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LWÓW PERIOD OF S. ULAM'S MATHEMATICAL CREATIVITY

LWOWSKI OKRES TWÓRCZOŚCI MATEMATYCZNEJ S. ULAMA

Abstract

We provide an outline of Stanisław Ulam's results obtained in the framework of the widely understood Lvov school of mathematics.

Keywords: Stanisław Ulam, Lvov School of mathematics, Borsuk-Ulam theorem, Kuratowski-Ulam theorem, topological groups and semigroups

Streszczenie

W artykule przedstawiono zarys wyników Stanisława Ulama, które uzyskał w ramach szeroko rozumianej działalności Lwowskiej Szkoły Matematycznej.

Słowa kluczowe: Stanisław Ulam, Lwowska Szkoła Matematyczna, twierdzenie Borsuka-Ulana, twierdzenie Kuratowskiego-Ulana, grupy i półgrupy topologiczne

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Lvov period of the life of outstanding Polish mathematician Stanislaw Ulam lasted from his birth (1909) until 1939, when he moved to the United States. The aim of this note is to provide a review of Ulam's mathematical results obtained in Lvov.



Fig. 1. Stanislaw Ulam (1909–1984)

We will start with a short biographical information. Born in a Jewish family in Lvov, Ulam entered the Polytechnical school in 1927. As a student, he was influenced by K. Kuratowski and started to work in set theory.

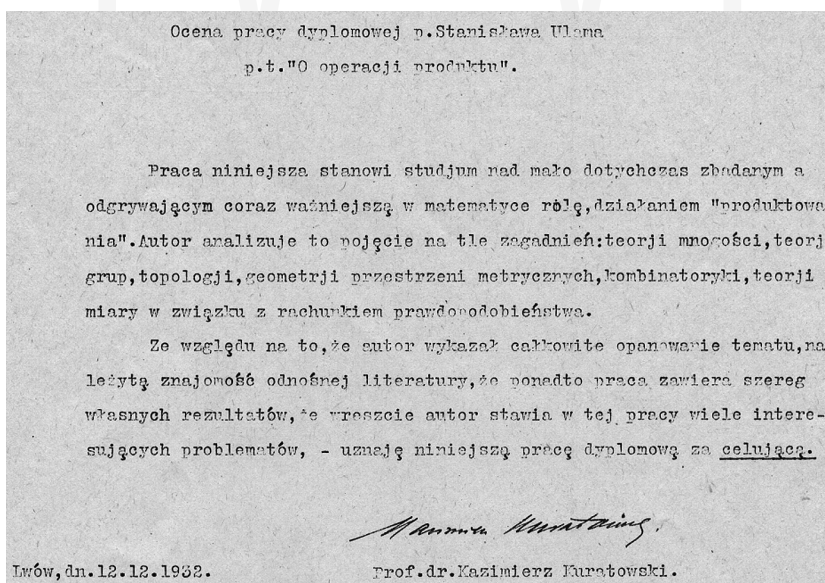


Fig. 2. Kuratowski's report on Ulam's diploma thesis

His first mathematical paper was published in *Fundamenta Mathematicae* in 1929. It contains one of Ulam's results obtained in students years. In his report about Ulam's diploma thesis, Kuratowski remarked that the author provides an analysis of the notion of product on the background of set theory, group theory, topology, geometry of metric spaces, combinatorics, measure theory in connection to probability theory.

In 1933 Ulam earned his PhD at the Lvov Polytechnical School (now Lviv Polytechnical University).

He was one of the most active members of the Lvov School of mathematics. Ulam participated in sessions of Lvov mathematicians in the Scottish café (Kawiarnia Szkocka, in Polish) and contributed to the Scottish book, a list of open problems formulated by these mathematicians.



Fig. 3. Modern picture of the Scottish café; it is again named “Szkocka”

In 1935, Ulam accepted J. von Neumann's invitation and moved to Princeton. According to his published memoirs, Ulam spent in Lvov all the summers in the years 1936–39. When in Lvov, he contacted local mathematicians and continued collaborating with them. However, his joint paper with Oxtoby [19] gives Cambridge, USA, as his affiliation.

In 1944, Ulam joined the Manhattan project. Before that he worked for the University of Wisconsin at Madison. After the war, Ulam became a professor at the University of Southern California in Los Angeles. In 1946 he returned to Los Alamos, where he kept his position until 1965. During these years Ulam was a visiting professor at various universities: Harvard, MIT, the University of California, San Diego, the University of Colorado at Boulder. Since 1967.

Ulam was a Graduate Research Professor at the University of Florida in the period 1974–1984. In 1998/99, the Department of Mathematics at the University of Florida initiated the Ulam colloquium in memory of Stanisław Ulam.

Ulam died in 1984 in Santa Fe.

There is vast literature devoted to Stanisław Ulam and his mathematical achievements (see, e.g. [7, 8, 18]). We believe, however, that the material of the present note, which concentrates on the Lvov period of activity, is not completely covered by other publications.

Borsuk-Ulam theorem. This well-known result was conjectured by Ulam and proved by Karol Borsuk [4]. It states that for any continuous map of an n -dimensional sphere into an n -dimensional Euclidean space there are two antipodal points with coinciding images.

Satz II⁷⁾. *Ist $f \in R^{nS_n}$ (d. h. bildet f die Sphäre S_n auf einen Teil von R^n ab), so gibt es einen derartigen Punkt $p \in S_n$, dass $f(p) = f(p^*)$ ist.*

⁷⁾ Dieser Satz wurde als Vermutung von S. t. Ulam aufgestellt.

