Delta$ where $y_1, ..., y_m$ are the free individual variables occurring in $[\hat{d}]\forall x\alpha, \Gamma \Rightarrow \Delta$, [2]

6a. $\Gamma \Rightarrow \Delta, [\hat{d}]\exists x\alpha, [\hat{d}]\alpha[y_1/x], ..., [\hat{d}]\alpha[y_m/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}]\exists x\alpha$ where $y_1, ..., y_m$ are the free individual variables occurring in $\Gamma \Rightarrow \Delta, [\hat{d}]\exists x\alpha$,

6b. $[\hat{d}]\alpha[z/x], [\hat{d}]\exists x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}]\exists x\alpha, \Gamma \Rightarrow \Delta$ where z is a fresh free individual variable,

7a. $\Gamma \Rightarrow \Delta, [\hat{d}][b\ ;\ c]\alpha, [\hat{d}][b][c]\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}][b\ ;\ c]\alpha$,

7b. $[\hat{d}][b\ ;\ c]\alpha, [\hat{d}][b][c]\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}][b\ ;\ c]\alpha, \Gamma \Rightarrow \Delta$,

8a. $\Gamma \Rightarrow \Delta, [\hat{d}][b^{--}]\alpha, [\hat{d}][b]\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}][b^{--}]\alpha$,

8b. $[\hat{d}][b^{--}]\alpha, [\hat{d}][b]\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}][b^{--}]\alpha, \Gamma \Rightarrow \Delta$,

9a. $\Gamma \Rightarrow \Delta, [\hat{d}][(b\ ;\ c)^{-}]\alpha, [\hat{d}][c^{-}\ ;\ b^{-}]\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, [\hat{d}][(b\ ;\ c)^{-}]\alpha$,

9b. $[\hat{d}][(b\ ;\ c)^{-}]\alpha, [\hat{d}][c^{-}\ ;\ b^{-}]\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $[\hat{d}][(b\ ;\ c)^{-}]\alpha, \Gamma \Rightarrow \Delta$.

A *decomposition tree of* S is a tree obtained through repeated decompositions of S.

[2]Strictly speaking, if $[\hat{d}]\forall x\alpha, \Gamma \Rightarrow \Delta$ has no free individual variable, then we adopt any free variable in L. Such a condition is also adopted in (6a).

Remark that in every decomposition of S, i.e., S_0 or $S_0; S_1$, if S is unprovable in FLS $-$ (cut), then so is S_0 or S_1. Observe moreover that the steps of the procedure where new fresh variables are introduced play a central role for constructing the domain U of a canonical model.

Lemma 3.17. *Suppose that* $\Gamma \Rightarrow \Delta$ *is an unprovable sequent in* FLS $-$ (cut). *Then, there exists an unprovable, saturated and possibly infinite sequent* $\Gamma^\omega \Rightarrow \Delta^\omega$ *such that* $\Gamma \subseteq \Gamma^\omega$ *and* $\Delta \subseteq \Delta^\omega$*.*

Proof. Suppose that $\Gamma \Rightarrow \Delta$ is an unprovable sequent in FLS $-$ (cut). We construct $\Gamma^\omega \Rightarrow \Delta^\omega$ from $\Gamma \Rightarrow \Delta$ as follows.

(1) We apply the decomposition procedure in Definition 3.16 to $\Gamma \Rightarrow \Delta$, in the following order:

$$\text{(1a)} \longrightarrow \text{(1b)} \longrightarrow \text{(2a)} \longrightarrow \cdots \longrightarrow \text{(9b)}$$

where the application of any step may be empty if the corresponding formula lacks in $\Gamma \Rightarrow \Delta$. In such a decomposition process, one of the decomposed elements of S is an unprovable sequent.

(2) We repeat the same procedure (1) without halting. Then, we obtain a finitely branching decomposition tree with infinitely many nodes.

