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Localizability of tachyonic particles and neutrinoless double beta decay

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Abstract The quantum field theory of superluminal (tachyonic) particles is plagued by a number of problems, which include the Lorentz non-invariance of the vacuum state, the ambiguous separation of the field operator into creation and annihilation operators under Lorentz transformations, and the necessity of a complex reinterpretation principle for quantum processes. Another unsolved question concerns the treatment of subluminal components of a tachyonic wave packet in the field-theoretical formalism, and the calculation of the time-ordered propagator. After a brief discussion on related problems, we conclude that rather painful choices have to be made in order to incorporate tachyonic spin-$\frac{1}{2}$ particles into field theory. We argue that the field theory needs to be formulated such as to allow for localizable tachyonic particles, even if that means that a slight unitarity violation is introduced into the S matrix, and we write down field operators with unrestricted momenta. We find that once these choices have been made, the propagator for the neutrino field can be given in a compact form, and the left-handedness of the neutrino as well as the right-handedness of the antineutrino follow naturally. Consequences for neutrinoless double beta decay and superluminal propagation of neutrinos are briefly discussed.

1 Introduction and overview

1.1 Tachyonic quantum mechanics

After the early attempts by Sudarshan et al. (Refs. [1–4]) and Feinberg (Refs. [5, 6]), tachyonic field theory has been scrutinized because a number of problematic issues were discovered and discussed at length in the literature [6–14]. Insightful reviews on the history of tachyonic quantum dynamics and tachyonic quantum field theory are given in Refs. [15–17]. *A priori*, Lorentz invariance does not imply that velocities greater than the speed of light are forbidden in nature. The Lorentz transformation singles out the speed of light as a limiting velocity, not as the maximum velocity allowed in nature. Indeed, superluminal Lorentz transformations have been discussed in some detail in the literature (e.g., in Refs. [1, 18]). A superluminal particle cannot be stopped, its velocity always remains greater than the speed of light, when measured from subluminal frames of reference. The Einstein addition theorem remains valid for superluminal particles. The velocity u' of a superluminal particle measured in a moving frame, $u' = (u - v)/(1 - u\,v)$, always is superluminal, $u' \notin (-1, 1)$, if $u > 1$ is superluminal in the rest frame, and the relative velocity of the frames fulfills $-1 < v < 1$, as a quick inspection shows. (We set the speed of light $c = 1$ in this article.) Tachyons are particles whose dispersion relation reads

$$E = \frac{m}{\sqrt{u^2 - 1}}, \qquad p = \frac{m\,u}{\sqrt{u^2 - 1}}, \qquad E^2 - p^2 = -m^2, \tag{1}$$

where $u = |\boldsymbol{u}| > 1$ is the (magnitude of the) velocity of the particle. The "rest frame" of a tachyon is the frame of infinite velocity, where the energy $E \to 0$ [5, 12]. Provided the mass term m is not in itself energy-dependent, a tachyon becomes faster as it loses energy, by virtue of the Lorentz-invariant dispersion relation (1). Note that $p = |\boldsymbol{p}| = E\ u > E$, because $u > 1$. While the relation $E^2 - p^2 = -m^2$ is obtained from the usual (tardyonic) dispersion relation $E^2 - p^2 = m^2$ by the simple replacement $m \to \mathrm{i}m$, the equations describing tachyonic particles do *not* necessarily follow from the wave equations for subluminal particles by the same, simple replacement (see also Appendix A). Rather, the equation

$$\left(\mathrm{i}\gamma^\mu \partial_\mu - \gamma^5 m\right)\psi(x) = 0 \tag{2}$$

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has been proposed for tachyonic spin-$\frac{1}{2}$ particles [19–22]. Here, the γ^μ are the Dirac matrices, and $\partial_\mu = \partial/\partial x^\mu$ is the derivative with respect to space-time coordinates, whereas $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ is the fifth current matrix. In Hamiltonian form, the equation reads

$$H_5\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}), \qquad H_5 = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta\gamma^5 m, \tag{3}$$

where $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$ and $\beta = \gamma^0$. This Hamiltonian is not Hermitian but pseudo-Hermitian [23–32] and has the property

$$H_5(\boldsymbol{r}) = P H_5^+(-\boldsymbol{r})P^{-1} = \mathcal{P} H_5^+(\boldsymbol{r})\mathcal{P}^{-1}, \tag{4}$$

where $P = \gamma^0$ is the matrix representation of parity and $\mathcal{P}$ is the full parity transformation. The pseudo-Hermitian property makes the Hamiltonian usable for practical calculations [24] and ensures that its energy eigenvalues are real or come in complex-conjugate pairs. In view of $(\beta\gamma^5)^2 = -\mathbb{1}_{4\times 4}$, the matrix multiplying the tachyonic mass term in (3) is multiplied by a matrix representation of the imaginary unit rather than by the imaginary unit itself. The quantum dynamics induced by the Hamiltonian (3) have been studied in Ref. [22] and imply that all resonance energies in the spectrum of the Hamiltonian H_5 fulfill the tachyonic dispersion relation (1).

1.2 Tachyonic quantum fields

The main problems in the construction of a tachyonic field theory are as follows. Under a Lorentz boost with velocity $\boldsymbol{v}$, as emphasized by Feinberg [5], the energy of a particle transforms as

$$E' = \gamma(E - \boldsymbol{p}\cdot\boldsymbol{v}), \quad \gamma = \frac{1}{\sqrt{1-v^2}}, \tag{5}$$

into the moving frame, in analogy to $t' = \gamma(t - \boldsymbol{v}\cdot\boldsymbol{r})$ (we assume $v < 1$). Under certain conditions, we thus transform $E > 0$ into $E' < 0$ upon Lorentz transformation, if the particle is tachyonic ($p > E$). Therefore, some of the particle annihilation operators transform into antiparticle creation operators under Lorentz transformations; the separation of the field operator into creation and annihilation contributions is no longer Lorentz invariant. The sign change in E occurs precisely if and only if the time interval of the tachyon world line also changes sign under a Lorentz transformation. Therefore, if we accept the fact that the vacuum undergoes a Lorentz transformation, then the calculated cross sections actually are Lorentz invariant [5, 6]. This is tied to the Feinberg–Sudarshan reinterpretation principle [2, 5]. Note that in ordinary quantum field theory, the identification of antiparticles as negative-energy particles traveling backward in time (equivalent to positive-energy antiparticles traveling forward in time, with the values of all additive physical quantities reversed) is also tied to a reinterpretation principle. The difference between the reinterpretation principle for tardyonic versus tachyonic particles is not in the reversal of the arrow of time; it is solely in the fact that in the case of tachyonic particles and antiparticles, the separation into creation and annihilation parts is not Lorentz invariant.

