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GRAPHS WITH EVERY PATH OF LENGTH k IN A HAMILTONIAN CYCLE

GRAFY Z DOWOLNĄ ŚCIEŻKĄ DŁUGOŚCI k ZAWARTĄ W PEWNYM CYKLU HAMILTONOWSKIM

Abstract

In this paper we prove that if G is a $(k+2)$ -connected graph on $n \geq 3$ vertices satisfying $P(n+k)$:

$$d_G(x, y) = 2 \Rightarrow \max\{d(x), d(y)\} \geq \frac{n+k}{2}$$

for each pair of vertices x and y in G , then any path $S \subset G$ of length k is contained in a hamiltonian cycle of G .

Keywords: cycle, graph, hamiltonian cycle, matching, path

Streszczenie

W pracy udowodniono, że w $(k+2)$ -spójnym grafie G o $n \geq 3$ wierzchołkach, który spełnia warunek $P(n+k)$:

$$d(x, y) = 2 \Rightarrow \max\{d(x), d(y)\} \geq \frac{n+k}{2}$$

dla dowolnej pary wierzchołków x i y , każda ścieżka $S \subset G$ długości k jest zawrta w pewnym cyklu hamiltonowskim grafu G .

Słowa kluczowe: cykl, cykl hamiltonowski, graf, skojarzenie, ścieżka

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1. Introduction

We consider only finite graphs without loops and multiple edges. By V or $V(G)$ we denote the vertex set of the graph G and respectively by E or $E(G)$, the edge set of G . By $d_x(G)$ or $d(x)$, we denote *the degree of a vertex x in the graph G* and by $d(x, y)$ or $d_G(x, y)$, *the distance between x and y in G .*

Definition 1.1 (cf [10]). *Let k, s_1, \dots, s_ℓ be positive integers. We call S a path system of length k , if the connected components of S are paths:*

$$\begin{aligned} P^1 : & \quad x_0^1 x_1^1 \dots x_{s_1}^1, \\ & \quad \vdots \\ P^\ell : & \quad x_0^\ell x_1^\ell \dots x_{s_\ell}^\ell \end{aligned}$$

and $\sum_{i=1}^{\ell} s_i = k$.

Let S be a path system of length k and let $x \in V(S)$. We shall call x an *internal vertex* if x is an internal vertex (cf [3]) in one of the paths P^1, \dots, P^ℓ .

If q denotes the number of internal vertices in a path system S of length k then $0 \leq q \leq k - 1$. If $q = 0$, then S is a *k -matching* (i.e. a set of k independent edges).

Let H be a subgraph of G . By $G \setminus H$ we denote the graph obtained from G by the deletion of the edges of H .

Definition 1.2. *The graph F is said to be an H -edge cut-set of G if $F \subset E(H)$ and $G \setminus F$ is not connected.*

Definition 1.3. *The graph F is said to be a minimal H -edge cut-set of G if F is an H -edge cut-set of G which has no proper subset being an edge cut-set of G .*

Definition 1.4 (cf [7]). *Let G be a graph on $n \geq 3$ vertices and $k \geq 0$. G is *k -edge-hamiltonian* if for every path system P of length at most k there exists a hamiltonian cycle of G containing P .*

Let G be a graph and $H \subset G$ a subgraph of G . For a vertex $x \in V(G)$, we define the set $N_H(x) = \{y \in V(H) : xy \in E(G)\}$. Let H and D be two subgraphs of G . $E(D, H) = \{xy \in E(G) : x \in V(D) \text{ and } y \in V(H)\}$. For a set of vertices A of a graph G , we define the graph $G(A)$ as the subgraph induced in G by A . In the proof, we will only use oriented cycles and paths. Let C be a cycle and $x \in V(C)$, then x^- is the predecessor of x and x^+ is its successor.

