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STANISŁAW ZAREMBA (1863–1942) AND HIS RESULTS IN THE FIELD OF DIFFERENTIAL EQUATIONS

STANISŁAW ZAREMBA (1863–1942) I JEGO WYNIKI W DZIEDZINIE RÓWNAŃ RÓŻNICZKOWYCH

Abstract

The subject of the paper is presentation of the publications of Stanisław Zaremba in the field of partial differential equations. We present selected results in detail. Zaremba published about 120 works. Among them more than 60 were devoted to partial differential equations.

Keywords: *differential equations*, *partial differential equations*, *publications of Stanisław Zaremba*

Streszczenie

Artykuł poświęcony jest prezentacji publikacji Stanisława Zaremby w dziedzinie równań różniczkowych cząstkowych. Wybrane rezultaty zaprezentowano bliżej. Zaremba opublikował około 120 prac, z których ponad 60 jest poświęconych równaniom różniczkowym cząstkowym.

Słowa kluczowe: *równania różniczkowe zwyczajne*, *równania różniczkowe cząstkowe*, *publikacje Stanisława Zaremby*

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1. Stanisław Zaremba (03.12.1863–23.11.1942)

Stanisław Zaremba was born on 3th December 1863 in a village Romanówka [1]. In 1881 he finished high school (the German gymnasium in St. Petersburg) and next studied engineering at the Institute of Technology in St. Petersburg (getting an engineering diploma in 1886). Then in 1887 he went to Paris, where he studied mathematics for his doctorate at the Sorbonne in 1899, advised by Darboux and Picard.

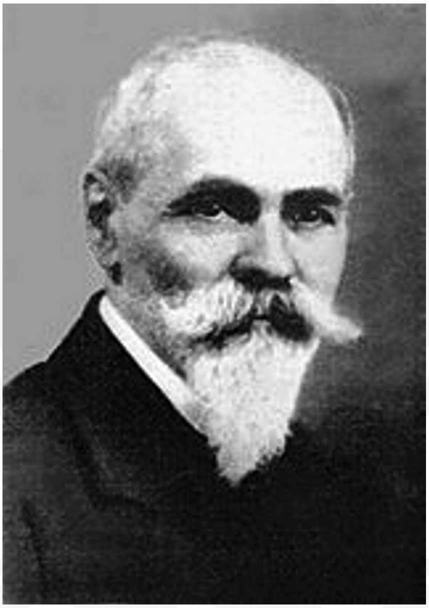

Fig. 1. Stanisław Zaremba (03.12.1863–23.11.1942)

As a topic for his dissertation Zaremba chose the ideas introduced by Riemann in 1861. His doctoral thesis *Sur un problème concernant l'état calorifique d'un corp homogène indéfini* was presented in 1889. At that time Zaremba got in touch with many mathematicians of the French school. He maintained these ties, engaging in a wide international cooperation after returning to Poland. In particular he collaborated with Painlevé and Goursat. Before 1900 Zaremba taught in secondary schools in France. At that time he concentrated hard on his research. He published his results in French mathematical journals, which resulted in his work becoming well known and highly respected by leading French mathematicians such as Poincaré and Hadamard. Zaremba's publications concerned mainly partial differential equations. These publications played a very important role in the world development of mathematical sciences. In 1900 he came back to Cracow being nominated *extraordinary professor* and in 1905 the *ordinary professor* (*full professor*) at the Jagiellonian University. He worked in Jagiellonian University until 1935 (when he retired). In that year he obtained the title of the *honorary professor of the Jagiellonian University*, a very unusual titular dignity. Zaremba died (at the age of 79) on 23th November 1942.

2. List of Stanisław Zaremba's publications in the field of differential equations

(On the basis of [2], where about 118 publications of S. Zaremba are mentioned)