If one does not separate the field operator into annihilation and creation parts, then one has to write it down in terms of annihilation operators only [2, 3]. This could seem to be attractive because the separation in this case is Lorentz invariant, but one then has to perform considerable effort in the calculation of a propagator, even in the case of scalar tachyonic particles [2, 3]. We therefore prefer to use the concept originally developed in Refs. [5, 6], even if the Lorentz covariance of the vacuum is extremely painful from the point of view of axiomatic field theory. We note that the Lorentz-mediated conversion of creation into annihilation operators is restricted to a small kinematic domain. Using (1), we have $E' = m(1 - uv)/\sqrt{u^2 - 1}$ and therefore $E' < 0$ if and only if $u > 1/v$. For two laboratories on Earth traveling with respect to each other at a relative velocity of $v \approx 10^{-6}$, this means that the creation/annihilation paradox affects only those tachyonic states with $u > 10^6$, i.e., tachyonic particles with energies $E \lesssim m/10^3$. For a hypothetical tachyonic neutrino with a tachyonic mass of 1 eV, this affects neutrino states with energies $E \lesssim 10^{-3}$ eV, which are clearly irrelevant for laboratory-based measurements (see also the discussion below in Sect. 4 for further insight into this problem).

1.3 Localizability

Another conceptual problem has never been satisfactorily addressed in the literature: namely, if we try to solve either the tachyonic Klein–Gordon or the tachyonic Dirac equation using a plane-wave *ansatz* of the form $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$, then the tachyonic energy

$$E = \sqrt{\boldsymbol{k}^2 - m^2 - \mathrm{i}\epsilon} \tag{6}$$

becomes imaginary for $|\boldsymbol{k}| < m$ (we here anticipate the $\mathrm{i}\epsilon$ prescription *à la façon Feynmanienne*). If one restricts the domain of allowed momenta to the region $|\boldsymbol{k}| > m$, then the plain-wave solutions proportional to $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$ do not form a complete set of eigenstates of the Hamiltonian any more. Fourier transformation becomes problematic, and also, a tachyonic wave packet proportional to a Dirac-δ (eigenstate of the position operator) cannot be expanded into plain-wave eigenstates any longer. This has led to criticism [7, 8]. We here argue that the states with $|\boldsymbol{k}| \le m$ should *not* be excluded from the theory. In the mathematical sense, the states with $|\boldsymbol{k}| \le m$ constitute resonances of the tachyonic Hamiltonian with complex resonance eigenvalues (resonances for $\mathrm{Im}\,E > 0$ and antiresonances for $\mathrm{Im}\,E < 0$). If one allows the resonances, then the domain of allowed wave vectors is no longer restricted. One can then localize (and measure the position of) a tachyonic particle, and furthermore, field anticommutators and propagators assume their usual, localized form. The decay (in time) of the resonances is thus interpreted as a genuine property of the tachyonic field when observed from a subluminal reference frame. The observation of the tachyonic wave packet from a subluminal frame leads to the elimination (evanescence) of subluminal components from the propagating tachyonic wave packets; the components with $|\boldsymbol{k}| \leq m$ constitute evanescent waves. Such a situation is not uncommon in physics. E.g., a particle bound in a cubic anharmonic oscillator potential is propagated via resonances with complex-valued resonance energies [33]; the energies become complex because a classical particle in a cubic potential escapes to infinity in finite time. The corresponding matter waves are evanescent. We have no better interpretation but to assert that for a particle visiting us "from the other side of the light barrier," subluminal components in wave packets must be suppressed (evanescent) because they are incompatible with the genuine superluminal nature of the tachyonic particle. The inclusion of the iϵ damps the particle solutions (positive energy, forward in time) and the antiparticle solutions (negative energy, backward in time) consistently so as to dampen the wave amplitude accordingly.

1.4 Implications of the tachyonic formulation

The formalism detailing all above mentioned concepts is outlined below, for the case of a tachyonic spin-$\frac{1}{2}$ particle. The question then is, given we are willing to invest so much effort in formulating a quantum field theory of tachyons, and given that we have to give up a number of esteemed axioms of field theory for incorporating them, what is the return of the investment? We can state that (i) the tachyonic Dirac equation seems to have exactly the right symmetries for the description of neutrinos [22], (ii) the tachyonic Dirac theory has the correct massless limit (shown below), and (iii) the suppression of right-handed helicity neutrino states, and left-handed helicity antineutrino states, follows naturally from the tachyonic theory because the suppressed states have negative norm. Furthermore, (iv) we need some form of a tachyonic field theory if the MINOS [34] and OPERA [35] data should stand the test of time, and if new measurements should confirm low-energy neutrino data [36–42] which show an apparent trend toward negative values for the neutrino mass square. (The neutrino mass square exhibits a clear trend toward negative values in all measurements, with the magnitude of the mass square increasing with the energy of the neutrino. The latter point is not a subject of the discussion in the current article.)

We proceed as follows. In Sect. 2, we discuss the spinor solutions of the tachyonic Dirac equation. Quantum field theory of tachyons is discussed in Sect. 3, where we also calculate the propagator for tachyonic spin-$\frac{1}{2}$ particles based on the time-ordered product of field operators. Neutrinoless double beta decay is discussed in Sect. 5. Finally, conclusions are reserved for Sect. 6. In particular, we here attempt to improve on an earlier work [12] to quantize the tachyonic spin-$\frac{1}{2}$ theory. The Fourier transform of our propagator is related to the inverse of the Hamiltonian, as it should. Units with $\hbar = c = \epsilon_0 = 1$ are used throughout the paper.