Definition 1.5 (cf [2]). *Let W be a property defined for all graphs of order n and let k be a non-negative integer. The property W is said to be *k -stable* if whenever $G + xy$ has property W and $d(x) + d(y) \geq k$ then G itself has property W .*

J.A. Bondy and V. Chvátal [2] proved the following theorem, which we shall need in the proof of our main result:

Theorem 1.1. *Let n and k be positive integers with $k \leq n - 3$. Then the property of being k -edge-hamiltonian is $(n + k)$ -stable.*

In 1960, O. Ore [9] proved the following:

Theorem 1.2. *Let G be a graph on $n \geq 3$ vertices. If for all nonadjacent vertices $x, y \in V(G)$ we have*

$$d(x) + d(y) \geq n$$

then G is hamiltonian.

Geng-Hua Fan [4] has shown:

Theorem 1.3. *Let G be a 2-connected graph on $n \geq 3$ vertices. If G satisfies*

$$P(n) : \quad d(x, y) = 2 \Rightarrow \max\{d(x), d(y)\} \geq \frac{n}{2}$$

for each pair of vertices x and y in G , then G is hamiltonian.

The condition for degree sum in Theorem 1.2 is called an *Ore condition* or an *Ore type condition* for graph G and the condition $P(k)$ is called a *Fan condition* or a *Fan type condition* for graph G .

Later, many *Fan type theorems* and *Ore type theorems* are shown.

Now we shall present Las Vergnas [8] condition $\mathcal{L}_{n,s}$.

Definition 1.6. *Let G be graph on $n \geq 2$ vertices and let s be an integer such that $0 \leq s \leq n$. G satisfies Las Vergnas condition $\mathcal{L}_{n,s}$ if there is an arrangement x_1, \dots, x_n of vertices of G such that for all i, j if*

$$1 \leq i < j \leq n, \quad i + j \geq n - s, \quad x_i x_j \notin E(G),$$

$$d(x_i) \leq i + s \quad \text{and} \quad d(x_j) \leq j + s - 1$$

then $d(x_i) + d(x_j) \geq n + s$.

Las Vergnas [8] proved the following theorem:

Theorem 1.4. *Let G be a graph on $n \geq 3$ vertices and let $0 \leq s \leq n - 1$. If G satisfies $\mathcal{L}_{n,s}$ then G is s -edge hamiltonian.*

Note that condition $\mathcal{L}_{n,s}$ is weaker than Ore condition.

Later Skupień and Wojda proved that the condition $\mathcal{L}_{n,s}$ is sufficient for a graph to have a stronger property (for details see [10]). Wojda [11] proved the following Ore type theorem:

Theorem 1.5. *Let G be a graph on $n \geq 3$ vertices, such that for every pair of nonadjacent vertices x and y*

$$d(x) + d(y) > \frac{4n - 4}{3}.$$

Then every matching of G lies in a hamiltonian cycle.

In 1996, G. Gancarzewicz and A. P. Wojda proved the following Fan type theorem:

Theorem 1.6. *Let G be a 3-connected graph of order $n \geq 3$ and let M be a k -matching in G . If G satisfies $P(n + k)$:*

$$d(x, y) = 2 \Rightarrow \max\{d(x), d(y)\} \geq \frac{n + k}{2}$$

for each pair of vertices x and y in G , then M lies in a hamiltonian cycle of G or G has a minimal odd M -edge cut-set.

In this paper we find a Fan type condition under which every path of length k in a graph G lies in a hamiltonian cycle.

For notation and terminology not defined above a good reference should be [3].

2. Result

Theorem 2.1. *Let G be a graph on $n \geq 3$ vertices and let S be a path of length k in G . If the graph G is l -connected, where $l = \min\{k + 2, n - 1\}$ and satisfies $P(n + k)$:*

$$d(x, y) = 2 \Rightarrow \max\{d(x), d(y)\} \geq \frac{n + k}{2} \tag{2.1}$$

for each pair of vertices x and $y \in V(G)$, then S lies in a hamiltonian cycle of G .

