

NAUKI PODSTAWOWE

CZASOPISMO TECHNICZNE
TECHNICAL TRANSACTIONS

FUNDAMENTAL SCIENCES

WYDAWNICTWO

POLITECHNIKI KRAKOWSKIEJ

1-NP/2012

ZESZYT 18

ROK 109

ISSUE 18

YEAR 109

MONIKA HERZOG*

APPROXIMATION OF FUNCTIONS OF TWO
VARIABLES FROM EXPONENTIAL WEIGHT SPACESAPROKSYMACJA FUNKCJI DWÓCH ZMIENNYCH
Z WYKŁADNICZYCH PRZESTRZENI WAGOWYCH

Abstract

In this paper we study approximative properties of modified Szasz-Mirakyan operators for functions of two variables from exponential weight spaces. We present theorems giving a degree of approximation by these operators for exponential bounded functions.

Keywords: linear positive operators, Bessel function, modulus of continuity, degree of approximation

Streszczenie

W artykule przedstawiono aproksymacyjne własności zmodyfikowanych operatorów typu Szasa-Mirakjana dla funkcji dwóch zmiennych z wykładniczych przestrzeni wagowych. Wyprowadzono twierdzenia podające rząd aproksymacji funkcji ograniczonych wykładniczo przez operatory tego typu.

Słowa kluczowe: dodatni operator liniowy, funkcja Bessela, modul ciągłości, rząd aproksymacji

* Dr Monika Herzog, Instytut Matematyki, Wydział Fizyki, Matematyki i Informatyki, Politechnika Krakowska.

1. Introduction

Let us denote by $C(R_0)$ a set of all real-valued functions continuous on $R_0 = [0; +\infty)$. In paper [1] we investigated operators of Szasz-Mirakyan type defined as follows

$$A_n^v(f; x) = \begin{cases} \frac{1}{I_\nu(nx)} \sum_{k=0}^{\infty} \frac{\left(\frac{nx}{2}\right)^{2k+v}}{\Gamma(k+1)\Gamma(k+v+1)} f\left(\frac{2k}{n}\right), & x > 0; \\ f(0), & x = 0, \end{cases}$$

where Γ is the Euler-gamma function and I_ν the modified Bessel function defined by the formula ([6], p. 77)

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+v}}{\Gamma(k+1)\Gamma(k+v+1)} \quad (1)$$

We studied approximative properties of these operators in exponential weight spaces

$$E_p = \left\{ f \in C(R_0) : w_p f \text{ is uniformly continuous and bounded on } R_0 \right\},$$

where w_p was the exponential weight function defined as follows:

$$w_p(x) = e^{-px}, \quad p \in R_+ \quad (2)$$

for $x \in R_0$.

In the spaces we introduced the norm:

$$\|f\|_p = \sup \left\{ w_p(x) |f(x)| : x \in R_0 \right\} \quad (3)$$

and we established ([1], Corollary 1) that operators A_n^v are linear, positive, bounded and transform the space E_p into E_r for some $r > p$.

Notice that a certain modification of A_n^v give a linear, positive and bounded operator L_n^v transforming the space E_p into E_r (Theorem 2.1), namely:

$$L_n^v(f; x) = \begin{cases} \frac{1}{I_\nu(nx)} \sum_{k=0}^{\infty} \frac{\left(\frac{nx}{2}\right)^{2k+v}}{\Gamma(k+1)\Gamma(k+v+1)} f\left(\frac{2k}{n+p}\right), & x > 0; \\ f(0), & x = 0. \end{cases} \quad (4)$$