1. *Sur un problème concernant l'état calorifique d'un corps solide homogène idéfini (*Dissertation in Sorbone – advisored by Darboux and Picard) Paris, chez Gauttier--Villars, 1899.
2. *Note concernant l'intégration d'une équation aux dérivées partialles*, C. R. d l' Ac. de Paris, 1890.
3. *Note concernant l'intégration d'une équation aux dérivées partialles,* Annales de l' École Normale, 3ieme Série 7, fasc. 3, mai 1890.
4. *Sur la réduction du nombree des périodes d'une fonction périodique*, Bull. de la Société Mathématique de France, 1894.
5. *Contribution a la théorie de la fonction de Green*, Bull. de la Société Mathématique de France, 1896.
6. *Note sur la methode des approximations successives de M. Picard*, C. R. d *l'*Ac. de Paris, 1897.
7. *Sur la méthode des approximations successives de M. Picard*, Journal de Mathématiques Pures et Appliquées 5ieme Série 3, fasc. 3, 1897.
8. *Sur la méthode des approximations successives de M. Picard,* Journal de Liouville, 1897.
9. *Note sur le problème de Dirichlet*, C. R. d *l'* Ac. de Paris, 1897.
10. *Note sur le problème de Dirichlet*, Annales de l' École Normale Supérieure, 1897.
11. *Sur le problème de Dirichlet*, Annales de l' École Normale Supérieure, 1897, 3ieme Série 14, juillet, 1897.
12. *Zastosowanie metody Picarda do równań różniczkowych trzech zmiennych*, Prace Mat.--Fiz., T. IX,1898.
13. *O zasadzie Dirichleta*, Prace Mat.-Fiz., T. IX, 1898.
14. *Sur l'équation aux dérivées partialles* $\Delta u + \xi u + f = 0$, Bull. de la Socciété Mathématique de France, 1898.
15. *Note sur un theoreme de M. Poincaré*, C. R. d *l'* Ac. de Paris,1899.
16. *Sur l'équation aux dérivées partialles* $\Delta u + \xi u + f = 0$ *et sur les fonctions harmoniques*, Annales de l' École Normale, 3ieme Série 16, octobre, 1899.
17. *Note sur le développement d'une fonction arbitraire en une série procedant suivant les fonctions harmoniques*, C. R. d *l'* Ac. de Paris, 1899.
18. *Sur le développement d'une fonction arbitraire en une série procédant suivant les fonctions harmoniques*, Journal de Liouville, 1890.
19. *O równaniu o pochodnych cząstkowych* $\Delta u + \xi u + f = 0$ *i o funkcjach harmonicznych*, Prace Mat.-Fiz., T. XI,1900.
20. *O rozwinięciu dowolnej funkcji w szereg (funkcji) harmonicznych*, Prace Mat.-Fiz., T. XI, 1900.

21. *Sur le développement d'une fonction arbitraire en une série procédant suivant les founctions harmoniques*, Journal de Mathématiques Pures et Appliquées, 1900.
22. *Contribution a la théorie de l'équation aux dérivées partialles* $\Delta v + \xi v = 0$, Annales de la Faculté des Sciences de l' Université de Toulouse, 1900.
23. *Contribution a la théorie des'équations de la Physique*, C. R. d *l'* Ac. de Paris, 1901.
24. *O tak zwanych funkcjach zasadniczych w teoryi równań fizyki matematycznej*, Rozprawy Wydz. Mat.-Przyrodniczego Akademii Umiejętności, Seria III, T. I, 1901.
25. *Sur les fonctions dites fondamentales la théorie des équations de la physique*, Bulletin International de l'Academie des Sciences de Cracovie, 1901.
26. *Note sur l' integration de l'équation aux dérivées partialles* $\Delta u + \xi u = 0$, C. R. d *l'* Ac. de Paris, 1901.
27. *O teoryi równania Laplace'a i o metodach Neumanna i Robina*, Rozprawy Wydz. Mat.--Przyrodniczego Akademii Umiejętności, Seria III, T. I, 1901.
28. *Sur la théorie de l'équation de Laplace at les méthodes de Neumann et de Robin*, Bulletin International de l'Academie des Sciences de Cracovie, 1901.
29. *Przyczynek do teoryi pewnego równania fizyki matematycznej*, Rozprawy Wydz. Mat.--Przyrodniczego Akademii Umiejętności, Seria III, T. I, 1901.
30. *Contribution a la théorie d' une équation de la physique*, Bulletin International de l'Academie des Sciences de Cracovie, 1901.
31. *Sur les méthodes de la moyenne arithmétique de Neumann et de Robin de la cas d'une frontiére non connexe*, Bulletin International de l'Academie des Sciences de Cracovie, 1902.
32. *Sur l' intégration de l'équation* $\Delta u + \xi u = 0$, Journal de Mathématiques Pures et Appliquées, 1902.
33. *Sur l' intégration de l'équation* $\Delta u + \xi u = 0$, Journal de Liouville, 1902.
34. *Détermination du cas ou les fonctions fondamentales de M. Poincaré sont déductibles de celles de M. Roy et de celles de M. Stekloff*, Bulletin International de l'Academie des Sciences de Cracovie, 1902.
35. *Sur une généralisation de la théorie classique de la viscosité*, Bulletin International de l'Academie des Sciences de Cracovie, 1903.
36. *Contribution a la théorie des fonctions fondamentales*, Annales de l' École Normale, 1903.
37. *O metodach średniej arytmetycznej Neumanna i Robina w przypadku, gdy ograniczenie nie jest spójne*, Rozprawy Wydz. Mat.-Przyrodniczego Akademii Umiejętności, 1903.
38. *Les fonctions fondamentales de M. Poincaré et la méthode de Neumann pour frontière compasèe des polygenes curvilignes*, Journal de Mathématiques Pures et Appliquées, 1904.
39. *Contribution à la théorie d' une équation fonctionelle de la physique*, Rendiconti del Circolo Matematico di Palermo, 1904.