Note that $1 \leq k \leq n - 1$ and since an $(n - 1)$ -connected graph of order n is a complete graph K_n , which is obviously k -edge hamiltonian for any k the result is interesting when $k < n - 3$.

For $k = 1$, the path S is a 1-matching and we have a special case of Theorem 1.6 (the graph is 3-connected, so in this case we can not have a minimal S -edge cut set).

Unfortunately, in Theorem 2.1 we can not decrease the connectivity of graph G . We can consider a vertex x and a complete graph K_m , $m \geq 3$. In the complete graph K_m we choose a path $S : s_1 \dots s_{k+1}$ of length $k = m - 2$. There is only one vertex $y \in K_m$ not contained in S .

Let G be a graph of order $n = m + 1$ obtained from two complete graphs $K_1 = \{x\}$ and K_m by adding edges xs_i , for $i \in \{1, \dots, k + 1\}$ The path S is a path of length k contained in G which is not contained in any hamiltonian cycle of the $(k + 1)$ -connected graph G , see Figure (1).

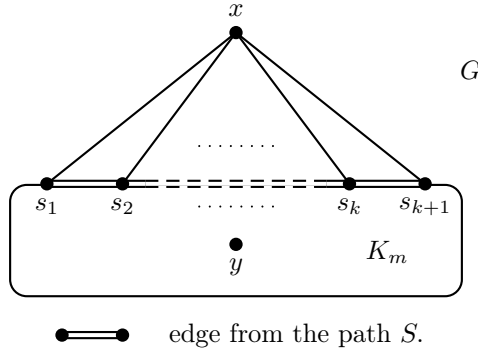


Fig. 1: A $(k + 1)$ -connected graph G with no hamiltonian cycle through the path $S : s_1 \dots s_{k+1}$.

Note that we can replace the vertex x by a complete graph K_ℓ , $\ell \geq k + 1$. Let $\{x_1, \dots, x_{k+1}\} \subset V(K_\ell)$ and let G be a graph of order $n = m + l$ obtained from two complete graphs K_ℓ and K_m by adding edges $x_i s_i$, for $i \in \{1, \dots, k + 1\}$. The path S is a path of length k contained in G which is not contained in any hamiltonian cycle of the $(k + 1)$ -connected graph G , see Figure (2).

3. Proof

Proof of Theorem 2.1:

Take G and S as in the assumptions of Theorem 2.1.

Consider the nonempty set

$$A = \left\{ x \in V(G) : d_x(G) \geq \frac{n + k}{2} \right\}.$$

Note that if x and y are nonadjacent vertices of A , then the graph obtained from G by the addition of the edge xy also satisfies the assumptions of the theorem. Therefore, and by Theorem 1.1 we may assume that:

$$xy \in E(G) \quad \text{for any } x, y \in A \quad \text{and } x \neq y. \quad (3.1)$$

By (3.1), A induces a complete subgraph $G(A)$ of the graph G .

In fact, since the property of being k -edge-hamiltonian is $(n + k)$ -stable, we can replace G with its $(n + k)$ -closure.

Let G_A be a graph obtained from G by deletion of vertices of the graph $G(A)$ (i.e. vertices from the set A).

Now take D , a connected component of the graph G_A .

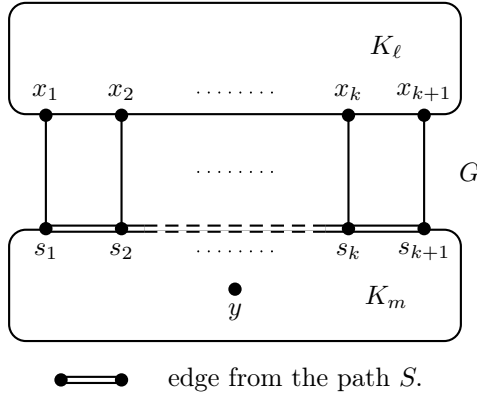


Fig. 2: A $(k + 1)$ -connected graph G with no hamiltonian cycle through the path $S : s_1 \dots s_{k+1}$.