(3) By Kӧnig's lemma, we have an infinite path of this decomposition tree as follows.

$$\Gamma_0 \Rightarrow \Delta_0 \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \cdots \ \infty$$

where $\Gamma_0 \Rightarrow \Delta_0$ is $\Gamma \Rightarrow \Delta$. In this sequence of the sequents on the infinite path, we have that $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ and $\Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \cdots$.

(4) We put $\Gamma^\omega := \bigcup_{i=0}^{\infty} \Gamma_i$ and $\Delta^\omega := \bigcup_{i=0}^{\infty} \Delta_i$. Note that $\Gamma^\omega \cap \Delta^\omega = \emptyset$.

Then, we have that $\Gamma \subseteq \Gamma^\omega$ and $\Delta \subseteq \Delta^\omega$, and can verify that $\Gamma^\omega \Rightarrow \Delta^\omega$ is an unprovable, saturated (infinite) sequent. □

Lemma 3.18. *Let* $\Gamma \Rightarrow \Delta$ *be an unprovable sequent in* FLS $-$ (cut)*, and* $\Gamma^\omega \Rightarrow \Delta^\omega$ *be an unprovable saturated (infinite) sequent constructed from* $\Gamma \Rightarrow \Delta$ *by Lemma 3.17. We define the canonical model* $A := \langle U, \{I^{\hat{d}}\}_{\hat{d} \in \text{SE}} \rangle$ *by*

1. $U := \{z \mid z$ *is a free individual variable occurring in* $\Gamma^\omega \Rightarrow \Delta^\omega\}$,
2. $p^{I^{\hat{d}}} := \{(z_1, ..., z_m) \mid [\hat{d}]p(z_1, ..., z_m) \in \Gamma^\omega\}$.

Then, for any formula α*, and any* $\hat{d} \in \mathrm{SE}$*,*

$$[([\hat{d}]\alpha \in \Gamma^{\omega} \textit{ implies } A \models_{\hat{d}} \underline{\alpha}) \textit{ and } ([\hat{d}]\alpha \in \Delta^{\omega} \textit{ implies not-}(A \models_{\hat{d}} \underline{\alpha}))]$$

where $\underline{\alpha}$ *is obtained from* α *by replacing every individual variable* x *occurring in* α *by the name* $\underline{x}$*.*

Proof. By induction on the complexity of α.

• Base step:

Case ($\alpha \equiv p(z_1, z_2, ..., z_m)$: atomic formula): By the definition of the canonical model.

• Induction step: We show some cases.

Case ($\alpha \equiv \forall x\beta$): We show only that $[\hat{d}]\forall x\beta \in \Gamma^{\omega}$ implies $A \models_{\hat{d}} \forall x\underline{\beta}$, since the other case (concerning Δ^{ω}) can be shown in the dual. Suppose $[\hat{d}]\forall x\beta \in \Gamma^{\omega}$. Then we obtain that $[\hat{d}]\beta[y_i/x] \in \Gamma^{\omega}$ for all $y_i \in U$, by Definition 3.15 (s9). By induction hypothesis, we obtain that $A \models_{\hat{d}} \underline{\beta}[\underline{y_i}/x]$ for all $y_i \in U$. This means $A \models_{\hat{d}} \forall x\underline{\beta}$.

Case ($\alpha \equiv \exists x\beta$): We show only that $[\hat{d}]\exists x\beta \in \Gamma^{\omega}$ implies $A \models_{\hat{d}} \exists x\underline{\beta}$, since the other case (concerning Δ^{ω}) can be shown in the dual. Suppose $[\hat{d}]\exists x\beta \in \Gamma^{\omega}$. Then, we obtain that $[\hat{d}]\beta[z/x] \in \Gamma^{\omega}$ for some $z \in U$, by Definition 3.15 (s11). By induction hypothesis, we obtain that $A \models_{\hat{d}} \underline{\beta}[\underline{z}/x]$ for some $z \in U$. This means $A \models_{\hat{d}} \exists x\underline{\beta}$.