In the present paper we introduce an analogy of the presented theorems for $f \in E_p$, but now we consider the bivariate version of the operator L_n^v

$$L_{n,m}^{\nu,\mu}(f;x,y)= \quad (5)$$

$$\left\{ \begin{array}{l} \frac{1}{I_{\nu}(nx)} \frac{1}{I_{\mu}(my)} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\left(\frac{nx}{2}\right)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)} \frac{\left(\frac{my}{2}\right)^{2j+\mu}}{\Gamma(j+1)\Gamma(j+\mu+1)} f\left(\frac{2k}{n+p}, \frac{2j}{m+q}\right), x>0, y>0; \\ \frac{1}{I_{\nu}(nx)} \sum_{k=0}^{\infty} \frac{\left(\frac{nx}{2}\right)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)} f\left(\frac{2k}{n+p}, 0\right), x>0, y=0; \\ \frac{1}{I_{\mu}(my)} \sum_{j=0}^{\infty} \frac{\left(\frac{my}{2}\right)^{2j+\mu}}{\Gamma(j+1)\Gamma(j+\mu+1)} f\left(0, \frac{2j}{m+q}\right), y>0, x=0; \\ f(0,0), x=y=0, \end{array} \right.$$

for $n, m \in N, \nu, \mu \in R_0$ and $f \in E_{p,q}$, where:

$$E_{p,q} = \left\{ f \in C(R_0^2) : w_{p,q} f \text{ is uniformly continuous and bounded on } R_0^2 \right\},$$

and $w_{p,q}$ is the exponential weight function

$$w_{p,q}(x,y) = e^{-(px+qy)}, \quad p, q \in R_+ \quad (6)$$

for $(x,y) \in R_0^2$.

It is easy to verify that $E_{p,q}$ is a normed space with the norm:

$$\|f\|_{p,q} = \sup \{ w_{p,q}(x,y) |f(x,y)|; (x,y) \in R_0^2 \} \quad (7)$$

Moreover, we use the weighted modulus of continuity defined as follows:

$$\omega(f, E_{p,q}; t, s) = \sup \left\{ \|\Delta_{h,d} f\|_{p,q} : h \in [0, t], d \in [0, s] \right\} \quad (8)$$

where:

$$\Delta_{h,d} f(x,y) = f(x+h, y+d) - f(x,y)$$

for $(x,y), (h,d) \in R_0^2$.

The note was inspired by the results of [3–5] which investigate approximation problems for bivariate operators.

We shall present theorems giving a degree of approximation of function $f \in E_{p,q}$ by operators $L_{n,m}^{\nu,\mu}$.

2. Auxiliary results

At the beginning we recall preliminary results which we immediately obtain from paper [1] and definition (4).

Lemma 2.1 ([1], Lemma 8) For each $v \in R_0$ there exists a positive constant $M(v)$ such that for all $n \in N$ and $x \in R_0$ we have:

$$\left| \frac{I_{v+1}(nx)}{I_v(nx)} \right| \leq M(v),$$

$$nx \left| \frac{I_{v+1}(nx)}{I_v(nx)} - 1 \right| \leq M(v).$$

By elementary calculations we get

Lemma 2.2 For each $n \in N$, $v \in R_0$, $p \in R_+$ and $x \in R_0$.

$$L_n^v(1; x) = 1, \quad L_n^v(t; x) = x \frac{n}{n+p} \frac{I_{v+1}(nx)}{I_v(nx)} = \frac{n}{n+p} A_n^v(t; x),$$

$$L_n^v(t^2; x) = x^2 \left(\frac{n}{n+p} \right)^2 \frac{I_{v+2}(nx)}{I_v(nx)} + x \frac{2n}{(n+p)^2} \frac{I_{v+1}(nx)}{I_v(nx)} = \left(\frac{n}{n+p} \right)^2 A_n^v(t; x),$$

$$L_n^v(t-x; x) = \frac{n}{n+p} \left(A_n^v(t; x) - \frac{p}{n} x \right),$$

$$L_n^v((t-x)^2; x) = \left(\frac{n}{n+p} \right)^2 \left(A_n^v((t-x)^2; x) - 2x \frac{p}{n} A_n^v(t-x; x) + \left(\frac{p}{n} x \right)^2 \right)$$