Suppose that there exist two nonadjacent vertices in D . Since D is connected, we have two vertices x and y in D such that $d_G(x, y) = 2$ and by the assumption that G satisfies $P(n + k)$, we have $x \in A$ or $y \in A$, which is a contradiction.

We have proved that every component of G_A is a complete graph K_ℓ , $\ell \in I$, joined with $G(A)$ by at least $k + 2$ edges.

Claim 3.1. *If $K_{\iota_0}, K_{\iota_1} \in \{K_\ell\}_{\ell \in I}$ are such that $\iota_0 \neq \iota_1$, then:*

$$N(K_{\iota_0}) \cap N(K_{\iota_1}) = \emptyset. \tag{3.2}$$

Proof of Claim 3.1:

Suppose that $N(K_{\iota_0}) \cap N(K_{\iota_1}) \neq \emptyset$. Then we have a vertex $y \in K_{\iota_0}$ and a vertex $y' \in K_{\iota_1}$ such that $d_G(y, y') = 2$ and by $P(n + k)$ either $y \in A$ or $y' \in A$. This contradicts the fact that K_{ι_0} and K_{ι_1} are two connected components of G_A . □

We have shown that the graph G consists of a complete graph $G(A)$ and of a family of complete components $\{K_\ell\}_{\ell \in I}$, of G_A , which do not have common neighbors in $G(A)$.

Since G is $(k + 2)$ -connected, we have the following:

Claim 3.2. *Every component $\{K_\ell\}_{\ell \in I}$, is joined with $G(A)$ by at least three edges such that end vertices of these edges are not internal vertices of the path S .*

We label vertices of path $S : s_1 s_2 \dots s_k s_{k+1}$.

Graph G consists of complete graph $G(A)$ and disjointed complete graphs $\{K_\iota\}_{\iota \in I}$, joined with $G(A)$ by at least three edges such that end vertices of these edges are not internal vertices of path S .

Firstly we consider the case when path S is contained in one complete graph (i.e. $G(A)$ or one graph $K_{\iota_0} \in \{K_\iota\}_{\iota \in I}$). In this case, by Claim3.2 we have a hamiltonian cycle through S .

Now we assume that S is not contained in the complete graph $G(A)$ or one graph $K_{\iota_0} \in \{K_\iota\}_{\iota \in I}$ and we can now define a cycle $C \subset G$ containing the path S and all vertices of $G(A)$.

We shall consider four cases:

1. Both end vertices of S are in $G(A)$ i.e. $s_1, s_{k+1} \in G(A)$.
2. Both end vertices of S are in the same component K_ι of G_A i.e. $s_1, s_{k+1} \in K_\iota$.
3. End vertices of S are in different components of G_A i.e. $s_1 \in K_{\iota_1}, s_{k+1} \in K_{\iota_2}, K_{\iota_1}, K_{\iota_2} \in \{K_\iota\}_{\iota \in I}$ are such that $\iota_1 \neq \iota_2$.
4. One end vertex of S is in $G(A)$ and the other end vertex is in a component K_ι of G_A . In this case, we can assume without loss of generality that $s_1 \in G(A)$ and $s_{k+1} \in K_\iota$.

If $C \subset G$ is a cycle in G , then by $G_{V \setminus C}$ we denote a graph obtained from G by deletion of vertices of cycle C .

Case 1: Both end vertices of S are in $G(A)$ i.e. $s_1, s_{k+1} \in G(A)$.

Note that even in this case, path S may pass through some components K_ι creating a kind of ears of the complete graph $G(A)$, on every incident graph K_ι . We can find an example of such ears on Figure (3).

