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Finite Simple Graphs and Their Associated Graph Lattices

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Abstract

In his 2005 dissertation, Antoine Vella explored combinatorical aspects of finite graphs utilizing a topological space whose open sets are intimately tied to the structure of the graph. In this paper, we go a step further and examine some aspects of the open set lattices induced by these topological spaces. In particular, we will characterize all lattices isomorphic to the open set lattices for finite simple graphs endowed with this topology, explore the structure of these lattices, and show that these lattices contain information necessary to reconstruct the graph and its complement in several ways.

1 Introduction

The content of this paper developed from Brian Frazier's initial exploration in [5]. Antoine Vella [12] explored the so-called *classical* topology on a finite graph $\mathcal{G} = (G, E)$; the open sets of this topology are those subsets of $G \cup E$ which contain only edges or are unions of edge-balls associated with vertices. The goal of this paper will be to establish a novel relationship between graphs and lattices using this topology. We utilize tools from graph theory, topology, and order theory and will assume the reader is for the most part conversant in these topics. There are many excellent texts available for those wishing greater information; we recommend Diestel [3] for graph theory and Munkres [8] for topology. Excellent general resources for order theory include Gratzer [7] and Davey and Priestley [2].

Before embarking on this project, it may be helpful to provide a summary of its main conclusions. The following paragraphs accomplish this, leaving precise definitions, technical details, context, and relevant concepts to be addressed in subsequent sections. The central idea is relatively straightforward: Given any finite graph $\mathcal{G} = (G, E)$, the lattice $\Omega(\mathcal{G})$ of open sets for the classical topology, partially ordered by subset inclusion, contains a wealth of information about the graph — encoded in an order-theoretic format.

The lattice $\Omega(\mathcal{G})$, when viewed in its entirety, is an intimidating structure; however, its structural complexity belies the fact that the lattice is built from very simple substructures. For example, the lattice $\Omega(\mathcal{G})$ is order-generated by its subposet of join-prime elements. This is not surprising since $\Omega(\mathcal{G})$ is a finite distributive lattice (see Gratzer [7] for example); however, it is also true that the subposet of join-prime elements is order-isomorphic to the (order-dual) of the incidence poset for the graph $\mathcal{G}$. (See Proposition 2.2, 2.4, and Corollary 2.5.) Consequently, the fact that the posets of join-prime and meet-prime elements of $\Omega(\mathcal{G})$ are order-isomorphic tells us that $\Omega(\mathcal{G})$ contains two subposets which can be used to reconstruct (a graph-isomorphic copy of) the graph $\mathcal{G}$. (See Corollary 2.8 and Lemma 2.9.)

Much of the complex structure presented in the lattice $\Omega(\mathcal{G})$ is contained in two Boolean sublattices which are (order-isomorphic to) the powersets of the edge and vertex sets of the graph $\mathcal{G}$. (See Definition 3.2 and Proposition 3.3). The subposet of $\Omega(\mathcal{G})$ which is *not* part of these Boolean sublattices (under mild restrictions on the graph $\mathcal{G}$) contains a *third* subposet of $\Omega(\mathcal{G})$ from which $\mathcal{G}$ can be reconstructed. (See Definition 3.8 and Theorem 3.10 in conjunction with Corollary 2.8.) There is even a subposet of $\Omega(\mathcal{G})$ which can be used to reconstruct (a graph-isomorphic copy of) the graph complement of $\mathcal{G}$. (See Theorem 2.10.)

Moreover, by focusing attention on those finite lattices order-generated by their join-prime elements, it is possible to characterize the class of finite lattices which are (order-isomorphic to) the lattice $\Omega(\mathcal{G})$ for some graph $\mathcal{G}$ — the requirement is simply that the join-prime elements form a graph poset. (See Definitions 2.1 and 2.6 along with Theorem 2.7.)

2 Graphs and Graph Lattices

For our purposes, a (simple) *graph* is an ordered pair $\mathcal{G} = (G, E)$ where G is a finite nonempty set whose elements are called *vertices* and E is a set of two-element subsets of G whose elements are called *edges*. Note that we do not consider edges to be directed, and we do not allow "loops" (one-element subsets of G) to be edges. Two vertices $u, v \in G$ are *adjacent* provided $\{u, v\} \in E$. Vertices cannot be self-adjacent; a vertex that is not adjacent to any other vertex is called an *isolated* vertex.

A vertex v is said to be *incident* to an edge e if $v \in e$. In this case, we say v is an *endvertex* (or simply "end") of e and say that e *joins* its end vertices. It is common to let uv denote an edge $\{u, v\}$. (Of course, it is understood that $uv = vu$ in this notation.)

Let $\mathcal{G} = (G, E)$ and $\mathcal{G}' = (G', E')$ be graphs. A mapping $f : (G \cup E) \longrightarrow (G' \cup E')$ is a *graph-homomorphism* provided the following conditions are met.

1. We have $f(G) \subseteq f(G')$ and $f(E) \subseteq f(E')$.
2. If $xy \in E$, then $f(x)f(y) \in E'$.

We point out that our definition of graph-homomorphism varies slightly from the standard (see Diestel [3] for example) in that it explicitly requires graph-homomorphisms to preserve edges. Readers familiar with graph-homomorphisms can easily see that this divergence from the norm is of no consequence; it does, however, make working with graph-homomorphisms in a topological context more convenient.

Let $\mathcal{G} = (G, E)$ and $\mathcal{G}' = (G', E')$ be graphs. A bijection $f : (G \cup E) \longrightarrow (G' \cup E')$ is a *graph-isomorphism* provided f is a graph-homomorphism with the property that $xy \in E$ if and only if $f(x)f(y) \in E'$. As is typical with isomorphisms, if two graphs are isomorphic, then they have the same structure (i.e. can be drawn to look identical to each other).

In a graph $\mathcal{G} = (G, E)$, the set of all edges incident to a vertex v is called the *edge neighborhood* of v and will be denoted by $E(v)$. The set $B(v) = E(v) \cup \{v\}$ is called the *edge-ball* of v. It is easy to see that the family

$$\mathbb{B}_G = \{\{e\} : e \in E\} \cup \{B(x) : x \in G\}$$

constitutes a basis for a topology on the set $G \cup E$. In graph-theory circles, this space is known as the *classical* topology for $\mathcal{G}$. It is worth noting that $U \subseteq G \cup E$ is open in the classical topology if and only if U satisfies one of the following conditions.

- We have $U \subseteq E$.

- If $x \in U \cap G$, then $B(x) \subseteq U$.

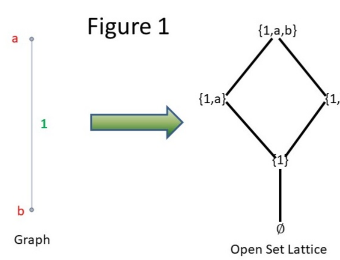
Vella [12] provides an extensive exploration of the combinatorical relationships between the graph $\mathcal{G}$ and its family open sets under the classical topology on $G \cup E$. Aside from notable exceptions provided by Thomassen and Vella [11] and Richter and Vella [9], these topological spaces have received little attention since.

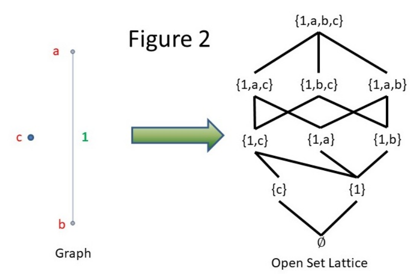
One reason these spaces have received little attention may stem from the fact that their structure is best understood in *order-theoretic* terms; therefore, we pause briefly to introduce some key ideas from the realm of order theory.

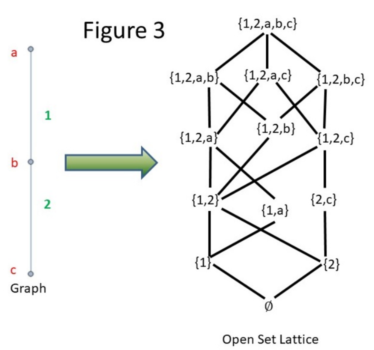
Suppose $\mathcal{P} = (P, \leq)$ is a poset (partially ordered set). We say $\mathcal{P}$ is *lower bounded* provided there exists some $\bot \in P$ such that $\bot \leq x$ for all $x \in P$. The notion of *upper bounded* poset is defined dually; and, of course, a poset is *bounded* provided it is both lower-bounded and upper-bounded. A poset $\mathcal{P}$ is called a *lattice* provided every pair of elements in P has a least upper bound and a greatest lower bound in $\mathcal{P}$. If $\mathcal{P}$ is a lattice, then it is common to let $x \vee y$ and $x \wedge y$ denote the least upper bound and greatest lower bound, respectively, for $x, y \in P$.