Lemma 2.3 For each $n \in N$, $v \in R_0$, $p \in R_+$ and $x \in R_0$.

$$L_n^v(e^{pt}; x) = \frac{I_v\left(nx \exp\left(\frac{p}{n+p}\right)\right)}{I_v(nx)} \exp\left(-\frac{vp}{n+p}\right),$$

$$L_n^v(te^{pt}; x) = \frac{nx}{n+p} \frac{I_{v+1}\left(nx \exp\left(\frac{p}{n+p}\right)\right)}{I_v(nx)} \exp\left(\frac{(1-v)p}{n+p}\right),$$

$$L_n^v(t^2 e^{pt}; x) = \left(\frac{nx}{n+p} \right)^2 \frac{I_{v+2}\left(nx \exp\left(\frac{p}{n+p}\right)\right)}{I_v(nx)} \exp\left(\frac{(2-v)p}{n+p}\right) + \frac{2}{n+p} \frac{nx}{n+p} \frac{I_{v+1}\left(nx \exp\left(\frac{p}{n+p}\right)\right)}{I_v(nx)} \exp\left(\frac{(1-v)p}{n+p}\right).$$

Similarly to Lemma 6 ([1]), using basic properties of modified Bessel function (1), we can prove

Lemma 2.4 For each $v \in R_0$ and $p \in R_+$ there exists a positive constant $M(v, p)$ such that for all $n \in N$ and $z \in R_0$ we have:

$$\frac{I_v\left(z \exp\left(\frac{p}{n+p}\right)\right)}{I_v(z)} \leq M(v, p) \exp\left(z \left(\exp\left(\frac{p}{n+p}\right) - 1 \right) - \frac{p}{2(n+p)}\right).$$

Lemma 2.5 For all $v \in R_0$ and $p \in R_+$ there exists a positive constant $M(v, p)$ such that for each $n \in N$ we have

$$\|L_n^v(1/w_p; \cdot)\|_p \leq M(v, p) \quad (9)$$

Proof. Pick $v \in R_0$ and $p \in R_+$. By definition (2) and Lemma 2.3 we get

$$w_p(x)L_n^v(1/w_p(t); x) = e^{-px} \exp\left(-\frac{vp}{n+p}\right) \frac{I_v\left(nx \exp\left(\frac{p}{n+p}\right)\right)}{I_v(nx)},$$

for $x \in R_0$ and $n \in N$.

Substituting $nx = z$ and applying Lemma 2.4 we get

$$\begin{aligned} w_p(x)L_n^v(1/w_p(t); x) &\leq M(v, p) \exp\left(-\frac{zp}{n}\right) \exp\left(-\frac{vp}{n+p}\right) \exp\left(z\left(\exp\left(\frac{p}{n+p}\right) - 1\right) - \frac{p}{2(n+p)}\right) = \\ &M(v, p) \exp\left(-\left(v + \frac{1}{2}\right)\frac{p}{n+p}\right) \exp\left(z\left(-\frac{p}{n} + \exp\left(\frac{p}{n+p}\right) - 1\right)\right) \leq M(v, p) \end{aligned}$$

because

$$\exp\left(\frac{p}{n+p}\right) - 1 = \sum_{k=1}^{\infty} \left(\frac{p}{n+p}\right)^k \frac{1}{k!} < \sum_{k=1}^{\infty} \left(\frac{p}{n+p}\right)^k = \frac{p}{n},$$

so we have the following estimation:

$$\exp\left(\frac{p}{n+p}\right) - 1 < \frac{p}{n} \quad (10)$$

for $n \in N$. From these inequalities and definition (3) we obtain (9).

An obvious consequence of the above lemma and definition (3) is

Theorem 2.1 For all $v \in R_0$ and $p \in R_+$ there exists a positive constant $M(v, p)$ such that for each $n \in N$ and $f \in E_p$ we have:

$$\|L_n^v(f; \cdot)\|_p \leq M(v, p) \|f\|_p.$$

Now we present the crucial lemma for the approximating theorems in the next section.