2 Solutions of the tachyonic equation

Starting from the solutions for the massless Dirac equation, we proceed to the calculation of the solutions of the tachyonic Dirac equation by an obvious generalization. The helicity basis is a convenient starting point for the calculations. The eigenfunctions of the operator $\boldsymbol{\sigma} \cdot \boldsymbol{k}$ are given by

$$a_+(\boldsymbol{k}) = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})\mathrm{e}^{\mathrm{i}\varphi} \end{pmatrix}, \tag{7a}$$

$$a_-(\boldsymbol{k}) = \begin{pmatrix} -\sin(\frac{\theta}{2})\mathrm{e}^{-\mathrm{i}\varphi} \\ \cos(\frac{\theta}{2}) \end{pmatrix}. \tag{7b}$$

Here, θ and φ are the polar and azimuthal angles of the wave vector $\boldsymbol{k}$, respectively. They fulfill the relation $\boldsymbol{\sigma} \cdot \boldsymbol{k} a_\pm(\boldsymbol{k}) = \pm|\boldsymbol{k}| a_\pm(\boldsymbol{k})$. The normalized positive-energy chirality and helicity eigenstates of the massless Dirac equation are

$$u_+(\boldsymbol{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} a_+(\boldsymbol{k}) \\ a_+(\boldsymbol{k}) \end{pmatrix}, \qquad u_-(\boldsymbol{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} a_-(\boldsymbol{k}) \\ -a_-(\boldsymbol{k}) \end{pmatrix}. \tag{8}$$

These eigenstates immediately lead to plane-wave solutions of the massless Dirac equation. Denoting by $\boldsymbol{p} = -\mathrm{i}\nabla$ the momentum operator, the massless Dirac Hamiltonian reads $H_0 = \boldsymbol{\alpha} \cdot \boldsymbol{p}$, where $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$ is the vector of Dirac α matrices. We use the Dirac matrices in the Dirac representation,

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix}. \tag{9}$$

The positive-energy solutions have the properties

$$\boldsymbol{\alpha} \cdot \boldsymbol{p} u_\pm(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} = |\boldsymbol{k}| u_\pm(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{10a}$$

$$\frac{\boldsymbol{\Sigma} \cdot \boldsymbol{p}}{|\boldsymbol{p}|} u_\pm(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} = \gamma^5 u_\pm(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} = \pm u_\pm(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{10b}$$

For the negative-energy solutions of the massless Dirac equation, we obtain the charge conjugate solutions of the ones for positive energy,

$$v_+(\boldsymbol{k}) = C\overline{u}_-(\boldsymbol{k})^{\mathrm{T}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -a_+(\boldsymbol{k}) \\ -a_+(\boldsymbol{k}) \end{pmatrix}, \tag{11a}$$

$$v_-(\boldsymbol{k}) = C\overline{u}_+(\boldsymbol{k})^{\mathrm{T}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -a_-(\boldsymbol{k}) \\ a_-(\boldsymbol{k}) \end{pmatrix}, \tag{11b}$$

where $C = \mathrm{i}\gamma^2\gamma^0$ is the charge conjugation matrix. The negative-energy states fulfill the relations

$$\boldsymbol{\alpha}\cdot\boldsymbol{p}\, v_\pm(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} = -|\boldsymbol{k}|v_\pm(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{12a}$$

$$-\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}v_\pm(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} = \gamma^5 v_\pm(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} = \pm v_\pm(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{12b}$$

The subscripts $\pm$ of the u and v spinors correspond to the chirality (eigenvalue of γ^5), which is equal to helicity for positive-energy eigenstates, and equal to the negative of the helicity for negative-energy eigenstates. Because of the relation $(\gamma^\mu k_\mu - \gamma^5 m)^2 = k^2 + m^2 = E^2 - \boldsymbol{k}^2 + m^2$, the generalization of these solutions to the massive tachyonic Dirac equation is rather straightforward. We find

$$\begin{aligned} U_+(\boldsymbol{k}) &= \frac{\gamma^5 m - \not{k}}{\sqrt{(E-|\boldsymbol{k}|)^2 + m^2}}u_+(\boldsymbol{k}) \\ &= \begin{pmatrix} \frac{m-E+|\boldsymbol{k}|}{\sqrt{2}\sqrt{(E-|\boldsymbol{k}|)^2+m^2}}a_+(\boldsymbol{k}) \\ \frac{m+E-|\boldsymbol{k}|}{\sqrt{2}\sqrt{(E-|\boldsymbol{k}|)^2+m^2}}a_+(\boldsymbol{k}) \end{pmatrix}, \end{aligned} \tag{13a}$$

$$\begin{aligned} U_-(\boldsymbol{k}) &= \frac{\not{k} - \gamma^5 m}{\sqrt{(E-|\boldsymbol{k}|)^2 + m^2}}u_-(\boldsymbol{k}) \\ &= \begin{pmatrix} \frac{m+E-|\boldsymbol{k}|}{\sqrt{2}\sqrt{(E-|\boldsymbol{k}|)^2+m^2}}a_-(\boldsymbol{k}) \\ \frac{-m+E-|\boldsymbol{k}|}{\sqrt{2}\sqrt{(E-|\boldsymbol{k}|)^2+m^2}}a_-(\boldsymbol{k}) \end{pmatrix}, \end{aligned} \tag{13b}$$

where $\not{k} = \gamma^\mu k_\mu$ is the Feynman slash. The massless limit $m \to 0$ is recovered by observing that $E \to |\boldsymbol{k}|$ for a particle approaching the light cone (called a luxon). So, $U_+(\boldsymbol{k}) \to u_+(\boldsymbol{k})$ and $U_-(\boldsymbol{k}) \to u_-(\boldsymbol{k})$ in the massless limit. The negative-energy eigenstates of the tachyonic Dirac equation are given as

$$\begin{aligned} V_+(\boldsymbol{k}) &= \frac{\gamma^5 m + \not{k}}{\sqrt{(E-|\boldsymbol{k}|)^2 + m^2}}v_+(\boldsymbol{k}) \\ &= \begin{pmatrix} \frac{-m-E+|\boldsymbol{k}|}{\sqrt{2}\sqrt{(E-|\boldsymbol{k}|)^2+m^2}}a_+(\boldsymbol{k}) \\ \frac{-m+E-|\boldsymbol{k}|}{\sqrt{2}\sqrt{(E-|\boldsymbol{k}|)^2+m^2}}a_+(\boldsymbol{k}) \end{pmatrix}, \end{aligned} \tag{14a}$$