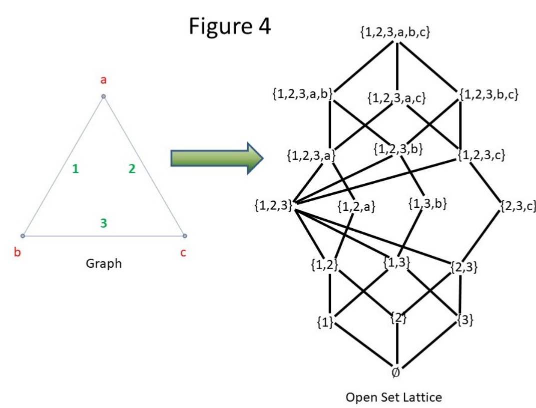
Let $\mathcal{G} = (G, E)$ be a graph. In the work to follow, we will let $\Omega(\mathcal{G})$ denote the poset of all subsets of $G \cup E$ which are open in the classical topology on $\mathcal{G}$, partially ordered by subset inclusion. For clarity, we will refer to the members of $\Omega(\mathcal{G})$ as the *graph-open* subsets of $\mathcal{G}$.

The poset $\Omega(\mathcal{G})$ clearly forms a bounded lattice. Indeed, the greatest lower bound and least upper bound of any family of open sets is simply its intersection and union, respectively. The following diagrams show the graph-open set lattices for several graphs.









If $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \preccurlyeq)$ are posets, then a function $f : P \longrightarrow Q$ is called an *order-homomorphism* provided $a \leq b$ implies $f(a) \preccurlyeq f(b)$. (It is common to say an order homomorphism is *strict* provided $a < b$ implies $f(a) \prec f(b)$.) A bijection $f : P \longrightarrow Q$ is an *order-isomorphism* provided $a \leq b$ if and only if $f(a) \preccurlyeq f(b)$. We should note that a bijection $f : P \longrightarrow Q$ is an order-isomorphism if and only if both f and its inverse function are order-homomorphisms.

Let $\mathcal{P} = (P, \leq)$ be any poset. A subset L of P is called a *lowerset* (or *order ideal*) of $\mathcal{P}$ provided $x \in L$ and $y \leq x$ together imply that $y \in L$. It is commonplace to let $\downarrow x = \{y \in P : y \leq x\}$ represent the *principal* lowerset *generated* by x. We will let $\texttt{Low}(\mathcal{P})$ denote the poset of a lowersets of $\mathcal{P}$, partially ordered by subset inclusion. It is easy to see that $\texttt{Low}(\mathcal{P})$ is a complete lattice (closed under arbitrary set-unions and set-intersections) in which every member is the union of a family of principal lowersets. Furthermore, it is a routine exercise to prove that $\mathcal{P}$ may be order-embedded in $\texttt{Low}(\mathcal{P})$ via the assignment $p \mapsto \downarrow p$. (See Davey and Priestley [2] for example.)

We say that X is an *upperset* (or *order filter*) of a poset $\mathcal{P} = (P, \leq)$ provided X is a lowerset in the order-dual of $\mathcal{P}$. It is common to let $\uparrow x$ denote the principal upperset of $\mathcal{P}$ generated by x.

In a poset $\mathcal{P} = (P, \leq)$, we say that $x \in P$ is *maximal* provided $\uparrow x = \{x\}$. Minimal elements in $\mathcal{P}$ are defined to be maximal elements in the order-dual of $\mathcal{P}$. We say that x *covers* $y \in P$ provided x and y are distinct, and $\uparrow y \cap \downarrow x = \{y, x\}$. For $x \in P$, we will let $\texttt{Cov}(x)$ denote the set of covers for x in $\mathcal{P}$. Note that $\texttt{Cov}(x)$ will be empty if x is a maximal member of $\mathcal{P}$.

In a poset $\mathcal{P} = (P, \leq)$, we say $A \subseteq P$ is an *antichain* provided the elements of A are pairwise incomparable. To be more precise, A is an antichain provided $x, y \in A$ and $x \leq y$ together imply $x = y$. We will say that a finite poset $\mathcal{P} = (P, \leq)$ is *bipartite* provided there exist disjoint nonempty antichains V_P and E_P such that $P = V_P \cup E_P$, and each member of E_P is covered by at least one member of V_P.

We caution that our definition of "bipartite poset" is somewhat different from the one commonly found in the literature. (See Erdös [4] for example.) However, the difference is primarily one of grouping in that we collect *all* maximal poset members into the antichain V_P. The antichain E_P may be a *proper* subset of the minimal poset members. If $\mathcal{P} = (V_P \cup E_P, \leq)$ is a bipartite poset, note that $\uparrow e$ contains at least two elements for every $e \in E_P$. If $x \in V_P$ is such that $\downarrow x = \{x\}$, then we will say that x is *isolated* in $\mathcal{P}$. The isolated members of $\mathcal{P}$ are, of course, precisely those members of V_P that are also minimal in $\mathcal{P}$.

Definition 2.1. We will say that a bipartite poset $\mathcal{P} = (V_P \cup E_P, \leq)$ is a *graph* poset provided the following conditions are met.

1. Every member of E_P is covered by exactly two members of V_P.

2. If $e, f \in E_P$ are distinct, then $\texttt{Cov}(e) \neq \texttt{Cov}(f)$.

If $\mathcal{G} = (G, E)$ is any graph, then there is a graph poset naturally associated with $\mathcal{G}$, namely the set $P_G = G \cup E$ endowed with the partial order $\sqsubseteq$ defined by $u \sqsubseteq v$ if and only if

- We have $u = v$ or

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- we have $u \in E$, $v \in G$ and $v \in u$.

Graph theorists commonly refer to the order-dual of the poset $\mathcal{P}_G = (P_G, \leq)$ as the *incidence* poset for the graph $\mathcal{G}$.

On the other hand, if $\mathcal{P} = (V_P \cup E_P, \leq)$ is any graph poset, then there is a natural way to associate a graph with $\mathcal{P}$. To begin, note that the conditions in Definition 2.1 guarantee there is a unique two-element subset of V_P associated with every member of E_P. With this in mind, let

$$\mathbb{E}_P = \{\texttt{Cov}(\epsilon) : \epsilon \in E_P\} \ .$$

We simply consider the structure $\mathcal{G}_P = (V_P, \mathbb{E}_P)$. (In other words, two distinct vertices x and y are adjacent if and only if $x \wedge y$ exists in $\mathcal{P}$.)

If $\mathcal{G} = (G, E)$, then it is not difficult to see that the graph $\mathcal{G}_{P_G} = (G, \mathbb{E}_{P_G})$ is graph-isomorphic to $\mathcal{G}$. Indeed, simply consider the mapping $f : (G \cup E) \longrightarrow (G \cup \mathbb{E}_{P_G})$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in G, \\ \{u, v\} & \text{if } x = \{u, v\} \in E \end{cases} .$$

On the other hand, if $\mathcal{P} = (V_P \cup E_P, \leq)$ is a graph poset, then it is also not difficult to see that the graph poset $\mathcal{P}_{G_P} = (V_P \cup \mathbb{E}_{G_P}, \sqsubseteq)$ is order-isomorphic to $\mathcal{P}$. Indeed, simply consider the mapping $g : (V_P \cup E_P) \longrightarrow (V_P \cup \mathbb{E}_{G_P})$ defined by

$$g(x) = \begin{cases} x & \text{if } x \in V_P, \\ \uparrow_{P_G} x & \text{if } x \in E_P \end{cases} .$$

It is worth noting that, aside from adjusting notation to fit the codomain structure, the functions f and g defined above are the same. We may consider graphs and finite graph posets to be essentially interchangeable structures.

The following proposition provides a simple but convenient alternative way to view the open set lattice for any graph.

Proposition 2.2. *If $\mathcal{G} = (G, E)$ is any graph, then $\Omega(\mathcal{G}) = \texttt{Low}(\mathcal{P}_G)$.*

Proof. Suppose that $U \in \Omega(\mathcal{G})$. If $U \subseteq E$, then U is an antichain (of minimal elements) in $\mathcal{P}_G$ and is therefore a lowerset of $\mathcal{P}_G$. Suppose $x \in U \cap G$ and suppose $y \in P_G$ is such that $y \leq x$. It follows that $y = x$ or that y is an edge in $B(x)$. In either case, $y \in U$; and we may conclude that U is a lowerset of $\mathcal{P}_G$.

Suppose that $U \in \texttt{Low}(\mathcal{P}_G)$. If $x \in U$, then we know $\downarrow x \subseteq U$. Since $\downarrow x = B(x)$ by construction, we know $U \in \Omega(\mathcal{G})$. $\square$

A member a of a lower-bounded poset $\mathcal{P} = (P, \leq)$ is called an *atom* of $\mathcal{P}$ provided $\downarrow p$ contains exactly two elements. We say a member c of an upper-bounded poset $\mathcal{P}$ is a *co-atom* (or dual atom) of $\mathcal{P}$ provided $\uparrow a$ contains exactly two elements. A lower-bounded poset $\mathcal{P}$ is *atomic* provided $\downarrow p$ contains an atom for every $p \in P - \{\bot\}$. Co-atomic posets are defined dually.