Case ($\alpha \equiv [b]\beta$ where b is an arbitrary sequence): We show only the case $[\hat{d}][b]\beta \in \Gamma^{\omega}$ implies $A \models_{\hat{d}} [b]\underline{\beta}$, since the other case (concerning Δ^{ω}) can be shown in the dual. Suppose $[\hat{d}][b]\beta \in \Gamma^{\omega}$. By induction hypothesis, we obtain $A \models_{\hat{d}\,;\,b} \underline{\beta}$. This means $A \models_{\hat{d}} [b]\underline{\beta}$. □

Theorem 3.19 (Strong completeness)**.** *For any sequent* S*, if* S *is valid, then* FLS − (cut) ⊢ S*.*

Proof. It is sufficient to show that if $\Gamma \Rightarrow \Delta$ is not provable in FLS − (cut), then there exists a model A such that $\Gamma \Rightarrow \Delta$ is not valid in A. Suppose that $\Gamma \Rightarrow \Delta$ is not provable in FLS − (cut). Then, by Lemma 3.18, we can construct a canonical model A with the condition in this lemma. Thus, we have $A \models_{\hat{d}} \underline{\gamma}$ and not-($A \models_{\hat{d}} \underline{\delta}$) for any $\gamma \in \Gamma \subseteq \Gamma^{\omega}$, any $\delta \in \Delta \subseteq \Delta^{\omega}$ and any $\hat{d} \in \mathrm{SE}$. Hence, we obtain "not-($A \models_{\hat{d}} \underline{\Gamma}_* \to \underline{\Delta}^*$)" for any $\hat{d} \in \mathrm{SE}$, and hence "not-($A \models_{\hat{d}} cl(\Gamma_* \to \Delta^*)$)" for any $\hat{d} \in \mathrm{SE}$. In particular, we can take $\emptyset$ as $\hat{d}$. Therefore, $\Gamma \Rightarrow \Delta$ is not valid in A. □

An alternative proof of Theorem 2.10 (Cut-elimination) is obtained by combining Theorem 3.19 with Theorem 3.14.

4. Remarks

4.1 Remarks on applications

Firstly, we explain that the sequence modal operator is useful for representing lists. Lists are known to be useful data structures in computer science. Lists have thus widely been studied by many researchers from the point of view of mathematical logic. For example, arithmetical first-order theories allowing encoding and decoding of lists were studied in [3]. [3] Logical representations of lists were proposed in [13] based on *separation logic* (SEL), and a modified version of such a list expression was also reconsidered in [7]. Lists are sometimes expressed as trees (i.e., sequences), and hence the sequence modal operator may be appropriate for expressing lists.

It is known that SEL, which permits reasoning about low-level imperative programs, is useful for specifying "shared mutable data structures" [13]. SEL has a novel logical connective $*_s$ for *separating conjunction*: A formula $\alpha *_s \beta$ holds for a heap (or addressable storage) h if there is a partition $\langle h_1, h_2 \rangle$ of h such that α holds for h_1 and β holds for h_2. It was shown in [13] that SEL is useful for representing list structures in which the relevant abstract values are sequences. The simplest list structure for representing sequences is the single-linked list. In order to represent such a list, the separating conjunction is used effectively to prohibit cycles within the list segment.

Let α be a sequence, and α_i be the i-th component of α. Suppose that an expression $i \mapsto \alpha$, which says "i point to α" , asserts that the heap contains one cell, at address i with contents α. Let $i \mapsto \alpha, j$ be the abbreviation of $(i \mapsto \alpha) *_s (i+1 \mapsto j)$. Then, by using SEL, we have the following expression for the singly-linked list with a list segment from i_0 to i_n representing the sequence α:

[3] It is pointed out that a relationship between FSL and the first-order or algebraic theories of sequences by Cégielski and Richard [3] may be worth studying, although the present paper does not discuss more about this topic.

$$\exists i_1, ..., i_{n-1}.\ (i_0 \mapsto \alpha_1, i_1) *_s (i_1 \mapsto \alpha_2, i_2) *_s \cdots *_s (i_{n-1} \mapsto \alpha_n, i_n)$$

where $i_0, ..., i_{n-1}$ are distinct, but i_n is not constrained.