Lemma 2.6 For all $v \in R_0$ and $p \in R_+$ there exists a positive constant $M(v, p)$ such that for each $n \in N$ and $x \in R_0$ we have:

$$w_p(x) \left| L_n^v \left(\frac{(t-x)^2}{w_p(t)}; x \right) \right| \leq M(v, p) \frac{x(x+1)}{n} \quad (11)$$

Proof. Let us fix $v \in R_0$ and $p \in R_+$.

By (3), linearity of the operator L_n^v and Lemma 2.3 it follows that:

$$\begin{aligned}
w_p(x) \left| L_n^v \left(\frac{(t-x)^2}{w_p(t)} ; x \right) \right| &= e^{-px} \left| L_n^v(t^2 e^{pt}; x) - 2x L_n^v(t e^{pt}; x) + x^2 L_n^v(e^{pt}; x) \right| = \\
e^{-px} \left| \left(\frac{nx}{n+p} \right)^2 \frac{I_{v+2}(nx \exp(\frac{p}{n+p}))}{I_v(nx)} \exp\left(\frac{p(2-v)}{n+p}\right) + \frac{2nx}{(n+p)^2} \frac{I_{v+1}(nx \exp(\frac{p}{n+p}))}{I_v(nx)} \exp\left(\frac{p(1-v)}{n+p}\right) \right. \\
&\quad \left. - \frac{2nx^2}{n+p} \frac{I_{v+1}(nx \exp(\frac{p}{n+p}))}{I_v(nx)} \exp\left(\frac{p(1-v)}{n+p}\right) + x^2 \frac{I_v(nx \exp(\frac{p}{n+p}))}{I_v(nx)} \exp\left(\frac{-pv}{n+p}\right) \right| = \\
e^{-px} \left(\frac{n}{n+p} \right)^2 \exp\left(\frac{-pv}{n+p}\right) \left| x^2 \frac{I_{v+2}(nx \exp(\frac{p}{n+p}))}{I_v(nx)} \exp\left(\frac{2p}{n+p}\right) + \frac{2x}{n} \frac{I_{v+1}(nx \exp(\frac{p}{n+p}))}{I_v(nx)} \exp\left(\frac{p}{n+p}\right) \right. \\
&\quad \left. - 2x^2 \frac{I_{v+1}(nx \exp(\frac{p}{n+p}))}{I_v(nx)} \exp\left(\frac{p}{n}\right) + x^2 \frac{I_v(nx \exp(\frac{p}{n+p}))}{I_v(nx)} + x^2 \left(\frac{p}{n}\right)^2 \frac{I_v(nx \exp(\frac{p}{n+p}))}{I_v(nx)} \right. \\
&\quad \left. - x^2 \frac{2p}{n} \frac{I_v(nx \exp(\frac{p}{n+p}))}{I_v(nx)} - 2x^2 \frac{p}{n} \frac{I_{v+1}(nx \exp(\frac{p}{n+p}))}{I_v(nx)} \exp\left(\frac{p}{n}\right) \right| \leq \\
e^{-px} \exp\left(\frac{-pv}{n+p}\right) \left(x^2 \left| \frac{\exp\left(\frac{2p}{n+p}\right) I_{v+2}(nx \exp(\frac{p}{n+p}))}{I_{v+1}(nx \exp(\frac{p}{n+p}))} - \exp\left(\frac{p}{n+p}\right) \right| \left| \frac{I_{v+1}(nx \exp(\frac{p}{n+p}))}{I_{v+1}(nx)} \right| \left| \frac{I_{v+1}(nx)}{I_v(nx)} \right| + \right. \\
&\quad \left. x^2 \left| 1 - \frac{\exp\left(\frac{p}{n+p}\right) I_{v+1}(nx \exp(\frac{p}{n+p}))}{I_v(nx \exp(\frac{p}{n+p}))} \right| \left| \frac{I_v(nx \exp(\frac{p}{n+p}))}{I_v(nx)} \right| + \frac{2x}{n} \exp\left(\frac{p}{n+p}\right) \left| \frac{I_{v+1}(nx \exp(\frac{p}{n+p}))}{I_{v+1}(nx)} \right| \left| \frac{I_{v+1}(nx)}{I_v(nx)} \right| \right. \\
&\quad \left. + x^2 \left(\frac{p}{n}\right)^2 \left| \frac{I_v(nx \exp(\frac{p}{n+p}))}{I_v(nx)} \right| + x^2 \frac{2p}{n} \left| 1 - \frac{\exp\left(\frac{p}{n+p}\right) I_{v+1}(nx \exp(\frac{p}{n+p}))}{I_v(nx \exp(\frac{p}{n+p}))} \right| \left| \frac{I_v(nx \exp(\frac{p}{n+p}))}{I_v(nx)} \right| \right).
\end{aligned}$$