If $\mathcal{G} = (G, E)$ is a graph, then its open set lattice $\Omega(\mathcal{G})$ is finite and contains at least two members; therefore $\Omega(\mathcal{G})$ is both atomic and co-atomic. There is a particularly simple characterization for both the atoms and the co-atoms of $\mathcal{G}$.

Proposition 2.3. *Let $\mathcal{G} = (G, E)$ be a graph.*

1. *The atoms of $\Omega(\mathcal{G})$ are the singleton edges and the singleton isolated vertices.*
2. *The co-atoms of $\Omega(\mathcal{G})$ have the form $E \cup (G - \{v\})$, where v is any member of G.*

Proof. Singleton edges and singleton isolated vertices are the only graph-open sets that can cover the empty set, which is the smallest member of $\Omega(\mathcal{G})$; hence, Claim (1) is trivial.

Now, let $v \in G$ and consider the set $C(v) = E \cup (G - \{v\})$. This set is clearly a member of $\Omega(\mathcal{G})$ and is covered by the set $G \cup E$ which is the largest element of the lattice. Hence, $C(v)$ is a co-atom.

On the other hand, suppose that θ is a co-atom of $\Omega(\mathcal{G})$. Suppose that x, y are distinct vertices missing from θ and consider the graph-open set $\chi = \theta \cup B(x)$. It is clear that $y \notin \chi$. Consequently, we have $\theta \subset \chi \subset G \cup E$ — contrary to assumption. We must conclude that θ is missing *at most* one vertex. Now suppose that θ is missing an edge, and let $e = xy$ be one edge missing from θ. Since θ is graph-open, we know that $x \notin \theta$ and $y \notin \theta$. We have shown this situation to be impossible; therefore we must conclude that θ is missing exactly one vertex. Thus, we know that $\theta = E \cup (G - \{v\})$ for some $v \in G$.

□

An element j of a lattice $\mathcal{L} = (L, \leq)$ is *join-prime* provided whenever $F \subseteq L$ is finite and $j \leq \bigvee F$, then $j \leq x$ for some $x \in F$. It is a routine exercise to prove a poset has a least element $\bot$ if and only if $\bigvee \emptyset$ exists in the poset. (Indeed, one can show $\bigvee \emptyset = \bot$ when either is assumed to exist.) With this in mind, the least element of a lattice (if it exists) cannot be join-prime. We will let $\mathtt{JP}(\mathcal{L})$ denote the subposet of join-prime elements for $\mathcal{L}$. Note that any atom of a lower-bounded lattice is also join-prime in that lattice. In a finite lattice, an element j is join-prime if and only if $\downarrow j - \{j\}$ contains a unique maximal element; in this sense, join-prime elements generalize the notion of "atom" in finite lattices. The following result is easily proven but will play a crucial role in much of the work to follow.

Proposition 2.4. *If $\mathcal{P} = (P, \leq)$ is a finite poset, then the join-prime members of $\mathtt{Low}(\mathcal{P})$ are precisely the principal lowersets of $\mathcal{P}$.*

If $\mathcal{G} = (G, E)$ is any graph, then the singleton edge sets and edge-balls of $\mathcal{G}$ are precisely the principal lowersets of $\mathtt{Low}(\mathcal{P}_G)$. With this in mind, we have the following result.

Corollary 2.5. *If $\mathcal{G} = (G, E)$ is a graph, then the join-prime members of $\Omega(\mathcal{G})$ are precisely the members of $\mathbb{B}_G$.*

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A lattice $\mathcal{L} = (L, \leq)$ is *distributive* provided $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$. It is worth noting that

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \iff x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \ .$$

Join-prime elements play a key role in understanding the structure of distributive lattices. (See Davey and Priestley [2] Chapter 8 for historically important examples.)

In the parlance of order-theory, a subposet X of a finite lattice $\mathcal{L} = (L, \leq)$ is *join-dense* in that lattice provided every member of the lattice is the join of a finite (possibly empty) subset of X. It is well-known that a finite lattice $\mathcal{L}$ is distributive if and only if $\mathtt{JP}(\mathcal{L})$ is join-dense in $\mathcal{L}$. (See Gratzer [7] pages 102 and 112 for example.) Every finite, distributive lattice $\mathcal{L}$ is order-isomorphic to the lowerset lattice of its poset of join-prime elements. The order isomorphism is provided by the function $f : L \longrightarrow \mathtt{JP}(\mathcal{L})$ defined by

$$f(a) = \downarrow_L a \cap \mathtt{JP}(\mathcal{L}) \ .$$

The proof is straightforward; see Gratzer [7] or Davey and Priestley [2] page 171 for details.

Definition 2.6. We will say that a finite lattice $\mathcal{L} = (L, \leq)$ is a *graph lattice* provided the following conditions are met.

1. The poset $\mathtt{JP}(\mathcal{L})$ is a graph poset.
2. The poset $\mathtt{JP}(\mathcal{L})$ is join-dense in $\mathcal{L}$.

Let $\mathcal{G} = (G, E)$ be any graph and let $\mathcal{P}_G = (G \cup E, \sqsubseteq)$ be its graph poset. We know that $\mathcal{P}_G$ is order-isomorphic to $\mathtt{JP}(\mathtt{Low}(\mathcal{P}_G))$; hence, $\mathtt{Low}(\mathcal{P}_G))$ is a graph lattice. Consequently, Proposition 2.2 tells us that $\Omega(\mathcal{G})$ is a graph lattice for any graph $\mathcal{G}$. It should come as no surprise that every graph lattice arises in this fashion.

Theorem 2.7. *Suppose $\mathcal{L} = (L, \leq)$ is a graph lattice. There exists a graph $\mathcal{G}_L$ such that $\Omega(\mathcal{G}_L)$ is order-isomorphic to $\mathcal{L}$.*

Proof. We know that $\mathcal{P}_L = (\mathtt{JP}(\mathcal{L}), \leq)$ is a graph poset. Let $V_L = \{j \in \mathtt{JP}(\mathcal{L}) : \uparrow j = \{j\}\}$ and let $E_L = \mathtt{JP}(\mathcal{L}) - V_L$. For convenience, let $\mathtt{Cov}(e)$ denote the set of covers for each $e \in E_L$. Consider the graph $\mathcal{G}_L = (V_L, \mathbb{E}_{P_L})$, and consider the mapping $f : \mathtt{JP}(\mathcal{L}) \longrightarrow \mathbb{B}_{G_L}$ defined by

$$f(x) = \begin{cases} B(x) & \text{if } x \in V_L, \\ \{\mathtt{Cov}(x)\} & \text{if } x \in E_L \end{cases} \ .$$

Since we do not allow multiple edges between vertices, it is easy to see that f is a bijection. Now, if $x, y \in V_L$ or $x, y \in E_L$, then it is clear that

$$x \leq y \iff x = y \iff f(x) = f(y) \ .$$

On the other hand, if $x \in V_L$ and $y \in E_L$, then it is clear that

$$y < x \iff x \in \texttt{Cov}(y) \iff \{\texttt{Cov}(y)\} \in B(x) \iff f(y) \subset f(x) \; .$$

We may conclude that the graph posets $(\mathbb{B}_{G_L}, \subseteq)$ and $\mathcal{P}_L$ are order-isomorphic. In light of this fact, it is easy to show that $\mathcal{L}$ is order-isomorphic to $\Omega(\mathcal{G}_L)$. Indeed, the order-isomorphism is accomplished via the mapping $\varphi : \mathcal{L} \longrightarrow \Omega(\mathcal{G}_L)$ defined by

$$\varphi(x) = \bigcup \{ f(j) : j \in \texttt{JP}(\mathcal{L}) \cap \downarrow x \} \; .$$

□

If $\mathcal{L}$ is a graph lattice, then we will call the graph $\mathcal{G}_L$ the graph *induced* by $\mathcal{L}$. Note that there is a bijection between the edges and isolated vertices of the induced graph $\mathcal{G}_L$ and the atoms of the lattice $\mathcal{L}$. Figures 5 and 6 illustrate the process of passing from a graph lattice $\mathcal{L}$ to the open set lattice for the induced graph $\mathcal{G}_L$.