Since $G(A)$ is a complete graph we have a cycle C containing the path S and all vertices of $G(A)$ performing the following conditions:

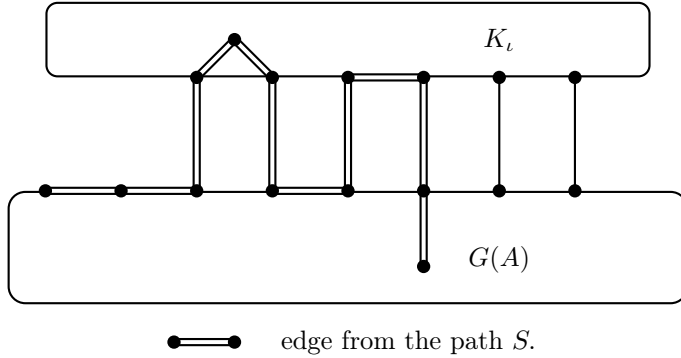


Fig. 3: Example of ears of the graph $G(A)$.

- C contains all edges of $E(S) \cap E(G(A))$ and all vertices of A . (3.3)

- If K_{ι_0} and K_{ι_1} are two different components of $G_V \setminus C$ then (3.4)

$$N(K_{\iota_0}) \cap N(K_{\iota_1}) = \emptyset.$$

- Let $x \notin V(C)$, $y \in V(C)$ and $xy \in E(G)$ then: (3.5)

if y is not an internal vertex of S , then $y \in A$,

if y^- is not an internal vertex of S , then $y^- \in A$,

if y^+ is not an internal vertex of S , then $y^+ \in A$.

Properties (3.3 — 3.5) will allow us to extend C to a hamiltonian cycle.

Note that this cycle C may not be contained in $G(A)$.

Subcase 1.1: Extending the cycle C through components K_ι incident with ears

Since graph G is $(k+2)$ -connected, we have at least $k+2$ edges joining K_ι with $G(A)$, so at least one of these edges say uc_i , $u \in V(K_\iota)$, $c_i \in V(G(A)) \setminus S$, is not incident with S . If a component K_ι is incident with an ear, at least one interior vertex of path S is contained in this ear, and we have an additional edge say $u'c_j$, $u' \in V(K_\iota)$, $c_j \in V(G(A)) \setminus S$, not incident with S joining K_ι with $G(A)$. Using these two edges uc_i and $u'c_j$, we can extend the cycle C through the remaining vertices of K_ι .

Without loss of generality, we can assume that on cycle C , the vertices are ordered in the following way: $s_1 \dots s_{k+1}c_{k+2} \dots c_i \dots c_j \dots s_1$.

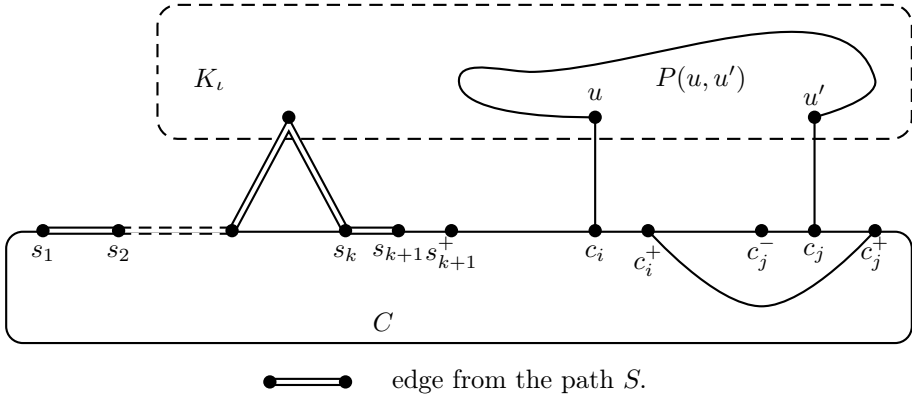


Fig. 4: Extension of the cycle C through component K_L incident with an ear.