$$\begin{aligned} V_-(\boldsymbol{k}) &= \frac{-\not{k} - \gamma^5 m}{\sqrt{(E-|\boldsymbol{k}|)^2 + m^2}}v_-(\boldsymbol{k}) \\ &= \begin{pmatrix} \frac{-m+E-|\boldsymbol{k}|}{\sqrt{2}\sqrt{(E-|\boldsymbol{k}|)^2+m^2}}a_-(\boldsymbol{k}) \\ \frac{m+E-|\boldsymbol{k}|}{\sqrt{2}\sqrt{(E-|\boldsymbol{k}|)^2+m^2}}a_-(\boldsymbol{k}) \end{pmatrix}. \end{aligned} \tag{14b}$$

In the massless limit, as before, we have $V_+(\boldsymbol{k}) \to v_+(\boldsymbol{k})$ and $V_-(\boldsymbol{k}) \to v_-(\boldsymbol{k})$. The states are normalized with respect to the condition

$$\begin{aligned} U_+^+(\boldsymbol{k})U_+(\boldsymbol{k}) &= U_-^+(\boldsymbol{k})U_-(\boldsymbol{k}) \\ &= V_+^+(\boldsymbol{k})V_+(\boldsymbol{k}) = V_-^+(\boldsymbol{k})V_-(\boldsymbol{k}) = 1. \end{aligned} \tag{15}$$

The corresponding positive- and negative-energy solutions of the massive tachyonic Dirac equation are given as

$$\Psi(x) = U_\pm(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}k\cdot x}, \quad k = (E, \boldsymbol{k}),\ E = \sqrt{\boldsymbol{k}^2 - m^2}, \tag{16a}$$

$$\left(\mathrm{i}\gamma^\mu\partial_\mu - \gamma^5 m\right)\Psi(x) = \left(\gamma^\mu k_\mu - \gamma^5 m\right)\Psi(x) = 0, \tag{16b}$$

$$\Phi(x) = V_\pm(\boldsymbol{k})\mathrm{e}^{\mathrm{i}k\cdot x}, \quad k = (E, \boldsymbol{k}),\ E = \sqrt{\boldsymbol{k}^2 - m^2}, \tag{16c}$$

$$\left(\mathrm{i}\gamma^\mu\partial_\mu - \gamma^5 m\right)\Phi(x) = \left(-\gamma^\mu k_\mu - \gamma^5 m\right)\Phi(x) = 0. \tag{16d}$$

Here, Ψ is for positive energy, and Φ is for negative energy. All above formulas are valid for

$$|\boldsymbol{k}| \geq m, \qquad E = \sqrt{\boldsymbol{k}^2 - m^2} \in \mathbb{R}. \tag{17}$$

For $|\boldsymbol{k}| < m$, one encounters resonances. We define the width Γ of a resonance of the tachyonic Dirac Hamiltonian as follows:

$$E = \pm\sqrt{\boldsymbol{k}^2 - m^2 - \mathrm{i}\epsilon} = \mp\mathrm{i}\frac{\Gamma}{2}, \tag{18a}$$

$$\Gamma = 2\sqrt{m^2 - \boldsymbol{k}^2}, \quad |\boldsymbol{k}| < m. \tag{18b}$$

The wave functions describing the resonances are as follows:

$$R_+(\boldsymbol{k}) = \begin{pmatrix} \frac{m+\frac{\mathrm{i}}{2}\Gamma+|\boldsymbol{k}|}{\sqrt{2}\sqrt{\boldsymbol{k}^2+m^2+\frac{1}{4}\Gamma^2}}a_+(\boldsymbol{k}) \\ \frac{m-\frac{\mathrm{i}}{2}\Gamma-|\boldsymbol{k}|}{\sqrt{2}\sqrt{\boldsymbol{k}^2+m^2+\frac{1}{4}\Gamma^2}}a_+(\boldsymbol{k}) \end{pmatrix}, \tag{19a}$$

$$R_-(\boldsymbol{k}) = \begin{pmatrix} \frac{m-\frac{\mathrm{i}}{2}\Gamma-|\boldsymbol{k}|}{\sqrt{2}\sqrt{\boldsymbol{k}^2+m^2+\frac{1}{4}\Gamma^2}}a_-(\boldsymbol{k}) \\ \frac{-m-\frac{\mathrm{i}}{2}\Gamma-|\boldsymbol{k}|}{\sqrt{2}\sqrt{\boldsymbol{k}^2+m^2+\frac{1}{4}\Gamma^2}}a_-(\boldsymbol{k}) \end{pmatrix}, \tag{19b}$$

$$E = -\frac{\mathrm{i}}{2}\Gamma = -\frac{\mathrm{i}}{2}\sqrt{m - \boldsymbol{k}^2}, \quad |\boldsymbol{k}| < m. \tag{19c}$$

The antiresonance eigenstates are

$$S_+(\boldsymbol{k}) = \begin{pmatrix} \frac{-m-\frac{\mathrm{i}}{2}\Gamma+|\boldsymbol{k}|}{\sqrt{2}\sqrt{\boldsymbol{k}^2+m^2+\frac{1}{4}\Gamma^2}}a_+(\boldsymbol{k}) \\ \frac{-m+\frac{\mathrm{i}}{2}\Gamma-|\boldsymbol{k}|}{\sqrt{2}\sqrt{\boldsymbol{k}^2+m^2+\frac{1}{4}\Gamma^2}}a_+(\boldsymbol{k}) \end{pmatrix}, \tag{20a}$$

$$S_-(\boldsymbol{k}) = \begin{pmatrix} \frac{-m+\frac{\mathrm{i}}{2}\Gamma-|\boldsymbol{k}|}{\sqrt{2}\sqrt{\boldsymbol{k}^2+m^2+\frac{1}{4}\Gamma^2}}a_-(\boldsymbol{k}) \\ \frac{m+\frac{\mathrm{i}}{2}\Gamma-|\boldsymbol{k}|}{\sqrt{2}\sqrt{\boldsymbol{k}^2+m^2+\frac{1}{4}\Gamma^2}}a_-(\boldsymbol{k}) \end{pmatrix}, \tag{20b}$$

$$E = \frac{\mathrm{i}}{2}\Gamma = \frac{\mathrm{i}}{2}\sqrt{m - \boldsymbol{k}^2}, \quad |\boldsymbol{k}| < m. \tag{20c}$$