40. *Solution generale du problème de Fourier*, Bulletin International de l'Academie des Sciences de Cracovie, 1905.
41. *Ogólne rozwiązanie zagadnienia Fouriera*, Rozprawy Wydz. Mat.-Przyrodniczego Akademii Umiejętności, 1905.
42. *Sur les fonction de Green et quelques-unes de ses applications*, Bulletin International de l'Academie des Sciences de Cracovie, 1906.
43. *L'equation biharmonique et une classe remarquable de fonctions fondamentales harmoniques*, Bulletin International de l'Academie des Sciences de Cracovie, 1907.
44. *Równania biharmonijne i pewien szczególny rodzaj funkcyj harmonijnych zasadniczych*, Wiadomości Matematyczne, XI, 1907 (cz. II sprawozdanie).
45. *Nowa metoda uzasadniania podstawowych własności funkcji Greena*, Wiadomości Matematyczne, XI, 1907.
46. *Sur l' intégration de l'équation biharmonique*, Bulletin International de l'Academie des Sciences de Cracovie, 1908.
47. *Sur le principe de Dirichlet*, Atti del IV Congresso Internazionale dei Matematici, Roma, 1908.
48. *Sur le princile de minimum*, Bulletin International de l'Academie des Sciences de Cracovie, 1909.
49. *Sur l' unicité de la solution du problème de Dirichlet*, Bulletin International de l'Academie des Sciences de Cracovie, 1909.
50. *Sur le calcul numérique des fonctions demandées dans le problème de Dirichlet et le problème hydrodynamique*, Bulletin International de l'Academie des Sciences de Cracovie, 1909.
51. *Le problème biharmonique restreint*, Annales de l'Ecole Normale, 1909.
52. *Sur le principe de Dirichlet*, Acta Mathematica 34, Stockholm, 1910.
53. *Sur la problème mixte relatif à équation de Laplace*, Bulletin International de l'Academie des Sciences de Cracovie, 1910.
54. *Sur un théoréme fondamental relatif à l'équation de Fourier*, Comptes Rendus du Congres International des Mathématiciens, Strassburg, 1920.
55. *Les fonctions réel es non analytiques et les solutions singulariéres des équations differentialles du premier ordre*, Annales de la Société Polonaise de Mathématiques, 5, 1922.
56. *Sur une forme remarquable de l' intégrale de l'équation des cordes vibrantes*, Nouvelles Annales de Mathématiques, Paris, 1923.
57. *Sur un problème toujours possible comprenant à titre de cas particuliers, le problème de Dirichlet et celui de Neumann*, Journal de Mathématiques Pures et Appliquées, 1927.
58. *Pogląd na współczesny stan teoryi potencjału*, Mathesis Polska, 1931.
59. *Un théoréme général relatif aux équation aux dérivées partialles du second ordre lineaires et du type hyperbolique*, Bulletin International de l'Academie des Sciences de Cracovie, 1934.