We can replace cycle C , by the following cycle C'

$$C' : s_1 \dots s_{k+1} s_{k+1}^+ \dots c_i P(u, u') c_j c_j^- \dots c_i^+ c_j^+ \dots s_1,$$

where $P(u, u') \subset K_L$ is a path joining u with u' containing all vertices of $K_L \setminus S$. Note that this cycle C' satisfies conditions (3.3 — 3.5). We can find an example of the cycle C' on Figure 4.

Case 2: Both end vertices of S are in the same component K_L of G_A i.e. $s_1, s_{k+1} \in K_L$.

Since the graph G is $(k+2)$ -connected, we have at least $k+2$ edges joining K_L with $G(A)$. In this case, at least two edges from the path $S : s_i s_{i+1}$ and $s_j s_{j+1}$ are joining K_L with $G(A)$, so at least two independent edges $uv, u'v', u, u' \in V(K_L), v, v' \in V(G(A))$, not incident with S joining K_L with $G(A)$.

Consider the following path:

$$P : v u s_1 \dots s_k P(s_{k+1}, u') v',$$

where $P(s_{k+1}, u') \subset K_L$ is a path joining s_{k+1} with u' containing all vertices of $K_L \setminus \{V(S) \cup \{u\}\}$. We can find an example of the path P on Figure 5.

The graph $G(A)$ is complete, so we can extend P to a cycle C containing all vertices of $G(A)$ and satisfying (3.3 — 3.5).

Note that as in Case 1 the path S may pass through some components K_i creating the kind of ears of the complete graph $G(A)$. Using the same argument as in the

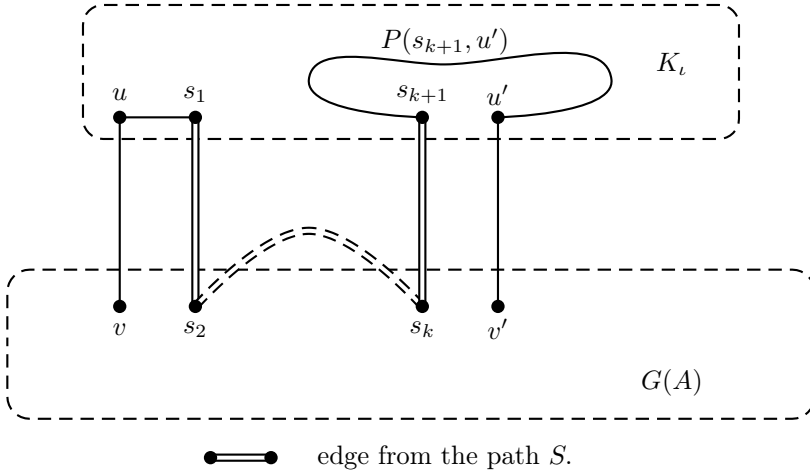


Fig. 5: Path P containing S with both end vertices in $G(A)$.

Subcase 1.1 we can extend the cycle C through components K_i incident with ears preserving the properties (3.3 — 3.5).

Case 3: End vertices of S are in different components of G_A i.e. $s_1 \in K_{l_1}$, $s_{k+1} \in K_{l_2}$, $K_{l_1}, K_{l_2} \in \{K_l\}_{l \in I}$ are such that $l_1 \neq l_2$.

Again, since graph G is $(k+2)$ -connected we have at least $k+2$ edges joining every component K_i with $G(A)$. In this case, for $i = l_1$ and $i = l_2$ at least one edge from the path S is joining K_i with $G(A)$, so we have at least two independent edges uv , $u \in V(K_{l_1})$, $v \in V(G(A))$, $u'v'$, $u' \in V(K_{l_2})$, $v' \in V(G(A))$, not incident with S joining respectively K_{l_1} and K_{l_2} with $G(A)$.