Fig. 4. Fragments of Borsuk's paper

The Borsuk-Ulam theorem is known to be equivalent to the Brouwer fixed point theorem as well as other statements. It has numerous variations as well as applications in topology and other parts of mathematics (see the book [16]). In particular, N. Alon [1] and D.B. West [2] applied the Borsuk-Ulam theorem and its generalization to solving the so called Necklace splitting problem. In [25] it is shown how this theorem can be used in constructing consensus halving, i.e. division of an object into two parts so that n people believe that these parts are equal.

T. Banach and I. Protasov [3] used this theorem for finding the number of decomposition of Abelian groups into non-symmetric parts.

Kuratowski-Ulam theorem. In [13] Kuratowski and Ulam proved a theorem which is a counterpart of the Fubini theorem for categories: if C is a set of the first category in the product of Baire spaces X and Y (the space Y is supposed to be separable), then the set $C \cap (x \times Y)$ is of the first category in $x \times Y$ for all x except the set of the first category.

These results of [13] were generalized in subsequent papers. In particular, in [10] the notion of Kuratowski-Ulam pair is introduced. Namely, such is a pair of spaces (X, Y) for which the conclusion of the Kuratowski-Ulam theorem holds. A space Y is a universally Kuratowski-Ulam space if (X, Y) is a Kuratowski-Ulam pair for every space X . The article [10] is devoted to properties of universal Kuratowski-Ulam spaces.

Measurable cardinals. In 1930, in his paper [26] published in *Fundamenta Mathematicae*, Ulam introduced the notion of a measurable cardinal. An uncountable cardinal number κ is measurable if there exists a κ -additive, non-trivial, 0-1-valued measure on the power set of κ . Here, κ -additivity means that the measure of the union of any disjoint family of cardinality $< \kappa$ is the sum of measures of its members. It is known that any measurable cardinal is inaccessible. The theory of large cardinals is now an important part of the set theory.

In [26] Ulam also introduced an object which is now called the Ulam matrix.

Remark that some consequences of one result of [26] are derived in [27]. In particular, it is proved in the latter paper that in every set of positive outer (Lebesgue) measure there exists an uncountable disjoint family of subsets of positive outer measure.

Topological groups. The theory of topological groups and semigroups was also developed in the Lvov school of mathematics. Stanisław Ulam published few papers

in this direction; most of them were coauthored by Józef Schreier. As Ulam remarked, the published joint paper were a result of collaboration taking place almost every day.

A basis of a topological (semi)group X is a subset S of X such that the sub(semi)group generated by S is dense in X . It is proved in [22] that the semigroup $C([0, 1]^m)$ of the continuous selfmaps of the m -dimensional cube $[0, 1]^m$, $m \geq 1$, has a basis consisting of five elements.

In [23] the following statement is proved: almost every (up to a set of the first category) pair of elements of a compact metrizable connected group is a basis of this group.

In [20] Ulam and Schreier consider the topological group S_∞ of bijections of the set N of natural numbers. The topology on S_∞ is generated by a complete metric. It is proved that a proper normal subgroup of S_∞ consists of finite permutations (i.e. those moving only finite number of elements of N). In other words, the group S_∞ is topologically simple.

One of the main results of the paper is finding a basis in S_∞ that consists of three elements. This paper is supplemented by a short note [24] in which it is shown that there is no outer automorphism of S_∞ .

The paper [21] deals with auto-homeomorphisms of the circle. One of the result of this note is that the group of order-preserving autohomeomorphisms is simple.

Some additional information concerning the activity of Józef Schreier in the theory of topological (semi)groups can be found in [11].

Borsuk-Ulam functor and geometric topology. In the paper [5] K. Borsuk and S. Ulam considered the subset $X(n)$ of nonempty subsets of cardinality $\leq n$ in a metric space X . The set $X(n)$ is endowed with the Hausdorff metric. Although Borsuk and Ulam did not notice this explicitly, the construction $X(n)$ determines a functor in the category of metric spaces and continuous maps.

Let $I = [0, 1]$. It is proved in the mentioned paper that $I(n)$ is an n -dimensional, locally connected Cantor manifold. Also, $I(n)$ is homeomorphic to the space I^n , for $n = 1, 2, 3$, while for $n > 3$ the space $I(n)$ cannot be embedded into I^n . Some formulated problems concerning spaces $X(n)$ were solved by another authors. In particular, Jaworowski [12] considered the problem of preserving ANRs by the functor of symmetric product. His proof, however, contained a gap. This was first noticed by V. Fedorchuk [9] who found a correct proof that uses the theory of Q -manifolds.

In the joint paper [14] with K. Kuratowski the authors study the number $\tau(A, B)$, where A, B are compact metric spaces, defined as follows:

$$\tau(A, B) = \min_{f(A)=B} \max_{f(x)=f(y)} d(x, y),$$

where d denotes the metric on A . If \mathbf{B} is a class of metric spaces, then $\tau(A, B)$ denotes the infimum of $\tau(A, B)$, for all spaces B in \mathbf{B} . For an n -dimensional space A and the class \mathbf{B} of smaller dimension the number $\tau(A, B)$ is proved to coincide with the Urysohn constant. Also if A is a non-unicoherent continuum and \mathbf{B} consists of unicoherent continua, then $\tau(A, B)$ is proved to be positive.

In [6], some invariants of maps with small preimages are found. In particular, it is shown that the non-contractibility of a map into a given ANR-space is such an invariant of a compact metric space.

Remarks. We do not pretend to be complete. In particular, we did not mention above the joint paper with Łomnicki [15] containing important results (although with an erroneous proof) concerning products of probability measures (e.g. [8]).

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