3. Selected scientific results of Stanisław Zaremba in the field of partial differential equations

This presentation is based on the papers mentioned in [7] and [5]: T. Ważewski, J. Szarski, *Stanisław Zaremba*, *Studia z dziejów katedr Wydziału Matematyki, Fizyki, Chemii Uniwersytetu Jagiellońskiego* (ed. S. Gołąb), Wydawnictwa Jubileuszowe – Tom XV, Uniwersytet Jagielloński, Kraków, 1964, pp. 103-117; the A. Pelczar elaboration [4] entitled: *Stanisław Zaremba* (*100th anniversary of taking up a chair at the Jagiellonian University*). *Prepared for the International Conference 90 years of the reproducing Kernel Property, Kraków, April 16–21, 2000 organised by the Chair of Functional Analysis of the Jagiellonian University.* The last one is also based on the the paper [5]. Because Ważewski, Szarski and Pelczar (world-famous specialists in the field of differential equations) had written about Zaremba's scientific results and characterised Zaremba's mathematical achievements, below we are going to cite *in extenso* a part of Pelczar's elaboration [4] on Zaremba's works connected with the subject of differential equations.

"... Among his important results there are those concerning the elliptic equation

$$\Delta u + \xi u + f = 0 \tag{1}$$

with boundary Dirichlet conditions as well as Neumann and Fourier type conditions. Some of these results were included into the canon of the fundamental knowledge on the theory of partial differential equations. Before talking about some details let us quote a sentence from the book [J. Mawhin, *Metody wariacyjne dla nieliniowych problemów Dirichleta,* (Polish version of the book *Problèmes de Dirichlet variationneles non linéaires*; translated by D.P. Idziak, A. Nowakowski, S. Walczak), Warszawa 1995] of Jean Mawhin: *According to Bouligand Zaremba's contribution to the development of the theory of the Dirichlet problem is the same as that of Poincaré and Lebesgue.*

In the paper [S. Zaremba, Sur le problème de Dirichlet, *Annales de l'École Normale* (3), 14, 1897, 251-258] properties of the Green function G for a Dirichlet problem in the three dimensional space is considered and it is shown that the function

$$u = \int_S \frac{\partial G}{\partial n} \sigma ds$$

is a solution to a given Dirichlet problem with the boundary condition described by a continuous function σ, a discussion of properties of u in the case of non-continuos σ is included as well.

In the paper [S. Zaremba, Sur l'équation aux dérivées partielles $\Delta u + \xi u + f = 0$ et sur les fonctions harmoniques, *Annales de l'École Normale,*(3)16, 1899, 427-463]. Zaremba discussed the equation (1) for $f = 0$ with the condition

$$\frac{\partial G}{\partial n} = hu, \tag{2}$$

where h is a non-negative constant and $\frac{\partial G}{\partial n}$ is the interior normal derivative. He proved that there exist a sequence of eigenvalues and corresponding sequence $\{U^k\}$ of orthonormal eigenfunctions. He proved also that if ξ is not an eigenvalue then the problem (1)–(2) (with $f = 0$) has exactly one solution. Moreover, every function satisfying the boundary condition (2) can be represented as a Fourier series with respect to the sequence $\{U^k\}$ of eigenfunctions. Zaremba developed some idea of Poincaré and used *generalised potentials* defined by replacing in the classical definition of Newtonian potential the function $\frac{1}{r}$ by the function $\frac{\exp(-\eta r)}{r}$, where η is a complex number such that $re\,\eta > 0$ and $\eta^2 + \xi = 1$. This notion of generalised potentials introduced by Zaremba turned out to be very useful in several other problems.

Analogous results for the homogenous problem: (1) with $f = 0$ and the boundary problem $u = 0$ are given in the paper [S. Zaremba, Sur le développement d'une fonction arbitraire en un série procédant suivant les fonctions harmoniques, *Journal de Mathématiques pures et appliquées,*(5),6, 1900, 47-72].

In [S. Zaremba, Contribution à la théorie de l'équation aux dérivées partielles $\Delta u + \xi u = 0$, *Annales de la Faculté des Sciences d l'Université de Toulouse,* (32), 3, 1900, 5-12]. Zaremba gave some conditions sufficient for derivatives of arbitrary order of solutions of Dirichlet problems for the homogenous equation (1) (with $f = 0$) in a domain D to be continuous in the closure of D. Importance of this result is underlined by Jean Mawhin in the preface to the Polish version of his book [J. Mawhin, *Metody wariacyjne dla nieliniowych problemów Dirichleta,* (Polish version of the book *Problèmes de Dirichlet variationneles non linéaires*; translated by D.P. Idziak, A. Nowakowski, S. Walczak), Warszawa 1995].