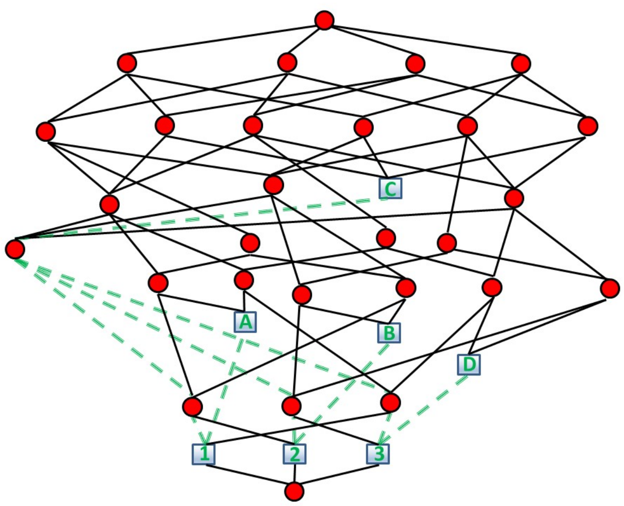

Figure 5 — A Graph Lattice $\mathcal{L}$

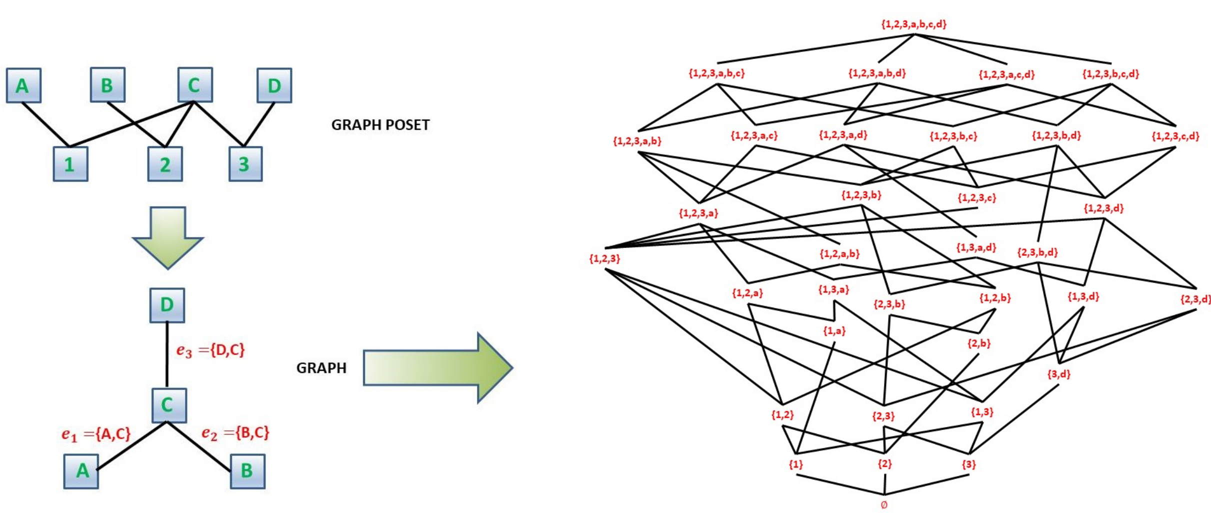

Figure 6 — Induced Graph $\mathcal{G}_L$ and Its Open Set Lattice $\Omega(\mathcal{G}_L)$

Corollary 2.8. *If $\mathcal{G} = (G, E)$ is a graph, then the induced graph $\mathcal{G}_{\Omega(\mathcal{G})}$ is graph-isomorphic to $\mathcal{G}$.*

Proof. Since the maximal members of $\mathtt{JP}(\Omega(\mathcal{G}))$ are precisely the edgeballs of $\mathcal{G}$, we know $V_{\Omega(\mathcal{G})} = \{B(x) : x \in G\}$. We also know by construction that

$$\mathbb{E}_{P_{\Omega(\mathcal{G})}} = \{\mathtt{Cov}(\{e\}) : \{e\} \in \mathtt{JP}(\Omega(\mathcal{G})) - V_{\Omega(\mathcal{G})}\} \ .$$

Of course, $\{e\} \in \mathtt{JP}(\Omega(\mathcal{G})) - V_{\Omega(\mathcal{G})}$ if and only if there exist $x, y \in G$ such that $e = \{x, y\}$; hence, we know $\mathtt{Cov}(\{e\}) = \{B(x), B(y)\}$. It follows that $B(x)$ is adjacent to $B(y)$ if and only if x is adjacent to y. In light of the previous discussion, the mapping $f : G \cup E \longrightarrow V_{\Omega(\mathcal{G})} \cup \mathbb{E}_{P_{\Omega(\mathcal{G})}}$ defined by

$$f(u) = \begin{cases} B(u) & \text{if } u \in G, \\ \mathtt{Cov}(\{u\}) & \text{if } u \in E \end{cases}$$

is the graph-isomorphism we seek. □

An element p of a lattice $\mathcal{L}$ is *meet-prime* provided it is join-prime in the order-dual of $\mathcal{L}$. In other words, p is meet-prime whenever $F \subseteq L$ is finite and $\bigwedge F \leq p$, then there exist $x \in F$ such that $x \leq p$. It is a routine exercise to prove a poset has a greatest element $\top$ if and only if $\bigwedge \emptyset$ exists in the poset. (Indeed, one can show $\bigwedge \emptyset = \top$ when either is assumed to exist.) With this in mind, the greatest element of a lattice (if it exists) cannot be meet-prime. Note that the concept of meet-prime element is order-dual to that of join-prime element. We will let $\mathtt{MP}(\mathcal{L})$ denote the subposet of meet-prime elements for the lattice $\mathcal{L}$.

In Theorem 2.7, we demonstrated that the subposet of join-prime elements for any graph lattice $\mathcal{L}$ induces a graph $\mathcal{G}_L$ whose open set lattice is in turn order-isomorphic to $\mathcal{L}$. We now introduce a special case of a result appearing in Snodgrass and Tsinakis [10] which proves that $\mathtt{MP}(\mathcal{L})$ can also be used to create the graph $\mathcal{G}_L$. We provide its proof for completeness, noting that we have merely adapted arguments appearing in the aforementioned paper.

Lemma 2.9. *If $\mathcal{L}$ is any finite lattice, then $\mathtt{JP}(\mathcal{L})$ is order-isomorphic to $\mathtt{MP}(\mathcal{L})$. The order-isomorphism is accomplished via the mappings $\phi : \mathtt{MP}(\mathcal{L}) \longrightarrow \mathtt{JP}(\mathcal{L})$ and $\zeta : \mathtt{JP}(\mathcal{L}) \longrightarrow \mathtt{MP}(\mathcal{L})$ defined by*

$$\phi(m) = \bigwedge\{x \in L : x \not\leq m\} \quad \textit{and} \quad \zeta(j) = \bigvee\{y \in L : j \not\leq y\} \ .$$

Proof. Let $a, b \in L$. We say that the ordered pair $(a, b) \in L \times L$ *splits* the lattice $\mathcal{L}$ provided $\downarrow a \cap \uparrow b = \emptyset$ and $\downarrow a \cup \uparrow b = L$. If (a, b) splits $\mathcal{L}$, then it is easy to see that a is meet-prime and b is join-prime in $\mathcal{L}$. Indeed, to see why a is meet-prime, suppose $x, y \in L$ are such that $x \wedge y \leq a$. If it were the case that $\{x, y\} \subseteq \uparrow b$, then we would know $x \wedge y \in \uparrow b$ as well. However, this is impossible, since $\uparrow a \cap \uparrow b = \emptyset$. Thus, we must conclude that $x \leq a$ or $y \leq a$. The proof that b is join-prime is similar.

Suppose $a \in \mathtt{MP}(\mathcal{L})$, and consider the element $\phi(a)$. Since a is meet-prime, it follows that $\phi(a) \not\leq a$; and, for each $x \in L$, we must have $x \not\leq a$ if and only if $\phi(a) \leq x$. In light of this observation, the pair $(a, \phi(a))$ splits $\mathcal{L}$; and we must conclude that $\phi(a)$ is join-prime in $\mathcal{L}$. If we instead assume that b is join-prime in $\mathcal{L}$, then similar reasoning demonstrates that the pair $(\zeta(b), b)$ splits $\mathcal{L}$; and we must conclude that $\zeta(b)$ is meet-prime in $\mathcal{L}$.

It follows that ϕ does indeed map $\texttt{MP}(\mathcal{L})$ to $\texttt{JP}(\mathcal{L})$ and ζ does indeed map $\texttt{JP}(\mathcal{L})$ to $\texttt{MP}(\mathcal{L})$. Furthermore, if $a \in \texttt{MP}(\mathcal{L})$, the fact that $(a, \phi(a))$ splits $\mathcal{L}$ tells us

$$\zeta(\phi(a)) = \bigvee\{y \in L : \phi(a) \not\leq y\} = \bigvee\{y \in L : y \leq a\} = a \ .$$

Likewise, if $b \in \texttt{JP}(\mathcal{L})$, then $\phi(\zeta(b)) = b$; and we may conclude that ϕ and ζ are mutually inverse mappings.