These states are normalized according to

$$\begin{aligned} R_+^+(\boldsymbol{k})R_+(\boldsymbol{k}) &= R_-^+(\boldsymbol{k})R_-(\boldsymbol{k}) \\ &= S_+^+(\boldsymbol{k})S_+(\boldsymbol{k}) = S_-^+(\boldsymbol{k})S_-(\boldsymbol{k}) = 1. \end{aligned} \tag{21}$$

The rest frame for tachyons is that of infinite speed [5, 12], which corresponds to $|\boldsymbol{k}| = m$ and $E \to 0$. The eigenstates simplify in this limit to read

$$U_+(\boldsymbol{k}) \to R_+(\boldsymbol{k}) \to \begin{pmatrix} a_+(\boldsymbol{k}) \\ 0 \end{pmatrix}, \quad |\boldsymbol{k}| \to m, \tag{22a}$$

$$U_-(\boldsymbol{k}) \to R_-(\boldsymbol{k}) \to \begin{pmatrix} 0 \\ -a_-(\boldsymbol{k}) \end{pmatrix}, \quad |\boldsymbol{k}| \to m. \tag{22b}$$

The negative-energy spinors tend to the following values:

$$V_+(\boldsymbol{k}) \to S_+(\boldsymbol{k}) \to \begin{pmatrix} 0 \\ -a_+(\boldsymbol{k}) \end{pmatrix}, \quad |\boldsymbol{k}| \to m, \tag{23a}$$

$$V_-(\boldsymbol{k}) \to S_-(\boldsymbol{k}) \to \begin{pmatrix} -a_-(\boldsymbol{k}) \\ 0 \end{pmatrix}, \quad |\boldsymbol{k}| \to m. \tag{23b}$$

One observes that the massless solutions (8) and (11) cannot be normalized according to the covariant expression $\overline{u}u = u^+\gamma^0 u$, because this expression vanishes for all four solutions indicated in (8) and (11). In the massive case, we calculate after some algebraic simplification,

$$\overline{U}_+(\boldsymbol{k})U_+(\boldsymbol{k}) = \frac{m}{|\boldsymbol{k}|}, \qquad \overline{U}_-(\boldsymbol{k})U_-(\boldsymbol{k}) = -\frac{m}{|\boldsymbol{k}|}, \tag{24a}$$

$$\overline{V}_+(\boldsymbol{k})V_+(\boldsymbol{k}) = -\frac{m}{|\boldsymbol{k}|}, \qquad \overline{V}_-(\boldsymbol{k})V_-(\boldsymbol{k}) = \frac{m}{|\boldsymbol{k}|}. \tag{24b}$$

By a multiplication with $(|\boldsymbol{k}|/m)^{1/2}$, we can thus obtain covariantly normalized spinors $\mathcal{U}_\sigma(\boldsymbol{k})$ and $\mathcal{V}_\sigma(\boldsymbol{k})$, which fulfill the following helicity-dependent normalizations:

$$\overline{\mathcal{U}}_\sigma(\boldsymbol{k})\mathcal{U}_\sigma(\boldsymbol{k}) = \mathcal{U}_\sigma^+(\boldsymbol{k})\gamma^0\mathcal{U}_\sigma(\boldsymbol{k}) = \sigma, \tag{25a}$$

$$\overline{\mathcal{V}}_\sigma(\boldsymbol{k})\mathcal{V}_\sigma(\boldsymbol{k}) = \mathcal{V}_\sigma^+(\boldsymbol{k})\gamma^0\mathcal{V}_\sigma(\boldsymbol{k}) = -\sigma, \tag{25b}$$

where $\sigma = \pm$ is the chirality (in the massless limit). We here indicate these solutions for the eigenstates with real energy eigenvalues, in the normalization (25), and start with the positive-energy solutions.

$$\mathcal{U}_+(\boldsymbol{k}) = \begin{pmatrix} \frac{m-E+|\boldsymbol{k}|}{2\sqrt{m}\sqrt{|\boldsymbol{k}|-E}} a_+(\boldsymbol{k}) \\ \frac{m+E-|\boldsymbol{k}|}{2\sqrt{m}\sqrt{|\boldsymbol{k}|-E}} a_+(\boldsymbol{k}) \end{pmatrix}, \tag{26a}$$

$$\mathcal{U}_-(\boldsymbol{k}) = \begin{pmatrix} \frac{m+E-|\boldsymbol{k}|}{2\sqrt{m}\sqrt{|\boldsymbol{k}|-E}} a_-(\boldsymbol{k}) \\ \frac{-m+E-|\boldsymbol{k}|}{2\sqrt{m}\sqrt{|\boldsymbol{k}|-E}} a_-(\boldsymbol{k}) \end{pmatrix}. \tag{26b}$$

The negative-energy spinors in the normalization (25) are given as

$$\mathcal{V}_+(\boldsymbol{k}) = \begin{pmatrix} \frac{-m-E+|\boldsymbol{k}|}{2\sqrt{m}\sqrt{|\boldsymbol{k}|-E}} a_+(\boldsymbol{k}) \\ \frac{-m+E-|\boldsymbol{k}|}{2\sqrt{m}\sqrt{|\boldsymbol{k}|-E}} a_+(\boldsymbol{k}) \end{pmatrix}, \tag{27a}$$

$$\mathcal{V}_-(\boldsymbol{k}) = \begin{pmatrix} \frac{-m+E-|\boldsymbol{k}|}{2\sqrt{m}\sqrt{|\boldsymbol{k}|-E}} a_-(\boldsymbol{k}) \\ \frac{m+E-|\boldsymbol{k}|}{2\sqrt{m}\sqrt{|\boldsymbol{k}|-E}} a_-(\boldsymbol{k}) \end{pmatrix}. \tag{27b}$$

In the following, we understand that for $|\boldsymbol{k}| < m$, the $\mathcal{U}_\pm$ and $\mathcal{V}_\pm$ are defined as the resonances and antiresonances (19) and (20), renormalized to the condition (25) in full analogy with (26) and (27).