The paper [S. Zaremba, Sur l'intégration de l'équation $\Delta u + \xi u = 0$, *Journal de Mathématiques pures et appliqueés*, (5), 8, 1902, 59-117] deals with the following problem. Let D be a bounded domain in the three dimensional real space, S be the intersections of the boundaries of D and $-D$, n the normal unit vector directed into the domain D. For a function u of three real variables and a given point $x^0 \in S$ we put

$$\left(\frac{\partial u}{\partial n}\right)_i = \lim \frac{1}{t}(u(x^0 + tn) - u(x^0) \quad \text{as} \quad t \to 0+,$$

$$\left(\frac{\partial u}{\partial n}\right)_i = \lim \frac{1}{t}(u(x^0 + tn) - u(x^0) \quad \text{as} \quad t \to 0-,$$

and for a function v and $x^0 \in S$, $(\nu)_i = \lim \nu(x)$ $x \to x^0$, $x \in D$ and $(\nu)_e = \lim \nu(x)$ $x \to x^0$, $x \notin D$.

We look for two solutions u and v of the homogenous equation (1) (that is with $f = 0$), which are generalised potentials of single layer and of double layer respectively, such that

$$\left(\frac{\partial u}{\partial n}\right)_e - \left(\frac{\partial u}{\partial n}\right)_i = \lambda\left[\left(\frac{\partial u}{\partial n}\right)_e + \left(\frac{\partial u}{\partial n}\right)_i\right] + 2\varphi,$$

$$(\nu)_e - (\nu)_i = \lambda[(\nu)_e + (\nu)_i] + 2\varphi,$$

Where λ is a parameter and φ is a given function defined on S. Zaremba proved that this problem has (under general, relatively weak regularity assumptions) a solution which is an analytic function of λ and has at most one essential singularity (at infinity) and single poles at points belonging to sequence of real numbers independent on the function φ. This permits to deal with the Neumann method assuming weak regularity conditions. An analogous problem on the plane is considered in the paper [S. Zaremba, Les fonctions fondamentales de M. Poincaré et la méthode de Naumann pour une frontière composée des polygones curvilignes, *Journal de Mathématiques pures et appliquées,* (5),10, 1904, 395-444] (without the assumption of the continuity of $\frac{\partial u}{\partial n}$).

In an earlier paper [S. Zaremba, Sur la méthode d'approximations successives de M. Picard, *Journal de Mathématiques pures et appliquées,* (5), 3, 1897, 311-329] Zaremba deals with successive approximations for solutions of a non-linear equation

$$\Delta u = f\left(x, y, z, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right).$$

This paper, as well as the paper [S. Zaremba, Contribution a la théorie de la fonction de Green, *Bulletin de la Société Mathématique de France,* 54, 1896, 19-24], is cited in [A. Sommerfeld, Randwertaufgaben in der Theorie der partiellen Differentialgleichungen, [in:] *Ecyklopädie der Mathematischen Wissenschaften,band II-1,* Leipzig 1907, 505-570] (p. 528) where a canon of the theory of elliptic partial differential equations is presented.

The paper [S. Zaremba, Le problème biharmonique restreint, *Annales de l'École Normale*, (3), 26, 1909, 337-404] is an extension of an unpublished note presented to the Paris Academy of Sciences and characterised by that Academy as *extrémement honorable.* A biharmonic problem considered there is such that we are looking for a solution u of the equation

$$\Delta^2 u = 0$$

considered in a domain D such that

$$u = \varphi \quad \text{and} \quad \frac{\partial u}{\partial x_i} = \frac{\partial \varphi}{\partial x_i}$$

on the boundary ∂D of the domain D,
where φ is a sufficiently regular function defined on ∂D (in particular the second power of the Laplacian of φ is assumed to be integrable) requesting certain natural regularity condition to be fulfilled by $u.$ Zaremba proved that in order to solve that problem it is sufficient to find a function v harmonic in D, such that ν^2 is integrable and for every harmonic function h such that h^2 is integrable on D the following equality is satisfied

$$u = \int_D \Delta\varphi h d\tau = \int_D v h d\tau.$$

Zaremba proved theorems on existence and uniqueness of solutions to that problem. He proved also that this problem can be solved by determining the minimum of an integral on *D*.

Basing on some result of the paper [S. Zaremba, Contribution à la théorie d'une équation fonctionelle de la physique, *Rendiconti del Circolo Matematico di Palermo,* 19, 1904, 140--150] Witold Wilkosz (1891–1941) proved a theorem on analyticity of harmonic functions (see [W. Wilkosz, Sur un point fondamental de la théorie du potentiel, *Comptes Rendus de l'Académie des Sciences de Cracovie,* 174, 1922, 435-437]).