Finally, suppose that $u, v \in \texttt{MP}(\mathcal{L})$ and suppose that $u \leq v$. If $x \not\leq v$, then it is certainly the case that $x \not\leq u$. Consequently, we know that $\phi(u) \leq \phi(v)$; and we may conclude that ϕ is an order-homomorphism. The proof that ζ is also an order-homomorphism is similar. $\square$

Lemma 2.9 tells us that if $\mathcal{L} = (L, \leq)$ is any graph lattice, then $\texttt{MP}(\mathcal{L})$ is a graph poset which is order isomorphic to the graph poset for $\mathcal{G}_L$. Consequently, we may also construct (a graph-isomorphic copy of) the graph $\mathcal{G}_L$ from $\texttt{MP}(\mathcal{L})$.

It is worth noting that the graph lattice $\Omega(\mathcal{G})$ for a graph $\mathcal{G} = (G, E)$ also contains information sufficient to construct the *graph complement* of $\mathcal{G}$. The graph complement $\mathcal{G}^c = (G, E^c)$ is defined by $\{x, y\} \in E^c$ if and only if $x, y \in G$ and $\{x, y\} \notin E$. We know $\{x, y\} \notin E$ if and only if $B(x) \cap B(y) = \emptyset$. For each $x, y \in G$, let $B(x, y) = B(x) \cup B(y)$ and consider the following sets

$$V_G = \{B(x) : x \in G\} \ , \qquad E^c_G = \{B(x, y) : B(x) \cap B(y) = \emptyset\} \ , \qquad P^c = V_G \cup E^c_G$$

$$\mathbb{E}^c_G = \{\{B(x), B(y)\} : B(x, y) \in E^c_G\} \ .$$

Theorem 2.10. *If $\mathcal{G} = (G, E)$ is a graph, then the pair $\mathcal{G}_{P^c} = (V_G, \mathbb{E}^c_G)$ is (graph isomorphic to) the graph complement of $\mathcal{G}$. Furthermore, $(\mathcal{P}^c, \subseteq)$ is the incidence poset for $\mathcal{G}_{P^c} = (V_G, \mathbb{E}^c_G)$.*

Proof. To see that $\mathcal{G}_{P^c}$ serves as the graph complement for $\mathcal{G}$, suppose $B(x), B(y) \in V_P$ and observe that

$$\{B(x), B(y)\} \in \mathbb{E}^c_G \iff B(x) \cap B(y) = \emptyset \iff \{x, y\} \notin E \ .$$

It is clear that the elements $B(x, y)$ are pairwise incomparable; hence, P^c is the union of two disjoint antichains since a graph must contain at least two vertices. It is possible that E^c_G is empty — this will occur if and only if $\mathcal{G}$ is a *complete* graph; that is, if and only if $E = \{\{x, y\} : x, y \in G\}$.

Suppose that $U \in E^c_G$. We know $U = B(x, y)$ for some $x, y \in G$; and it is clear that $B(x) \subset U$ and $B(y) \subset U$. Since every member of V_G is join-prime in $\Omega(\mathcal{G})$, it also follows that $B(z) \subset U$ implies $B(z) = B(x)$ or $B(z) = B(y)$. Consequently, U covers exactly two members of V_G. Suppose $V \in E^c_G$ is distinct from U. There exist $a, b \in G$ such that

$V = B(a, b)$, and we may assume $x \neq a$. Of course, this implies $\downarrow U \cap V_G \neq \downarrow V \cap V_G$; and we may conclude that $\mathcal{P}^c = (P^c, \subseteq)$ is the *order-dual* of a graph poset whenever E_G^c is nonempty.

□

Figures 7 and 8 together demonstrate how the graph complement of a graph $\mathcal{G}$ may be constructed from members of $\Omega(\mathcal{G})$ using Theorem 2.10.

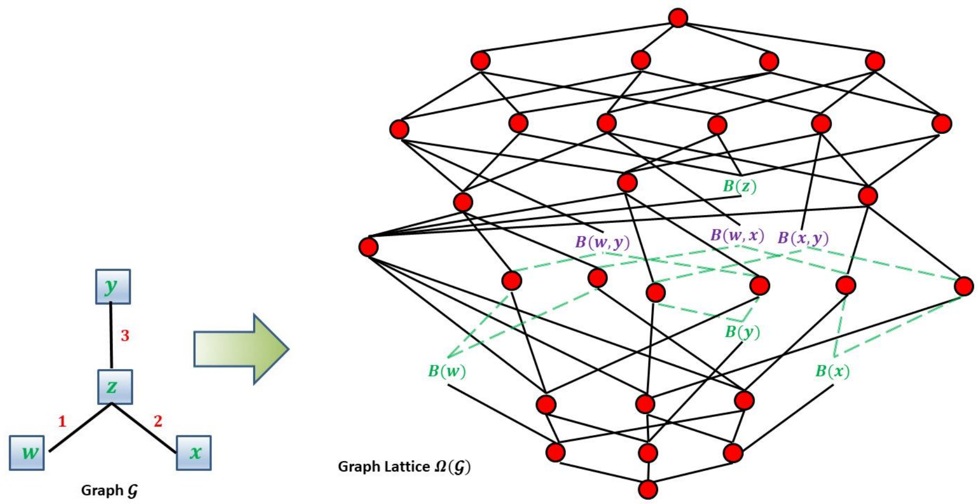

Figure 7 — A Graph $\mathcal{G}$ and Its Associated Graph Lattice $\Omega(\mathcal{G})$

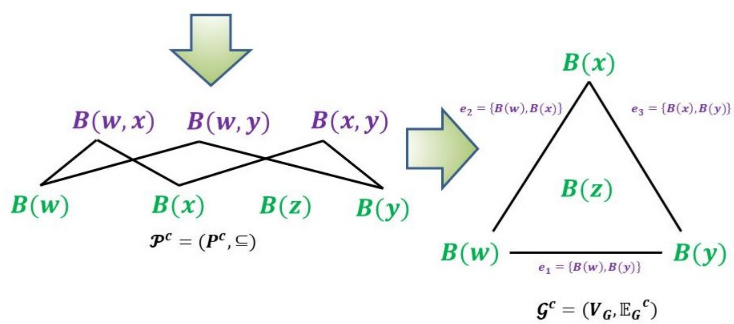

Figure 8 — Using a Subposet of $\Omega(\mathcal{G})$ to Construct the Graph Compelement of $\mathcal{G}$

3 The Structure of Graph Lattices

In this section, we explore some of the structural properties of graph lattices and tie these properties to their corresponding graphs. In light of the previous section, we can adopt the perspective that a graph lattice is (isomorphic to) the lowerset lattice of some graph poset, or we can adopt the perspective that a graph lattice is (isomorphic to) the open set lattice for a graph endowed with the classical topology. We will move freely between these perspectives in the work to follow.

If $\mathcal{P} = (V_P \cup E_P, \leq)$ is any graph poset, then we know that the join-prime members of $\texttt{Low}(\mathcal{P})$ are simply the principal lowersets of $\mathcal{P}$. Let us consider what Lemma 2.9 tells us about meet-prime elements in $\texttt{Low}(\mathcal{P})$. Suppose that U is meet-prime in $\texttt{Low}(\mathcal{P})$. This means that $U = \zeta(\downarrow p)$ for some $p \in V_P \cup E_P$ since the join-prime members of $\texttt{Low}(\mathcal{P})$ are precisely the principal lowersets of $\mathcal{P}$.

First, suppose that $P = \zeta(\downarrow x)$ for some $x \in E_P$. Of course, we know $\downarrow x = \{x\}$; hence we also know

$$U = \bigcup\{I \in \texttt{Low}(\mathcal{P}) : x \notin I\} = P - (x \cup \texttt{Cov}(x)) \ .$$

For simplicity, let $\epsilon(\texttt{Cov}(x)) = P - (\{x\} \cup \texttt{Cov}(x))$. Now, suppose that $U = \zeta(\downarrow y)$ for some $y \in V_P$. This tells us that

$$U = \bigcup\{I \in \texttt{Low}(\mathcal{P}) :\downarrow y \not\subseteq I\} = P - \{y\} \ .$$

We have now proven the following result.

Theorem 3.1. *Let $\mathcal{P}(V_P \cup E_P, \leq)$ be a graph poset. A member U of $\texttt{Low}(\mathcal{P})$ is meet-prime if and only if U is a co-atom of $\texttt{Low}(\mathcal{P})$ or $U = \epsilon(\texttt{Cov}(x))$ for some $x \in E_P$.*

An element x of a bounded lattice $\mathcal{L}$ is *complemented* provided there exist $y \in L$ such that $x \wedge y = \bot$ and $x \vee y = \top$, where $\bot$ and $\top$ denote the least and greatest elements, respectively, for $\mathcal{L}$. Complements in distributive lattices are necessarily unique. A bounded, distributive lattice in which every element has a complement is called a *Boolean* lattice. Some authors require Boolean lattices to contain at least two elements; we shall not use that requirement. Boolean lattices comprise one of the most important classes of lattices; we recommend Givant [6] as an excellent resource on this topic.