Consider the following path:

$$P : vP_1(u, s_1)s_2 \dots s_kP_2(s_{k+1}, u')v',$$

where $P_1(u, s_1) \subset K_{l_1}$ is a path joining u with s_1 containing all vertices of $K_{l_1} \setminus \{V(S) \cup \{u\}\}$ and $P_2(s_{k+1}, u') \subset K_{l_2}$ is a path joining s_{k+1} with u' containing all vertices of $K_{l_2} \setminus \{V(S) \cup \{u'\}\}$. See Figure 6.

The graph $G(A)$ is complete so we can extend P to a cycle C containing all vertices of $G(A)$ and satisfying (3.3 — 3.5).

Note that as in Case 1 path S may pass through several components K_i creating the kind of ears of the complete graph $G(A)$. Using the same argument as in Subcase

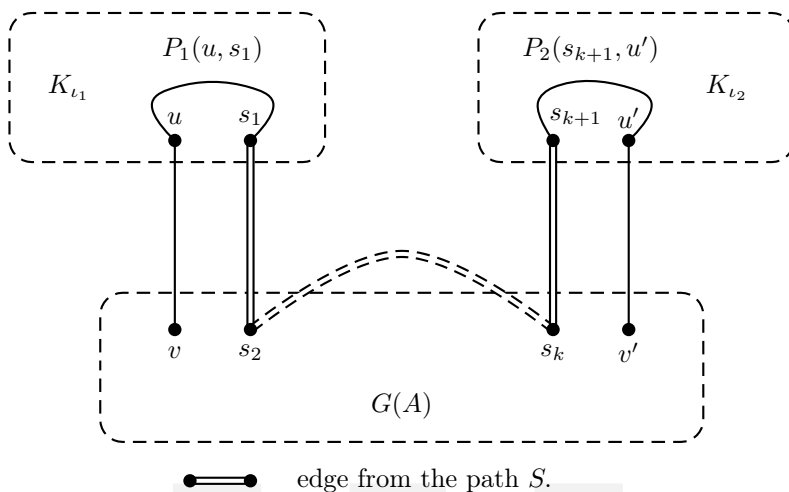


Fig. 6: Path P containing S with both end vertices in $G(A)$.

1.1, we can extend cycle C through components K_i incident with ears preserving the properties (3.3 — 3.5).

Case 4: One end vertex of s is in $G(A)$ and the other end vertex is in a component K_l of G_A . In this case, we can assume without loss of generality, that $s_1 \in G(A)$ and $s_{k+1} \in K_l$.

Since graph G is $(k+2)$ -connected, we have at least $k+2$ edges joining the component K_l with $G(A)$. In this case, at least one edge from the path S is joining K_l with $G(A)$, so we have at least one edge uv , $u \in V(K_l)$, $v \in V(G(A))$, not incident with S joining K_l with $G(A)$.

Consider the following path:

$$P: s_1 s_2 \dots s_k P(s_{k+1}, u) v,$$

where $P(s_{k+1}, u) \subset K_{l_0}$ is a path joining s_{k+1} with u containing all vertices of $K_l \setminus \{V(S) \cup \{u\}\}$, see Figure 7.

Both v and s_1 are in $G(A)$ and the graph $G(A)$ is complete, so we can extend P to a cycle C containing all vertices of $G(A)$ and satisfying (3.3 — 3.5).

Note that as in Case 1, path S may pass through several components K_i creating the kind of ears of the complete graph $G(A)$. Using the same argument as in Subcase