The Dirichlet problem with non-continuous boundary conditions was treated in the paper [S.Zaremba, Sur l'unicité de la solution du problème de Dirichlet, *Bulletin International de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, 561–564]; the main result of this paper is the first one of that type.

Several other papers were devoted to the theory of the Dirichlet problem (see for instance [S. Zaremba, Sur le calcul numérique des fonctions demandées dans le problème de Dirichlet et le problème hydrodynamique, *Bulletin International de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, (2), 125-195], [S. Zaremba, Sur le principe du minimum, *Bulletin Internationale de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, (7), 197-264], [S. Zaremba, Sur le principe de Dirichlet, *Acta Mathematica,* 34, 1911, 293-316], [S. Zaremba, Sur un problème toujours possible comprenant à titre de cas particuliers, le problème de Dirichlet et celui de Neumann, *Journal de Mathématiques pures et appliquées,*(9), 6, 1927, 127-163]). In papers [S. Zaremba, Sur le principe du minimum, *Bulletin Internationale de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, (7), 197-264], [S. Zaremba, Sur le principe de Dirichlet, *Acta Mathematica*, 34, 1911, 293-316], [S. Zaremba, Sur un problème toujours possible comprenant à titre de cas particuliers, le problème de Dirichlet et celui de Neumann, *Journal de Mathématiques pures et appliquées,*(9), 6, 1927, 127-163] Zaremba developed his beautiful and fruitful idea of solving instead of the original Dirichlet problem some other problem which has always solutions and which can be reduced to the Dirichlet problem if the last one has a solution. Paper [S. Zaremba, Sur le calcul numérique des fonctions demandées dans le problème de Dirichlet et le problème hydrodynamique, *Bulletin International de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, (2), 125-195] gives some numerical method of solving Dirichlet problems and – in a sense – extends the idea of the paper [S. Zaremba, L'équation biharmonique et une classe remarquable de fonctions fonda-mentales harmoniques, *Bulletin International de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1907, (3), 147-196] which will be referred to later on in another context.

The Dirichlet problem was also the subject of Zaremba presentation [S. Zaremba, Sur le principe de Dirichlet, *Atti del IV Congresso Internazionale dei Matematici (Roma, 6–11 Aprile 1908),* Vol. II, Communicazioni delle sezioni I e II, Roa 1909, 194-199] during the IV-th International Congress of Mathematicians in Rome in 1908. In papers [S. Zaremba, Sur le principe du minimum, *Bulletin Internationale de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, (7), 197-264] and [S. Zaremba, Sur le principe de Dirichlet, *Atti del IV Congresso Internazionale dei Matematici (Roma, 6–11*

Aprile 1908), Vol. II, Communicazioni delle sezioni I e II, Roma 1909, 194-199] Zaremba introduced generalised solutions into the direct method of variational calculus built up by Hilbert (see [8]). In [S. Zaremba, Sur le principe du minimum, *Bulletin Internationale de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1909, (7), 197-264] an example of a domain in which there is no solution of a linear Dirichlet problem; it was the first such example in the literature, as it is pointed out by Jean Mawhin in [J. Mawhin, *Metody wariacyjne dla nieliniowych problemów Dirichleta,* (Polish version of the book *Problèmes de Dirichlet variationneles non linéaires*; translated by D.P. Idziak, A. Nowakowski, S. Walczak] and by Pierre Dugac, Beno Eckman, Jean Mawhin and Jean-Paul Pier in the section "Guidelines 1900–1950" (see [*Development of Mathematics 1900–1950*, edited by Jean-Paul Pier, Birkhäuser Verlag, Basel-Boston-Berlin, 1994], p. 6) presenting a list of the most important results obtained in this period the paper [42] is cited in the bibliography. In the same place [*Development of Mathematics 1900–1950*, edited by Jean-Paul Pier, Birkhäuser Verlag, Basel-Boston-Berlin 1994] Zaremba is mentioned as the author of a *method of orthogonal projection in Dirichlet problem.*

In the paper [S. Zaremba, Sopra un teorema d'unicita relativo alla equazione della onde sferiche, *R.C. della Accademia dei Lincei,* (5), 24, 1915, 904-908] an equation of so-called spherical wave is considered. There is given a method of estimation of

$$u = \int \mathrm{grad}^2 u d\tau,$$

where u is a solution of that equation. The idea of Zaremba used in his method was applied later on by Friedrichs and Levy in order to get known (now) integral inequalities satisfied by general solution of hyperbolic equations. These inequalities have been generalised by Juliusz Schauder (and became some fundamental elements in the survey of the theory of hyperbolic equations).