3 Quantized theory

Up to now, we have only considered the relativistic quantum theory, not the field operator which corresponds to the tachyonic spin-$\frac{1}{2}$ field. According to the Feinberg–Sudarshan reinterpretation principle [1–5], problems associated with the violation of causality by the emergence of tachyonic particles can be overcome. Namely, in Ref. [2], the authors argue that a physically "sensible" theory is achieved by insisting that the only physical quantities are transition amplitudes and a negative-energy in (out) state is physically understood to be a positive-energy out (in) state. This statement is perhaps oversimplified and in need of a further explanation. It can be understood as follows. Suppose that observer A sees event $E1$ before $E2$, and observer B sees event $E2$ before $E1$, because the two events are separated by a space-like interval, and the Lorentz transform between the frames of observers A and B reverses the time ordering of events $E1$ and $E2$. Then, according to Ref. [1], the reversal of time ordering occurs precisely under the condition that the Lorentz transformation between the two frames also changes the sign of the energy. So, provided one reinterprets the negative-energy solutions of the tachyonic Dirac Hamiltonian propagating backward in time (the antiresonances included) as positive-energy solutions propagating forward in time, the creation and absorption of a particle can be reinterpreted without further problems if only the transition amplitude is unaffected by the reinterpretation. This point has been stressed in Refs. [1, 4] and is illustrated in Fig. 1.

In Ref. [5], both particle as well as antiparticle creation and annihilation operators are used in the field operators, and in order to ensure Lorentz covariance of the quantization conditions, anticommutators are used in order to quantize a scalar field theory. The use of anticommutators is dictated by the fact that under Lorentz transformations, the energy may change sign and therefore, the commutator of an annihilation and a creation operator would otherwise change

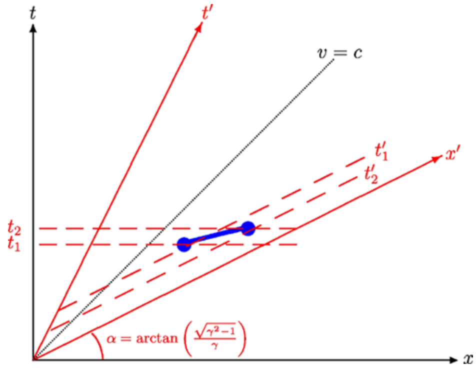

Fig. 1 (Color online) Illustration of the Feinberg–Sudarshan reinterpretation principle in a Minkowski diagram: The primed (moving) frame moves at velocity v with Lorentz factor $\gamma = 1/\sqrt{1-v^2}$. The x_1' axis is tilted by the angle $\alpha = \arctan(\sqrt{1-\gamma^{-2}})$ with respect to the x_1 axis, and the t' axis is tilted with respect to t by the same angle. Two (spacelike, tachyonic) events occur at $t_2 > t_1$ and transform to $t_2' < t_1'$ in the moving frame. According to the reinterpretation principle, the two events are seen by the moving observer as a negative-energy particle annihilation event (positive-energy antiparticle creation event) at space-time point (t_1', x_1'). The negative-energy particle is created at the point (t_2', x_2'), which is equivalent to stating that the positive-energy antiparticle is annihilated at (t_2', x_2'). According to reinterpretation, an antiparticle travels forward in time from (t_2', x_2') to (t_1', x_1') in the space–time coordinates of the moving frame. In the laboratory (rest) frame, the two events are seen as particle creation at (t_1, x_1) followed by annihilation at (t_2, x_2)

sign under a Lorentz transformation, which is inconsistent [see (4.8) and (4.9) of Ref. [5]]. However, for fermions, no such problem arises, and we may define the quantization conditions naturally, in terms of anticommutators. In Refs. [2, 3], the authors argue that the field operators for the tachyonic field should be formulated in terms of particle operators alone, which allows the authors to quantize the scalar theory using commutators. Concerning spin-$\frac{1}{2}$ particles, the authors of Ref. [12] write that in agreement with Refs. [1, 4], the antiparticle operators are not included in their field operator, since, for tardyons, "they represent an interpretation of the negative-energy states; thus the inclusion of both negative-energy states and antiparticle states is redundant. Any reinterpretation principle which views the negative-energy annihilation operators as antiparticle creation operators will exclude the possibility of an invariant vacuum state" [see text following (17) of Ref. [12]]. In the formalism of [2, 3], antiparticle operators have to be inserted by hand into the tachyonic field operator, as in (1.11) of Ref. [3]. *We explicitly accept the Lorentz-non-invariance of the vacuum state in the following derivation.* According to (5.7) of Ref. [5], the Lorentz transformation of the vacuum state, in this case, forces us to occupy all particle and antiparticle states whose energy changes sign under the Lorentz transformation, in the Lorentz-transformed vacuum state (see also Sect. 4 below).

In full analogy with (3.157) of Ref. [43], and (17) of Ref. [12], and in full agreement with the Feinberg–Sudarshan reinterpretation principle, we thus write the field operator for the tachyonic field as

$$\begin{aligned}\psi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{m}{E}\sum_{\sigma=\pm}\big[b_\sigma(k)\mathcal{U}_\sigma(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}k\cdot x} \\ + b_\sigma(-k)\mathcal{V}_\sigma(\boldsymbol{k})\mathrm{e}^{\mathrm{i}k\cdot x}\big], \\ k = (E, \boldsymbol{k}),\ E = E_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 - m^2 - \mathrm{i}\epsilon}.\end{aligned} \tag{28}$$