In the following, we consider to use LS instead of SEL. The "point-to" relation $i \mapsto \alpha$ in SEL is encoded in LS by the formula $[i]\alpha$. A separating conjunction formula $\alpha *_s \beta$ in SEL is encoded in LS by $[i]\alpha' \wedge [j]\beta'$ $(i \neq j)$. This expression may be appropriate, since a sequence is adequately representable based on the sequence modal operator. Then, the list structure discussed above is naturally encoded in LS by

$$\exists i_1, ..., i_{n-1}.\ ([i_0]\alpha_1 \wedge [i_0+1]i_1) \wedge ([i_1]\alpha_2 \wedge [i_1+1]i_2) \wedge \cdots \wedge ([i_{n-1}]\alpha_n \wedge [i_{n-1}+1]i_n)$$

where $i_0, ..., i_{n-1}$ are distinct, but i_n is not constrained.

Secondly, we explain that the sequence modal operator can be used as the next-time temporal operator X. As mentioned in Section 2, LS is regarded as a modified extension of Prior's next-time temporal logic [12]. Indeed, the following rule, which is similar to the rule (regu) of LS and FLS, appears in some temporal logics:

$$\frac{\Gamma \Rightarrow \Delta}{\mathrm{X}\Gamma \Rightarrow \mathrm{X}\Delta}\ (\mathrm{Xregu}).$$

Moreover, the following clause of the semantics of FLS:

for each literal sequence e, $A \models_{\hat{d}} [e]\alpha$ iff $A \models_{\hat{d}\ ;\ e} \alpha$

is similar to the following clause of the semantics of some temporal logics:

$(A, i) \models \mathrm{X}\alpha$ iff $(A, i+1) \models \alpha$.

An instance of the sequence modal operator, expressed as $[x]$, may thus be interpreted as X. A formula of the form $\overbrace{\mathrm{X}\cdots\mathrm{X}}^{n}\alpha$, which means "$\alpha$ holds after n-time units", is then interpreted as $\overbrace{[x]\cdots[x]}^{n}\alpha$ in LS and FLS.

Thirdly, we explain that LS and FLS can be used as an alternative proof system for dynamic logics (DLs) [6]. It is known that a Gentzen-type "cut-free" sequent calculus for DLs have not been obtained yet, although a tree-style cut-free sequent calculus for a Kleene-star-free version of DLs was proposed in [2]. Then, we can use LS and FLS as an alternative proof theory for DLs, since LS and FLS give a good proof theory for the program composition and program converse operators in DLs. ¿From the point of view of proof theory, LS and FLS have the following advantages: LS and FLS enjoy cut-elimination, and LS and the monadic fragment of FLS are decidable.

Finally in this subsection, it is mentioned that hierarchical tree structures can suitably be expressed using the sequence modal operator. Such an example is proposed in [8]. In [8], an extended full computation tree logic, CTLS*, is introduced as a Kripke semantics with a sequence modal operator. CTLS* is regarded as a tempotal extension of the sequence-indexed semantics of LS. It was shown in [8] that CTLS* can appropriately represent hierarchical tree structures for biological taxonomy where the sequence modal operator is applied to tree structures.

4.2 Remarks on some extensions

In the following, we explain that extending and modifying the completeness results for LS and FLS have some technical difficulties. We consider the sequence inference rules for $\cup$ (non-deterministic choice) and $\cdot^*$ (iteration) of the form:

$$\frac{[\hat{d}][b]\alpha, \Gamma \Rightarrow \Delta}{[\hat{d}][b \cup c]\alpha, \Gamma \Rightarrow \Delta}\ (\cup\text{left1}) \qquad \frac{[\hat{d}][c]\alpha, \Gamma \Rightarrow \Delta}{[\hat{d}][b \cup c]\alpha, \Gamma \Rightarrow \Delta}\ (\cup\text{left2})$$