Zaremba considered also several other problems. He discussed for example, as it has been mentioned already, problems of Neumann and Fourier. An important contribution to the theory of Fourier problem was presented in [S. Zaremba, Solution générale du problème de Fourier, *Bulletin International de l'Académie des Sciences, Classe des Sciences Mathématiques et Naturelles,* 1905, 69-168].

The Fourier equation

$$\Delta_{x,t} - \frac{\partial u}{\partial t} = 0, \quad u = u(x,t),$$

was the subject of Zaremba's presentation (see [S. Zaremba, Sur un théoreme fondamental relatif à l'èquation de Fourier, *Compte Rendus du Congrés International des Mathématiciens (Strasbourg 22–30 Septembre 1920)*, Toulouse 1921, 343-350]) during the International Congress of Mathematicians in Strasbourg in 1920.

Let us now present some special part of Zaremba's contribution to the development of the theory *of reproducing kernels* (see for instance [F. H. Szafraniec, The reproducing kernel Hilbert space and its multiplication operators, *Operator Theory: Advances and Applications*, 114, 2000, 253-263]). The best and probably the shortest way to do it is by referring to the Aronszajn paper [N. Aronszajn, Theory of reproducing kernels, *Trans. Amer. Mat. Soc.*, 68, 1950, 337-404]. He wrote: "Examples of kernels of the type in which we are interested have been known for a long time, since all the Green's functions of self-adjoint ordinary differential equations (as also some Green's functions – the bounded ones – of partial differential equations) belong to this type (…) There have been and continue to be two trends in the consideration of these kernels (…). The second trend was initiated during the first decade of the century in the work of S.Zaremba" [1. S. Zaremba, *L'équation biharmonique et une classe remarquable de fonctions fonda-mentales harmoniques, Bulletin International de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles,* 1907, (3), 147–196)*,* 2. S. Zaremba*, Sur le calcul numérique des fonctions demandées dans le problème de Dirichlet et le problème hydrodynamique, Bulletin International de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles, 1909, (2), 125-195*)] *on boundary value problems for harmonic and biharmonic functions. Zaremba was the first to introduce, in a particular case, the kernel corresponding to a class of functions, and to state its reproducing property* (…). *However, he did not develop any general theory, nor did give any particular name to the kernels he introduced.* In that way one links certain results of Zaremba with some important part of the modern theory of operators which shows how deep were those result being now more than ninety years old...".

Most of Zaremba's scientific results were obtained in connection with particular questions belonging to theoretical physics [3–6]. Therefore they are therefore not the subject of these considerations.

References

[1] Archiwum UJ, sygn. WF II 163 (*Katedry i Instytut Matematyczny z lat (1851–1945)*).

[2] Pawlikowska-Brożek Z., Węglowska D., *Bibliografia Stanisława Zaremby* (manuskrypt), Kraków 1991.

[3] Pelczar A., *Równania różniczkowe w Polsce. Zarys historii do połowy lat siedemdziesiątych XX wieku*, Wiadomości Matematyczne, Vol. XXXVII, Warszawa 2001, pp. 63-18.

[4] Pelczar A., *Stanisław Zaremba (100th anniversary of taking up a chair at the Jagiellonian University)*, Prepared for the International Conference 90 years of the reproducing Kernel Property, Kraków, April 16–21, 2000 organised by the Chair of Functional Analysis of the Jagiellonian University.

[5] Ważewski T., Szarski J., *Stanisław Zaremba*, Studia z dziejów katedr Wydziału Matematyki, Fizyki, Chemii Uniwersytetu Jagiellońskiego (pod redakcją Stanisława Gołąba), Wydawnictwa Jubileuszowe – Vol. XV, Uniwersytet Jagielloński, Kraków 1964, 103-117.

[6] Ważewski T., *Twórczość i zasługi naukowe Stanisława Zaremby*, Dodatek do „Rocznika Polskiego Towarzystwa Matematycznego", Kraków 1951, 8-20.

[7] Selected papers of S. Zaremba, The papers presented at the above *List of Stanisław Zaremba's publications in the field of differential equations* (see part 2 of this paper).