The second term describes the absorption of a negative-energy tachyonic particle that propagates backward in time which is equivalent to the emission of a positive-energy antiparticle propagating forward in time by the Feinberg–Sudarshan reinterpretation principle. We thus have

$$\begin{aligned}\psi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{m}{E}\sum_{\sigma=\pm}\big[b_\sigma(k)\mathcal{U}_\sigma(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}k\cdot x} \\ + d_\sigma^+(k)\mathcal{V}_\sigma(\boldsymbol{k})\mathrm{e}^{\mathrm{i}k\cdot x}\big],\end{aligned} \tag{29}$$

where d_σ^+ creates antiparticles. We postulate that the anticommutators of the tachyonic field operators read as follows (Fermi–Dirac statistics):

$$\big\{b_\sigma(k), b_\rho(k')\big\} = \big\{b_\sigma^+(k), b_\rho^+(k')\big\} = 0, \tag{30a}$$

$$\big\{d_\sigma(k), d_\rho(k')\big\} = \big\{d_\sigma^+(k), d_\rho^+(k')\big\} = 0, \tag{30b}$$

and the nonvanishing anticommutators are as follows:

$$\big\{b_\sigma(k), b_\rho^+(k')\big\} = (-\sigma)(2\pi)^3\frac{E}{m}\delta^3\big(\boldsymbol{k}-\boldsymbol{k}'\big)\delta_{\sigma\rho}, \tag{30c}$$

$$\big\{d_\sigma(k), d_\rho^+(k')\big\} = (-\sigma)(2\pi)^3\frac{E}{m}\delta^3\big(\boldsymbol{k}-\boldsymbol{k}'\big)\delta_{\sigma\rho}. \tag{30d}$$

This implies that positive-energy states corresponding to spinors with negative chirality and negative helicity (in the massless limit) have to be quantized according to normal Fermi–Dirac statistics, whereas the anticommutators of right-handed chirality field operators receive a minus sign. One is led to the σ-dependent anticommutators quite naturally. Namely, one obtains a compact representation for the tensor-valued dyadic sums over the spinor eigenstates of the tachyonic Dirac equations only if an additional σ-dependent prefactor is introduced [see (34) below]. The σ-dependence of the anticommutator is thus necessary if we postulate that the form of the propagator as obtained from the second-quantized formalism should be consistent with the covariant form of the Green function, as given below in (39).

Yet, if we impose the σ-dependent anticommutator, then the norm of the right-handed helicity (positive chirality) neutrino one-particle state is negative,

$$\langle 1_{k,\sigma}|1_{k,\sigma}\rangle = \langle 0|b_\sigma(k)b_\sigma^+(k)|0\rangle$$
$$= \langle 0|\{b_\sigma(k), b_\sigma^+(k)\}|0\rangle = (-\sigma)V\frac{E}{m}, \tag{31}$$

where $V = (2\pi)^3\delta^3(\mathbf{0})$ is the normalization volume in coordinate space. Thus, if the neutrino is described by the tachyonic Dirac equation, this implies that right-handed helicity neutrino states have negative norm and can be excluded from the physical spectrum if one imposes a Gupta–Bleuler type condition (see Chap. 3 of Ref. [43]). Furthermore, the norm of the antineutrino states which correspond to positive-chirality spinors (in the massless limit) also is negative,

$$\langle \overline{1}_{k,\sigma}|\overline{1}_{k,\sigma}\rangle = \langle 0|d_\sigma(k)d_\sigma^+(k)|0\rangle$$
$$= \langle 0|\{d_\sigma(k), d_\sigma^+(k)\}|0\rangle = (-\sigma)V\frac{E}{m}, \tag{32}$$

which implies that antineutrinos can only exist in the right-handed helicity state. We remember that right-handed chirality implies left-handed helicity for antiparticles [see (12b)]. It is then rather easy to calculate the matrix-values tachyonic field anticommutator, which reads

$$\begin{aligned}
&\{\psi(x), \overline{\psi}(y)\} \\
&\quad = \langle 0|\{\psi(x), \overline{\psi}(y)\}|0\rangle \\
&\quad = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{m}{E}\sum_{\sigma=\pm}\{\mathrm{e}^{-\mathrm{i}k\cdot(x-y)}(-\sigma)\mathcal{U}_\sigma(\boldsymbol{k})\otimes\overline{\mathcal{U}}_\sigma(\boldsymbol{k}) \\
&\qquad + \mathrm{e}^{\mathrm{i}k\cdot(x-y)}(-\sigma)\mathcal{V}_\sigma(\boldsymbol{k})\otimes\overline{\mathcal{V}}_\sigma(\boldsymbol{k})\},
\end{aligned} \tag{33}$$

where σ is the chirality and $\otimes$ denotes the tensor product in spinor space. The two relations

$$\sum_\sigma(-\sigma)\mathcal{U}_\sigma(\boldsymbol{k})\otimes\overline{\mathcal{U}}_\sigma(\boldsymbol{k})\gamma^5 = \frac{\not{k}-\gamma^5 m}{2m}, \tag{34a}$$

$$\sum_\sigma(-\sigma)\mathcal{V}_\sigma(\boldsymbol{k})\otimes\overline{\mathcal{V}}_\sigma(\boldsymbol{k})\gamma^5 = \frac{\not{k}+\gamma^5 m}{2m} \tag{34b}$$

are analogous to those that lead from (3.169) to (3.170) of Ref. [43]. Note that the factor $(-\sigma)$ in these equations stems from the quantization conditions (30); if the σ-dependent prefactor is missing, the right-hand sides of (34a) and (34b) cannot be expressed in compact form. With the help of (34), we can thus derive the following, compact result:

$$\{\psi(x), \overline{\psi}(y)\}\gamma^5 = (\mathrm{i}\not{\partial} - \gamma^5 m)\mathrm{i}\varDelta(x-y), \tag{35}$$

where $\varDelta(x-y)$ is the expression encountered in (3.55) and (3.56) of [43],

$$\mathrm{i}\varDelta(x-y) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2E}\left[\mathrm{e}^{-\mathrm{i}k\cdot(x-y)} - \mathrm{e}^{\mathrm{i}k\cdot(x-y)}\right], \tag{36}$$

which is centered on the tachyonic mass shell. It follows that the equal-time anticommutator reads as

$$\{\psi(x), \overline{\psi}(y)\}\gamma^5|_{x_0=y_0} = -\gamma^0\partial_0\varDelta(x-y)|_{x_0=y_0}$$
$$= \gamma^0\delta^3(\boldsymbol{r}-\boldsymbol{s}) \tag{37}$$