$$\frac{\Gamma \Rightarrow \Delta, [\hat{d}][b]\alpha \quad \Gamma \Rightarrow \Delta, [\hat{d}][c]\alpha}{\Gamma \Rightarrow \Delta, [\hat{d}][b \cup c]\alpha}\ (\cup\text{right})$$

$$\frac{[\hat{d}][c^k]\alpha, \Gamma \Rightarrow \Delta}{[\hat{d}][c^*]\alpha, \Gamma \Rightarrow \Delta}\ (*\text{left}) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, [\hat{d}][c^n]\alpha\ \}_{n\in\omega}}{\Gamma \Rightarrow \Delta, [\hat{d}][c^*]\alpha}\ (*\text{right}).$$

Then, we consider an extension ELS of the $\cdot^-$-less fragment of LS by adding the sequence inference rules displayed above. The modified sequence-indexed semantics may naturally be obtained from the semantics for the $\cdot^-$-less fragment of LS by adding the following valuation conditions:

1. $v_{\hat{d}}([b \cup c]\alpha) = t$ iff $v_{\hat{d}}([b]\alpha) = v_{\hat{d}}([c]\alpha) = t$,
2. $v_{\hat{d}\ ;\ (b\cup c)}(\alpha) = t$ iff $v_{\hat{d}\ ;\ b}(\alpha) = v_{\hat{d}\ ;\ c}(\alpha) = t$,

3. $v_{\hat{d}}([b^*]\alpha) = t$ iff $v_{\hat{d}}([b^n]\alpha) = t$ for all $n \in \omega$,

4. $v_{\hat{d}\ ;\ b^*}(\alpha) = t$ iff $v_{\hat{d}\ ;\ b^n}(\alpha) = t$ for all $n \in \omega$.

The following hold: for any valuations $v_{\hat{d}}$, any formula α and any sequence c,

$$v_{\hat{d}}([c]\alpha) = t \text{ iff } v_{\hat{d}\ ;\ c}(\alpha) = t.$$

The setting based on the modified valuation conditions looks natural and plausible, but some problems of non-uniqueness of interpretations arise. For this semantics, we now consider $v_{\emptyset}([b \cup c](p \rightarrow q)) = t$ where p and q are distinct propositional variables, and b and c are distinct atomic sequences. Then, we obtain two different interpretations:

1. $v_{\emptyset}([b \cup c](p \rightarrow q)) = t$ iff $v_{\emptyset}([b](p \rightarrow q))) = v_{\emptyset}([c](p \rightarrow q))) = t$ (by 1) iff $v_b(p \rightarrow q) = v_c(p \rightarrow q) = t$ iff $[v_b(p) = t$ implies $v_b(q) = t]$ and $[v_c(p) = t$ implies $v_c(q) = t]$.

2. $v_{\emptyset}([b \cup c](p \rightarrow q)) = t$ iff $v_{b \cup c}(p \rightarrow q) = t$ iff $v_{b \cup c}(p) = t$ implies $v_{b \cup c}(q) = t$ iff $v_b(p) = v_c(p) = t$ implies $v_b(q) = v_c(q) = t$ (by 2).

This situation, i.e., "(1) implies (2)" holds, but the converse does not hold, means that the valuations are not well defined, and hence the completeness theorem w.r.t. this semantics cannot be shown for ELS in the present setting. Since the inference rules for $\cdot^*$ are regarded as infinite versions of the inference rules for $\cup$, the same problem arises.

4.3 Relation to semilattice relevant logic

As mentioned in Section 1, the logics LS and FLS are regarded as a logic of monoids. The idea of formalizing monoids in a logic is not new. *Semilattice relevant logics*, which have semilattice-based semantics [14], were studied by many researchers. For more information on semilattice relevant logics, see e.g., [4, 9] and the references therein. In the following, we make a comparison between a semilattice relevant logic and LS.