Of course, every *finite* Boolean lattice containing at least two elements is atomic. It is well-known that every finite Boolean lattice is order-isomorphic to the powerset lattice of its set of atoms, partially ordered by subset inclusion. (This includes the one-element Boolean lattice as well, since the powerset of the empty set contains exactly one element.)

Definition 3.2. Let $\mathcal{G} = (G, E)$ be a graph. In the work to follow, we will let $B_\bot$ represent the powerset of E, and we will let $B_\top = \{E \cup X : X \subseteq G\}$.

Let $\mathcal{L} = (L \leq)$ be a lattice and let $I \subseteq L$ be nonempty. Recall that $I \in \texttt{Low}(\mathcal{L})$ is an *ideal* of $\mathcal{L}$ provided I contains an upper bound for each of its finite (possibly empty) subsets. A subset F of L is a *filter* of L provided F is an ideal in the order dual of $\mathcal{L}$.

Proposition 3.3. *If $\mathcal{G} = (G, E)$ is a graph, then the following statements are true.*

1. *We have $B_\bot \cap B_\top = \{E\}$.*

2. *The posets $(B_\top, \subseteq)$ and $(B_\bot, \subseteq)$ are Boolean lattices.*

3. *The sublattice $(B_\bot, \subseteq)$ is an ideal of $\Omega(\mathcal{G})$.*

4. The sublattice $(B_\top, \subseteq)$ is a filter of $\Omega(\mathcal{G})$.

Proof. Claim (1) follows from Definition 3.2. It is a well-known fact that the powerset of any set is a Boolean lattice — see Givant and Halmos [6]. By construction, $(B_\perp, \subseteq)$ is the powerset of E; and $(B_\top, \subseteq)$ is order-isomorphic to the powerset of G. It is worth noting that $(B_\perp, \subseteq)$ is atomic if and only if E is nonempty; in this case, the atoms of $(B_\perp, \subseteq)$ are the singleton edge sets. Since G is assumed nonempty, the poset $(B_\top, \subseteq)$ is always atomic, and the atoms of $(B_\top, \subseteq)$ are all sets of the form $E \cup \{x\}$ such that $x \in G$.

It is easy to see that $(B_\perp, \subseteq)$ is an ideal of $\Omega(\mathcal{G})$. Since $(B_\perp, \subseteq)$ is a sublattice, we need only prove it is a lowerset of $\Omega(\mathcal{G})$. To this end, suppose that $\chi \in \Omega(\mathcal{G})$ is such that $\chi \subseteq \alpha$ for some $\alpha \in B_\perp$. Of course, this tells us that χ is an edge-only set and is therefore a member of $B_\perp$ by construction.

It is also easy to see that $(B_\top, \subseteq)$ is a filter of $\Omega(\mathcal{G})$. Again, since $(B_\top, \subseteq)$ is a sublattice, we need only prove it is an *upperset* of $\Omega(\mathcal{G})$. To this end, suppose that $\chi \in \Omega(\mathcal{G})$ is such that $\beta \subseteq \chi$ for some $\beta \in B_\top$. By construction, we know that $E \subseteq \beta$, so we know that $\chi = E \cup X$ for some $X \subseteq G$. Therefore, $\chi \in B_\top$ by construction. □

For ease of reading and in a common abuse of notation, we will usually identify the posets $(B_\perp, \subseteq)$ and $(B_\top, \subseteq)$ with their underlying sets.

Let $\mathcal{G} = (G, E)$ be a graph. We will call the lattice $B_\perp \cup B_\top$ the *Boolean cone* of $\Omega(\mathcal{G})$. We will refer to the poset $\texttt{Sus}(\Omega(\mathcal{G})) = \Omega(\mathcal{G}) - (B_\perp \cup B_\top)$ (partially ordered by subset inclusion) as the collection of *suspended* elements for $\Omega(\mathcal{G})$. A graph with empty edge set will have no suspended elements; indeed, the open set lattice of such a graph is simply the powerset of its vertices.

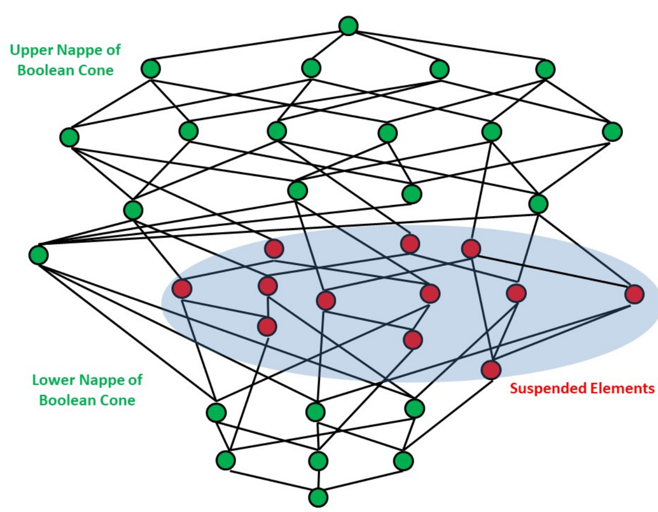

Figure 9 Anatomy of a Graph Lattice

In light of Theorem 3.1, every meet-prime element of $\Omega(\mathcal{G})$ that is not a co-atom must be a suspended element. Now, an element $U \in \Omega(\mathcal{G})$ that is not a co-atom will be meet-prime if and only if there exist $x, y \in G$ such that $U = (G - \{x, y\}) \cup (E - \{xy\}) = \epsilon(\texttt{Cov}(xy))$. It follows at once that the sets $\epsilon(\texttt{Cov}(xy))$ such that $xy \in E$ are maximal suspended elements in $\Omega(\mathcal{G})$.

Proposition 3.4. *Let $\mathcal{G} = (G, E)$ be a graph and suppose $U \in \Omega(\mathcal{G})$. The set U is maximal in the poset $\texttt{Sus}(\Omega(\mathcal{G}))$ if and only if $U = \epsilon(\texttt{Cov}(xy))$ for some $xy \in E$.*

Proof. Suppose that $U \in \Omega(\mathcal{G})$ is a maximal suspended element. If there exist $e, f \in E - U$, then $V = \{e\} \cup U$ would be a suspended element properly containing U — contrary to assumption. Hence, we must conclude that $E - U = \{e\}$. If we let $e = uv$, then we must conclude that $u, v \notin U$. It follows that $U = (E - \{e\}) \cup (G - X)$ for some X that contains u and v.

Suppose $y \in X - \{u, v\}$. Since $e \in B(u) \cap B(v)$, we know that $e \notin B(y)$. Therefore $\{y\} \cup U$ is a suspended member of $\Omega(\mathcal{G})$ properly containing U — contrary to assumption. □

Characterizing the minimal suspended elements requires a bit more care. Let $\mathcal{G} = (G, E)$ be a graph, and let $x \in G$. We will say that x is a *center* for $\mathcal{G}$ provided $E(x) = E$. If a graph contains no edges, then every vertex serves as a center for the graph. The graph $\mathcal{K}_2 = (\{x, y\}, \{xy\})$ appearing in Figure 1 is the only graph with nonempty edge set that contains more than one center. Any graph with nonempty edge set that is not $\mathcal{K}_2$ can contain at most one center since any edge can be incident to exactly two vertices. Graphs that contain a center are sometimes called *stars*. (See Diestel [3] for example.) Figures 1,3, and 6 present graphs that contain a center.

Proposition 3.5. *Suppose $\mathcal{L} = (L, \leq)$ is a graph lattice, and suppose $\mathcal{G}_L = (V_L, \mathbb{E}_{P_L})$ is the graph associated with $\mathcal{L}$ described in Theorem 2.7. The Boolean lattice $B_\top$ contains members of $\texttt{JP}(\mathcal{L})$ if and only if one of the following statements is true.*

1. *The graph $\mathcal{G}_L$ has empty edge set.*

2. *The graph $\mathcal{G}_L$ is a star.*

Proof. If $\mathcal{G}_L$ has empty edge set, then $\Omega(\mathcal{G}_L)$ is simply the powerset of the vertex set, and $\Omega(\mathcal{G}_L) = B_\top$. If $\mathcal{G}_L$ is a star, then there is a vertex x that serves as a center for $\mathcal{G}_L$. The edge-ball $B(x)$ must contain the edge set for $\mathcal{G}_L$ and therefore corresponds to a join-prime member of $\mathcal{L}$ contained in $B_\top$.