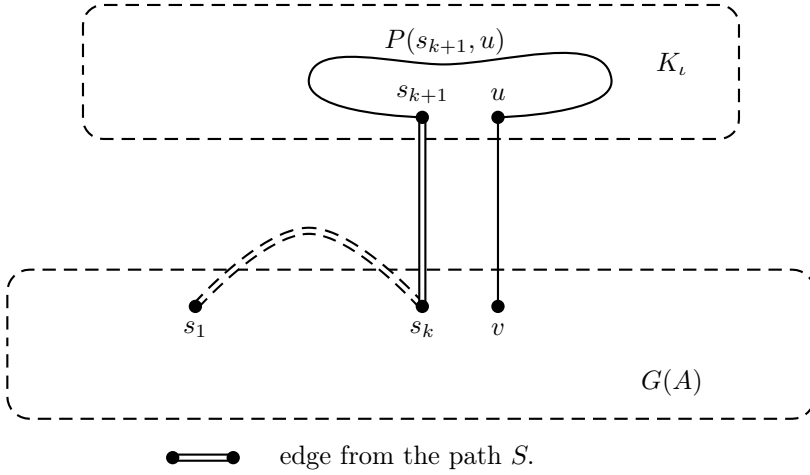


Fig. 7: Path P containing S with both end vertices in $G(A)$.

1.1, we can extend the cycle C through components K_i incident with ears preserving the properties (3.3 — 3.5).

In all cases we have defined a cycle C containing S and now we shall extend this cycle to a hamiltonian cycle.

Extending the cycle C to a hamiltonian cycle

We have already a cycle C satisfying conditions (3.3 — 3.5) and containing the path S , all vertices of $G(A)$, all vertices from components K_i containing vertices of the path S .

Consider component K_l not included in cycle C . This component does not contain any edge from S and since the graph G is $(k+2)$ -connected we have at least $k+2$ edges joining K_l with $G(A)$, so at least one of these edges say $uc_i, u \in V(K_l), c_i \in V(G(A))$, is not incident with S and at least one edge say $u'c_j, u' \in V(K_l), c_j \in V(G(A)) \setminus S$, not incident with internal vertices of S , joining K_l with $G(A)$. In the worst case $c_j = s_1$ or $c_j = s_{k+1}$.

Using these two edges uc_i and $u'c_j$, we can extend cycle C through the remaining vertices of K_l .

We consider the case $c_j = s_1$ and without loss of generality we can, assume that on cycle C , the vertices are ordered in the following way:

$$s_1 \dots s_{k+1} c_{k+2} \dots c_i \dots s_1 .$$

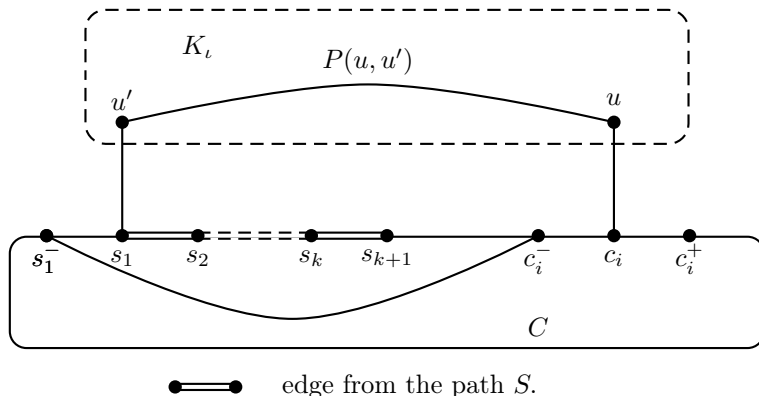


Fig. 8: Extension of cycle C through a component K_l not incident with S .

Note that since uc_i is not incident with S we have $c_i^-c_i, c_ic_i^+ \notin E(S)$ and we can replace cycle C with the following cycle C'

$$C' : u's_1 \dots s_{k+1}s_{k+1}^+ \dots c_i^-s_1^- \dots c_i^+c_iP(u, u'),$$

where $P(u, u') \subset K_l$ is a path joining u with u' containing all vertices of K_l , see Figure 8.

Note that this cycle C' satisfies conditions (3.3 — 3.5), $V(C) \subset V(C')$ and $E(S) \subset E(C')$.

The case $c_j = s_{k+1}$ is similar.

Applying this argument for all other components K_l we can extend C to a hamiltonian cycle containing the path S and the proof is complete. \square

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