Now applying Lemmas 2.1, 2.4, estimation (10) we get (11).

The definition of the operator $L_{n,m}^{v,\mu}$ implies

$$L_{n,m}^{v,\mu}(f; x, y) = L_n^v(f_1; x) L_m^\mu(f_2; y) \quad (12)$$

for all functions of the form $f(x, y) = f_1(x)f_2(y)$ where $f_1 \in E_p$ and $f_2 \in E_q$, $p, q \in R_+$.

In particular we get

$$\begin{aligned}
L_{n,m}^{v,\mu}(1; x, y) &= 1, \\
L_{n,m}^{v,\mu}(1/w_{p,q}; x, y) &= L_n^v(1/w_p; x) L_m^\mu(1/w_q; y).
\end{aligned}$$

From the above facts and Lemma 2.5 we derive

Lemma 2.8 For all $\nu, \mu \in R_0$ and $p, q \in R_+$ there exists a positive constant $M(\nu, \mu, p, q)$ such that for each $n, m \in N$ we have:

$$\left\| L_{n,m}^{\nu,\mu} \left(1/w_{p,q}; \cdot \right) \right\|_{p,q} \leq M(\nu, \mu, p, q).$$

Lemma 2.9 For all $\nu, \mu \in R_0$ and $p, q \in R_+$ there exists a positive constant $M(\nu, \mu, p, q)$ such that for each $n, m \in N$ we have:

$$\left\| L_{n,m}^{\nu,\mu} (f; \cdot) \right\|_{p,q} \leq M(\nu, \mu, p, q) \|f\|_{p,q}.$$

Proof. Applying definition (7), linearity of the operator and connection (12) we get

$$\begin{aligned} w_{p,q}(x,y) \left| L_{n,m}^{\nu,\mu} (f(t,s); x,y) \right| &\leq w_{p,q}(x,y) L_{n,m}^{\nu,\mu} (|f(t,s)|; x,y) = \\ &w_{p,q}(x,y) L_{n,m}^{\nu,\mu} \left(w_{p,q}(t,s) f(t,s) \frac{1}{w_{p,q}(t,s)}; x,y \right) \leq \\ &\|f\|_{p,q} w_p(x) w_q(y) L_n^\nu (1/w_p(t); x) L_m^\mu (1/w_q(s); y) \leq M(\nu, p) M(\mu, q) \|f\|_{p,q} \end{aligned}$$

Hence the operator $L_{n,m}^{\nu,\mu}$ transforms the space $E_{p,q}$ into $E_{p,q}$.

3. Approximation theorems

The following theorem estimates a weighted error of approximation for functions belonging to the space $E_{p,q}^1 = \{f \in E_{p,q} : f' \in E_{p,q}\}$.