Conversely, suppose that there exist join-prime elements in the set $B_\top$. If $B_\perp = \{\perp\}$, then every join-prime member of $\mathcal{L}$ must correspond to a vertex in $\mathcal{G}_L$; and we must conclude $\mathcal{G}_L$ has empty edge set. Suppose that $\{\perp\} \subset B_\perp$, and let $x \in \texttt{JP}(\mathcal{L}) \cap B_\top$. The atoms of $B_\perp$ correspond to the edges of the graph $\mathcal{G}_L$; hence we know that x does not correspond to a subset of $\mathbb{E}_{P_L}$. This tells us that x corresponds to $B(u)$ for some vertex $u \in V_L$. Since $B_\perp \subseteq \downarrow x$, we must conclude $B(u)$ is contains the edge set for $\mathcal{G}_L$. Hence, we know that $\mathcal{G}_L$ is a star. □

Proposition 3.6. *For a graph $\mathcal{G} = (G, E)$ with nonempty edge set, the following claims are equivalent for a vertex x.*

1. *The vertex x is not a center for $\mathcal{G}$.*

2. *The edge-ball $B(x)$ is a minimal suspended element in $\Omega(\mathcal{G})$.*

Proof. To prove that Claim (1) implies Claim (2), suppose $x \in G$ is not a center. Since $x \in B(x)$, we know $B(x) \notin B_\perp$. Since $E(x) \neq E$, we also know that $B(x) \notin B_\top$. Consequently, we may conclude that $B(x)$ is a suspended element. Any proper open subset of $B(x)$ contains only edges; hence, $B(x)$ must be a minimal suspended element.

To prove that Claim (2) implies Claim (1), suppose $B(x)$ is a minimal suspended element. Our assumption implies $B(x) \notin B_\top$. Hence we know that $E(x) \neq E$; and we must conclude that x is not a center. □

In light of the previous result, for any graph $\mathcal{G}$ with nonempty edge set, the minimal suspended elements of $\Omega(\mathcal{G})$ are *precisely* those edge-balls that are not generated by a center of the graph. (Recall that any member of $\texttt{Sus}(\Omega(\mathcal{G}))$ must contain an edge-ball; hence a suspended element which is not itself an edge-ball cannot be minimal.)

Lemma 3.7. *Let $\mathcal{G} = (G, E)$ be a graph with nonempty edge set. If G contains at least four elements, then the following claims are true.*

1. *No edge-ball is a maximal suspended element in $\Omega(\mathcal{G})$.*

2. *The poset $\texttt{Sus}(\Omega(\mathcal{G})$ is not an antichain.*

Proof. If x is a center for $\mathcal{G}$, then $B(x)$ is not a suspended element; and there is nothing to show. Suppose $x \in G$ is not a center. If $B(x)$ is a maximal member of $\Omega(\mathcal{G})$, then by Proposition 3.4, there exist $y, z \in G$ such that $B(x) = (G - \{y, z\}) \cup (E - \{yz\})$. Since x is the only vertex in $B(x)$, we are forced to conclude that $G = \{x, y, z\}$ — contrary to assumption.

We now establish Claim (2). Since G contains at least four members, we know G contains a vertex x that is not a center. By Proposition 3.6, $B(x)$ is a suspended element; hence we know $\texttt{Sus}(\Omega(\mathcal{G})$ is nonempty. By Claim (1), we also know that $B(x)$ is not maximal in $\texttt{Sus}(\Omega(\mathcal{G})$; consequently, we must conclude $\texttt{Sus}(\Omega(\mathcal{G})$ is not an antichain. □

Figures 1,2,3, and 4 present all of the graphs with nonempty edge set that contain at most three elements. In each case, the suspended elements of $\Omega(\mathcal{G})$ form an antichain. In light of Lemma 3.7, these are the only graphs having this property.

A graph lattice is a very complex entity; yet much of its complexity has little to do with the structure of the underlying graph. Indeed, the Boolean cone of a graph lattice is entirely determined simply by the *number* of vertices and edges contained in the graph. Any two graphs having the same number of edges will generate isomorphic lower nappes $B_\perp$ in their open set lattices; and any two graphs having the same number of vertices will generate isomorphic upper nappes $B_\top$ in their open set lattices. It therefore seems reasonable to focus attention upon the suspended elements and determine how these elements are constructed and whether they encode information sufficient to reconstruct the graph.

Let $\mathcal{G} = (G, E)$ be a graph. Let $\texttt{MaxSus}(\Omega(\mathcal{G}))$ represent the antichain of all maximal suspended elements in $\Omega(\mathcal{G})$, and let $\texttt{MinSus}(\Omega(\mathcal{G}))$ represent the antichain of all minimal suspended elements in $\Omega(\mathcal{G})$. Note that we will have $\texttt{MaxSus}(\Omega(\mathcal{G})) = \texttt{MinSus}(\Omega(\mathcal{G}))$ precisely when $\mathcal{G}$ has empty edge set or contains at most three vertices.

Definition 3.8. We will say a poset $\mathcal{P} = (P, \leq)$ is an *anti-graph* poset provided the following conditions are met.

1. There exist disjoint, nonempty antichains U_P and D_P such that $P = U_P \cup D_P$ and D_P contains at least four elements.

2. If $e, f \in U_P$ are distinct, then $\downarrow e \cap D_P \neq \downarrow f \cap D_P$.

3. If $e \in U_P$, then $D_P - \downarrow e$ contains exactly two elements.

4. If $x \in D_P$, then $U_P \cap \uparrow x$ is nonempty.

For completeness, we note that an anti-graph poset with n minimal elements is really just the order-dual of the incidence poset for an $(n-2)$-uniform *hypergraph*.

Of course, every anti-graph poset can be associated with a graph poset (and hence a graph) in a natural way. Suppose $\mathcal{P} = (U_P \cup D_P, \leq)$ is an anti-graph poset. Define a partial ordering $\sqsubseteq$ on $U_P \cup D_P$ as follows: For all $x, y \in U_P \cup D_P$, let $x \sqsubseteq y$ if and only if one of the following conditions holds:

1. We have $x = y$.

2. We have $x \in U_P$, $y \in D_P$, and $y \not\leq x$.

Let $\mathcal{P}_G = (U_P \cup D_P, \sqsubseteq)$. To see that $\mathcal{P}_G$ is a graph-poset, first suppose that $e \in U_P$. By Condition (3) of Definition 3.8, we know $D_P - \downarrow e$ contains exactly two elements; hence, we know that e is covered by exactly two elements in the poset $\mathcal{P}_G$. Now, suppose $e, f \in U_P$ are distinct. Condition (2) of Definition 3.8 guarantees that e and f have distinct covering sets in $\mathcal{P}_G$.

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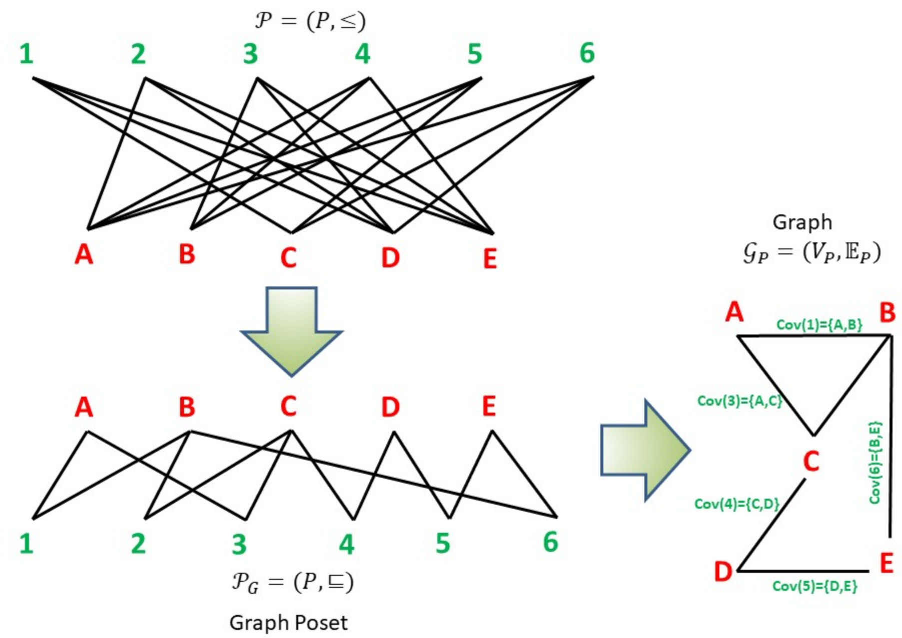

Figure 10 The Graph Poset and Graph Associated with an Anti-Graph Poset

Proposition 3.9. *Let $\mathcal{G} = (G, E)$ be a graph with nonempty edge set. If G contains at least four elements and $\mathcal{G}$ is not a star, then $(\texttt{MaxSus}(\mathcal{G}) \cup \texttt{MinSus}(\mathcal{G}), \subseteq)$ is an anti-graph poset.*

Proof. Let $D_P = \texttt{MinSus}(\mathcal{G})$ and let $U_P = \texttt{MaxSus}(\mathcal{G})$. We know that $D_P = \{B(x) : x \in G\}$ contains at least four elements by Proposition 3.6, and by Proposition 3.4, we know that $U \in U_P$ if and only if $U = \epsilon(\texttt{Cov}(xy))$ for some $xy \in E$. We also know that $U_P \cap D_P = \emptyset$ by Lemma 3.7. Consequently, if $U \in U_P$, we know that $B(x)$ and $B(y)$ are the only members of D_P that are not subsets of U. Suppose $V \in U_P$ and suppose $U \neq V$. We know that $V = (G - \{u, v\}) \cup (E - \{uv\})$; and since we do not allow multiple edges incident to the same pair of vertices, we may assume $x \neq u$. Consequently, we know that U covers $B(u)$ and V covers $B(x)$; and it follows that U and V do not cover the same subset of D_P.