A *label* is a finite set of positive integers. If a is a label and α is a formula, then the expression $a : \alpha$ is called a *labeled formula*. A label a is regarded as an element of a semilattice $\langle M, \cup, \emptyset \rangle$ with the identity $\emptyset$. Lower-case letters $a, b, ...$ are used for labels, Greek lower-case letters $\alpha, \beta, ...$ are used for labeled formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used for finite (possibly empty) multisets of labeled formulas. An expression of the form $\Gamma \Rightarrow \Delta$ is called a *labeled sequent*.

A sequent calculus $\mathrm{LR}^{\cup}$ for a semilattice relevant logic is then presented below [4, 9].

The initial sequents of $\mathrm{LR}^{\cup}$ are of the form:

$$a : \alpha \Rightarrow a : \alpha.$$

The structural inference rules of $\mathrm{LR}^{\cup}$ are of the form:

$$\frac{\Gamma \Rightarrow \Delta, a : \alpha \quad a : \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \ (\text{cut}^r)$$

$$\frac{\Gamma \Rightarrow \Delta}{a : \alpha, \Gamma \Rightarrow \Delta} \ (\text{we-left}^r) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, a : \alpha} \ (\text{we-right}^r)$$

$$\frac{a : \alpha, a : \alpha, \Gamma \Rightarrow \Delta}{a : \alpha, \Gamma \Rightarrow \Delta} \ (\text{co-left}^r) \quad \frac{\Gamma \Rightarrow \Delta, a : \alpha, a : \alpha}{\Gamma \Rightarrow \Delta, a : \alpha} \ (\text{co-right}^r).$$

The logical inference rules of $\mathrm{LR}^{\cup}$ are of the form:

$$\frac{\Gamma \Rightarrow \Delta, b : \alpha \quad a \cup b : \beta, \Sigma \Rightarrow \Pi}{a : (\alpha \rightarrow \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi} \ (\rightarrow\text{left}^r)$$

$$\frac{\{x\} : \alpha, \Gamma \Rightarrow \Delta, a \cup \{x\} : \beta}{\Gamma \Rightarrow \Delta, a : \alpha \rightarrow \beta} \ (\rightarrow\text{right}^r)$$

with the proviso: the positive integer x does not appear in the lower sequent,

$$\frac{a : \alpha, \Gamma \Rightarrow \Delta}{a : \alpha \wedge \beta, \Gamma \Rightarrow \Delta} \ (\wedge\text{left1}^r) \quad \frac{a : \beta, \Gamma \Rightarrow \Delta}{a : \alpha \wedge \beta, \Gamma \Rightarrow \Delta} \ (\wedge\text{left2}^r)$$

$$\frac{\Gamma \Rightarrow \Delta, a : \alpha \quad \Gamma \Rightarrow \Delta, a : \beta}{\Gamma \Rightarrow \Delta, a : \alpha \wedge \beta} \ (\wedge\text{right}^r) \quad \frac{a : \alpha, \Gamma \Rightarrow \Delta \quad a : \beta, \Gamma \Rightarrow \Delta}{a : \alpha \vee \beta, \Gamma \Rightarrow \Delta} \ (\vee\text{left}^r)$$

$$\frac{\Gamma \Rightarrow \Delta, a : \alpha}{\Gamma \Rightarrow \Delta, a : \alpha \vee \beta} \ (\vee\text{right1}^r) \quad \frac{\Gamma \Rightarrow \Delta, a : \beta}{\Gamma \Rightarrow \Delta, a : \alpha \vee \beta} \ (\vee\text{right2}^r).$$

A logic of monoids can be obtained from $\mathrm{LR}^{\cup}$ by replacing the "integer set labels" by "integer sequence labels" and replacing the union $\cup$ by the concatenation ;. The resulting logic is then called here a *monoid relevant logic*.