Theorem 3.1 For all $\nu, \mu \in R_0$, $p, q \in R_+$ and for each function $g \in E_{p,q}^1$ there exists a positive constant $M(\nu, \mu, p, q)$ such that for all $n, m \in N$ and $(x,y) \in R_0^2$ we have:

$$\begin{aligned} w_{p,q}(x,y) \left| L_{n,m}^{\nu,\mu} (g; x,y) - g(x,y) \right| &\leq \\ M(\nu, \mu, p, q) &\left(\|g'_x\|_{p,q} \frac{x+1}{\sqrt{n}} + \|g'_y\|_{p,q} \frac{y+1}{\sqrt{m}} \right). \end{aligned}$$

The proofs of the above and the next theorems are analogous to the proofs of Theorems 3.1, 3.2 ([2]) so we omit it.

The following theorem gives a degree of approximation of functions by operators $L_{n,m}^{\nu,\mu}$.

Theorem 3.2 For all $\nu, \mu \in R_0$, $p, q \in R_+$ and for each function $f \in E_{p,q}$ there exists a positive constant $M(\nu, \mu, p, q)$ such that for all $n, m \in N$ and $(x,y) \in R_0^2$ we have

$$w_{p,q}(x,y) \left| L_{n,m}^{\nu,\mu} (f; x,y) - f(x,y) \right| \leq M(\nu, \mu, p, q) \omega \left(f, E_{p,q}; \frac{x+1}{\sqrt{n}}, \frac{y+1}{\sqrt{m}} \right).$$

Theorem 3.2 implies the following corollaries.

Corollary 3.3 If $\nu, \mu \in R_0$, $p, q \in R_+$ and $f \in E_{p,q}$ then for all $(x, y) \in R_0^2$

$$\lim_{n,m \rightarrow \infty} L_{n,m}^{\nu,\mu}(f; x, y) = f(x, y).$$

Moreover, the above convergence is uniform on every set $[x_1, x_2] \times [y_1, y_2]$ with $0 \leq x_1 < x_2$, $0 \leq y_1 < y_2$.

Corollary 3.4 For all $\alpha, \beta \in (0, 1]$, $\nu, \mu \in R_0$, $p, q \in R_+$ and for each $f \in \text{Lip}(E_{p,q}^{\nu,\mu}, \alpha, \beta)$ there exists a positive constant $M(\nu, \mu, p, q)$ such that for all $n, m \in N$ and $(x, y) \in R_0^2$ we have:

$$w_{p,q}(x, y) |L_{n,m}^{\nu,\mu}(f; x, y) - f(x, y)| \leq M(\nu, \mu, p, q) \left(\left(\frac{x+1}{\sqrt{n}} \right)^\alpha + \left(\frac{y+1}{\sqrt{m}} \right)^\beta \right),$$

$$\text{where } \text{Lip}(E_{p,q}, \alpha, \beta) = \left\{ f \in E_{p,q} : \omega(f, E_{p,q}; t, s) = O(t^\alpha + s^\beta) \right\}.$$

References

- [1] Herzog M., *Approximation of functions from exponential weight spaces by operators of Szasz-Mirakyan type*, Comment. Math. 43, 1, 2003, 77-94.
- [2] Herzog M., *Approximation of functions of two variables by modified Szasz-Mirakyan operators*, Comment. Math. 52, 1, 2012, 3-9.
- [3] Rempulska L., Skorupka M., *On some operators in weighted spaces of functions of two variables*, Ricerche Mat. 48, 1, 1999, 1-20.
- [4] Wachnicki E., *Approximation by bivariate Mazhar-Totik operators*, Comment. Math. 50, 2, 2010, 141-153.
- [5] Walczak Z., *On certain modified Szasz-Mirakjan operators for functions of two variables*, Demonstratio Math. 33, 1, 2000, 91-100.
- [6] Watson G.N., *Theory of Bessel functions*, Cambridge Univ. Press, Cambridge 1966.