Finally, consider $B(x)$ for any $x \in G$. We know that x is not a center; hence, $E(x) \neq E$. Let $e = uv \in E - E(x)$ and consider $\epsilon(\texttt{Cov}(uv)) = (G - \{u, v\}) \cup (E - \{uv\})$. Since $e \notin B(x)$, we know that $u \neq x$ and $v \neq x$. Consequently, $B(x) \subset \epsilon(\texttt{Cov}(uv))$; and we may conclude that $B(x)$ is covered by members of U_P. □

There is a graph lattice associated with every anti-graph poset $\mathcal{P} = (U_P \cup D_P, \leq)$—namely the open set lattice of the graph poset $\mathcal{P}_G = (U_P \cup D_P, \sqsubseteq)$. The graph associated with $\mathcal{P}_G$ has nonempty edge set and contains at least four vertices; therefore, we know its poset of maximal and minimal suspended subsets forms an anti-graph poset. It should come as no surprise that this poset is order-isomorphic to the original anti-graph poset. Approaching the proof of this fact directly from the graph lattice is notationally cumbersome, so we conclude this paper by providing a proof that constructs (an isomorphic copy of) the graph lattice directly from the anti-graph poset.

Suppose that $\mathcal{P} = (U_P \cup D_P, \leq)$ is an anti-graph poset, and let $\texttt{Pow}[U_P \cup D_P]$ represent the powerset of $U_P \cup D_P$. For each $x \in D_P$, let $A(x) = \{x\} \cup (U_P - \uparrow x)$, and let

$$L_P = \{A \in \texttt{Pow}[U_P \cup D_P] : x \in A \cap D_P \text{ implies } A(x) \subseteq A\} \ .$$

If $x \neq y$ in D_P, then $A(x) \cap A(y)$ contains at most one member. To see why, suppose that e and f are distinct members of $A(x) \cap A(y)$. It follows that $e, f \in U_P$. However, it also follows that $e, f \notin \uparrow x \cup \uparrow y$; hence, by Condition (3) of Definition 3.8, we must have

$$D_P - \downarrow e = \{x, y\} = D_P - \downarrow f \ .$$

This is impossible by Condition (1) of Definition 3.8.

Theorem 3.10. *If $\mathcal{P} = (U_P \cup D_P, \leq)$ is an anti-graph poset, then the poset $\mathcal{L}_P = (L_P, \subseteq)$ is a graph lattice whose poset of maximal and minimal suspended elements is order-isomorphic to $\mathcal{P}$.*

Proof. It is easy to see that $\mathcal{L}_P$ is a lattice in which meet and join are simply set-intersection and set-union, respectively. Consider the set

$$\mathbb{B}_P = \{\{e\} : e \in U_P\} \cup \{A(x) : x \in D_P\} \ .$$

The empty set is the least element of $\mathcal{L}_P$. Hence, the singletons $\{e\}$ where $e \in D_P$ are clearly atoms (and hence join-prime) in $\mathcal{L}_P$. It is also clear that each $A(x)$ is join-prime in $\mathcal{L}_P$. Indeed, suppose that $B, C \in L_P$ are such that $A(x) \subseteq B \cup C$. We may assume that $x \in B$, and it is clear that $A(x) \subseteq B$.

By construction, if $U \in L_P$ and $x \in U \cap D_P$, then $A(x) \subseteq U$. Therefore, if $U \not\subseteq U_P$, then there must exist $x_1, ..., x_n \in D_P$ and (possibly empty) $E_U \subseteq U_P$ disjoint from each $A(x_j)$ such that

$$U = E_U \cup A(x_1) \cup ... \cup A(x_n) \ .$$

It follows that the join-prime elements of $\mathcal{L}_P$ are join-dense in $\mathcal{L}_P$. Furthermore, this characterization of the elements in L_P makes it is clear we must have $\texttt{JP}(\mathcal{L}_P) = \mathbb{B}_P$.

To see that $\mathcal{L}_P$ is a graph lattice, it will therefore suffice to show that $\mathbb{B}_P$ is a graph poset. Let $E_P = \{\{e\} : e \in U_P\}$ and let $V_P = \{A(x) : x \in D_P\}$. It is clear that E_P and V_P are antichains. Let $e \in U_P$. By assumption, e fails to cover exactly two members of D_P; let x and y be these elements. Since $e \in (U_P - \texttt{Cov}(x)) \cap (U_P - \texttt{Cov}(y))$, it follows that $A(x)$ and $A(y)$ are the *only* covers for $\{e\}$ in the poset $\mathbb{B}_P$.

Suppose now that $\{e\}, \{f\}$ are distinct members of E_P. Let $x, y \in D_P$ be the elements e fails to cover, and let $u, v \in D_P$ be the elements f fails to cover. Since $\{x, y\} \neq \{u, v\}$ by assumption, we may suppose that $x \neq u$. It follows that $f \in A(u)$ but $e \notin A(u)$. Likewise, it follows that $e \in A(x)$ but $f \notin A(x)$. Therefore, the cover sets for $\{e\}$ and $\{f\}$ in $\mathbb{B}_P$ are not the same; and we may conclude that $\mathbb{B}_P$ is indeed a graph poset.

We have proven that $\mathcal{L}_P$ is a graph lattice. Let $\mathcal{G}_P = (\{A(x) : x \in D_P\}, \mathbb{E}_P)$ be the graph associated with $\mathcal{L}_P$ by Theorem 2.7. We know that $\{A(x), A(y)\} \in \mathbb{E}_P$ if and only if $A(x) \cap A(y)$ is nonempty. For each $\{A(x), A(y)\} \in \mathbb{E}_P$, let $A(x) \cap A(y) = \{e_{xy}\}$. By assumption, we know that $\mathcal{G}_P$ contains at least four vertices. Since U_P is nonempty, we know the edge set for $\mathcal{G}_P$ is also nonempty by Condition (4) of Definition 3.8. Consequently, by Lemma 3.7, we know the edge-balls of $\mathcal{G}_P$ are the minimal suspended members of $\Omega(\mathcal{G}_P)$. Note that the edge-ball generated by $A(x)$ is the set

$$B(A(x)) = \{A(x)\} \cup \{e \in \mathbb{E}_P : A(x) \in e\}\ .$$

The assignment $B(A(x)) \mapsto x$ is therefore a bijection from the set $\texttt{MinSus}(\Omega(\mathcal{G}_P))$ to the set D_P. Now, suppose U is a maximal suspended member of $\Omega(\mathcal{G}_P)$. There exist $x, y \in D_P$ such that $U = \epsilon(\texttt{Cov}(A(x)A(y))) = (G_P - \{A(x), A(y)\}) \cup (\mathbb{E}_P - \{A(x)A(y)\})$. The assignment $U \mapsto e_{xy}$ is a bijection from the set $\texttt{MaxSus}(\Omega(\mathcal{G}_P))$ to the set U_P. Consequently, consider the function $\varphi : \texttt{Sus}(\Omega(\mathcal{G}_P)) \longrightarrow P$ defined by

$$\varphi(x) = \begin{cases} x & \text{if } U = B(A(x)), \\ e_{xy} & \text{if } U = \epsilon(\texttt{Cov}(A(x)A(y))) \end{cases}\ .$$

For each $e \in U_P$, let U_e denote the pre-image of e in $\texttt{MaxSus}(\Omega(\mathcal{G}_P))$ under the function φ. There exists a unique pair $\{x, y\} \subseteq D_P$ such that $U_e = (G_P - \{A(x), A(y)\}) \cup (\mathbb{E}_P - \{A(x)A(y)\})$; and we know $e = e_{xy}$. With this in mind, note that the edge $A(u)A(v) \in U_e$ if and only if $\{u, v\} \neq \{x, y\}$. Consequently, it follows that $A(u) \subset U_e$ if and only if $e \notin A(u)$.

Now, suppose that $U \in \texttt{MaxSus}(\Omega(\mathcal{G}_P))$ and $A(x) \in \texttt{MinSus}(\Omega(\mathcal{G}_P))$. There exist unique $e \in U_P$ such that $U = U_e$. Observe

$$A(x) \subseteq U_e \iff e \notin A(u) \iff e \in \uparrow_P x \iff x < e\ .$$

The function φ therefore defines an order-isomorphism between the poset $\mathcal{P}$ and the poset of maximal and minimal suspended elements of $\Omega(\mathcal{G}_P)$.

□

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