Remark that some characteristic "relevant" properties of $\mathrm{LR}^{\cup}$ are derived from the label conditions on the rules ($\to$left^r) and ($\to$left^r). We moreover consider to simplify the label conditions on these rules by

$$\frac{\Gamma \Rightarrow \Delta, a:\alpha \quad a:\beta, \Sigma \Rightarrow \Pi}{a:(\alpha{\to}\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\to\mathrm{left}^s) \quad \frac{a:\alpha, \Gamma \Rightarrow \Delta, a:\beta}{\Gamma \Rightarrow \Delta, a:\alpha{\to}\beta}\ (\to\mathrm{right}^s).$$

The resulting logic having ($\to$left^s) and ($\to$right^s) is regarded as a *monoid (non-relevant) logic* (ML). Now, we replace the expression $a:\alpha$ in ML by the expression $[a]\alpha$ in LS. Then, we can obtain a subsystem of LS.

In conclusion, LS and FLS are also important as a logic of monoids, since LS and FLS include a non-relevant and monoid version of the well-known semilattice relevant logic.

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References

[1] M. Bezem, T. Langholm and M. Walicki, *Completeness and decidability in sequence logic*, Proceedings of the 14-th International Conference on Logic for programming, artificial intelligence, and reasoning (LPAR 2007), N. Dershowitz and A. Voronkov (Eds.), Lecture Notes in Computer Science **4790** (2007), pp. 123–137.

[2] R.A. Bull, *Cut elimination for propositional dynamic logic without* *, Zeitscr. f. Math. Logik und Grundlagen d. Math., **38** (1992), pp. 85–100.

[3] P. Cégielski and D. Richard, *On arithmetical first-order theories allowing encoding and decoding of lists*, Theoretical Computer Science **22 (1-2)** (1999), pp. 55–75.

[4] S. Giambrone and A. Urquhart, *Proof theories for semilattice logics*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **33** (1987), pp. 433–439.

[5] E.A. Emerson, *Temporal and modal logic*, In: Handbook of Theoretical Computer Science, Formal Models and Semantics (B), Jan van Leeuwen (Ed.), Elsevier and MIT Press, 1990, pp. 995–1072.

[6] D. Harel, D. Kozen and J. Tiuryn, Dynamic logic (Foundations of Computing Series), The MIT Press, 2000.

[7] N. Kamide, *Dynamic non-commutative logic*, Journal of Logic, Language and Information **19 (1)** (2010), pp. 33–51.

[8] N. Kamide and K. Kaneiwa, *Extended full computation-tree logic with sequence modal operator: Representing hierarchical tree structures*, Proceedings of the 22nd Australasian Joint Conference on Artificial Intelligence (AI'09), Lecture Notes in Artificial Intelligence **5866** (2009), pp. 485–494.

[9] R. Kashima, *On semilattice relevant logics*, Mathematical Logic Quarterly **49(4)** (2003), pp. 401–414.

[10] T. Kutsia and B. Buchberger, *Predicate logic with sequence variables and sequence function symbols*, Proceedings of the 3rd International Conference on Mathematical Knowledge Management (MKM 2004), A. Asperti, G. Bancerek and A. Trybulec (Eds.), Lecture Notes in Computer Science **3119** (2004), pp. 205–219.

[11] J. Lambek, The mathematics of sentence structure, *The American Mathematical Monthly* 65, pp. 154-170, 1958.

[12] A.N. Prior, Past, present and future, Oxford: Clarendon Press, 1967.

[13] J.C. Reynolds, Separation logic: A logic for shared mutable data structures, Proceedings of the 17th Annual IEEE Symposium on Logic in Computer Science, 2002, pp. 55–74.

[14] A. Urquhart, *Semantics for relevant logics*, Journal of Symbolic Logic **37(1)** (1972), pp. 159–169.

[15] M. Walicki, M. Bezem and W. Szajnkenig, A strongly complete logic of dense time intervals, Proceedings of the Workshop on Logics for Resource-Bounded Agents, ESSLLI, Malaga, Spain, 2006.

[16] H. Wansing, *The logic of information structures*, Lecture Notes in Artificial Intelligence **681** (1993), 163 pages.

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