with the full, unfiltered Dirac-δ function and $x = (t, \boldsymbol{r})$ as well as $y = (t, \boldsymbol{s})$. This is contrast to previous ansatzes for tachyonic field theories [2, 5], where the available momenta were restricted to the domain $|\boldsymbol{k}| \geq m$. We here allow the resonances and antiresonances given in (19) and (20) to cover the region $|\boldsymbol{k}| < m$. Our compact result is obtained because the integration is over all $\boldsymbol{k} \in \mathbb{R}^3$, and the time derivative "pulls down" the resonance and antiresonance energies from the arguments of the exponentials. In passing, we note that because antiresonances propagate backwards in time, whereas resonances propagate forward in time, the imaginary parts of the resonance and antiresonance energies given in (19) and (20) consistently lead to evanescent waves in the corresponding directions of the arrow of time.

Furthermore, with the help of (29) and (34), after a short calculation, one obtains the tachyonic (T) time-ordered propagator which has the representation

$$\langle 0|T\psi(x)\overline{\psi}(y)\gamma^5|0\rangle = \mathrm{i}S_T(x-y), \tag{38a}$$

$$S_T(x-y) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}k\cdot(x-y)}\frac{\not{k}-\gamma^5 m}{k^2+m^2+\mathrm{i}\epsilon}. \tag{38b}$$

This is the equivalent of (3.174) of Ref. [43]. As usual, the time ordering in the time-ordered product in (38) is with regard to the field operators; the tensor product of the spinors is calculated according to (34). We have thus solved the problem of the quantization of our tachyonic field theory, under the introduction of the γ^5 matrix in our time-ordered product and the normalization of the spinors as in (25) and the anticommutator relations in (30). Because the couplings of the neutrino involve the chirality projector $(1-\gamma^5)/2$, the γ^5 matrix in (38) can be absorbed in a redefinition of the interaction Lagrangian. Due to the equality $\gamma^5(1\pm\gamma^5)/2 = \pm(1\pm\gamma^5)/2$, the chirality projectors are invariant under multiplication by γ^5.

We can otherwise explore the connection of the tachyonic propagator with the Green function, in a possible analogy to the Dirac equation,

$$S_T = \gamma^0\frac{1}{E-H_5}, \tag{39}$$

where E is the energy argument of the Green function and H_5 is the tachyonic Hamiltonian (up to the iϵ prescription). An easy calculation shows that in momentum space, with H_5 replaced by $\boldsymbol{\alpha}\cdot\boldsymbol{k} + \beta\gamma^5 m$,

$$S_T(k) = \frac{1}{\not{k}-\gamma^5 m} = \frac{\not{k}-\gamma^5 m}{k^2+m^2}, \tag{40}$$

with $\not{k} = \gamma^\mu k_\mu$. If we introduce the i$\epsilon$ prescription as before, we find that

$$S_T(k) = \frac{1}{\not{k} - \gamma^5(m + \mathrm{i}\epsilon)} = \frac{\not{k} - \gamma^5 m}{k^2 + m^2 + \mathrm{i}\epsilon}. \tag{41}$$

This result, obtained by the inversion of the kinetic operator, is in full agreement with the result obtained in (38) from the quantized theory. Notably, the expression for our propagator is much more compact than that obtained from the formalism of Ref. [12], where expressions become rather involved even at intermediate steps and the authors do not even proceed to the calculation of the propagator [see (20) of Ref. [12]].

The compact structure of our propagator hinges upon the inclusion of the resonance and antiresonance eigenstates of the tachyonic Hamiltonian, which enables us to invert the kinetic operator for any $\boldsymbol{k}$, even for $|\boldsymbol{k}| < m$. It is instructive to consider the mixed space-frequency representation

$$S_T(E, \boldsymbol{r} - \boldsymbol{r}') = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} S_T(E, \boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}. \tag{42}$$

For the photon propagator in the Feynman prescription, it is known that the branch cuts of the mixed representation are defined to be along the lines $\sqrt{\omega^2 + \mathrm{i}\epsilon}$ with an infinitesimal ϵ, and with the branch cut of the square root along the positive real axis [44, 45]. For the tachyonic Hamiltonian, the branch cuts extend infinitesimally below the positive and above the negative real axis, and contain small sections representing the resonances and antiresonances, with $|\mathrm{Im}(E)| \le m$. As convenient for loop calculations, the branch cuts of the modified tachyonic Dirac propagator allow for a Wick rotation (see Fig. 2).

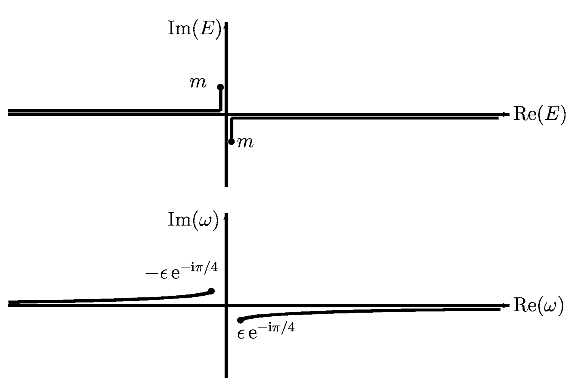

Fig. 2 Branch cuts of the tachyonic propagator (42) in the mixed frequency-position representation (*upper panel*) and (for comparison) branch cuts of the photon propagator (*lower panel*). See text for further explanations

4 Insight into Lorentz invariance

Let us consider the process shown in Fig. 1 in closer detail and choose the arrow of time as indicated in Fig. 3. In the laboratory frame, the particle moves from space-time point (t_1, x_1) to space-time point (t_2, x_2). We recall the form of the field operator

$$\psi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{m}{E} \sum_{\sigma=\pm} \big[b_\sigma(k) \mathcal{U}_\sigma(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}k\cdot x} + d_\sigma^+(k) \mathcal{V}_\sigma(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}k\cdot x} \big], \tag{43}$$

whose Dirac adjoint is

$$\overline{\psi}(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{m}{E} \sum_{\sigma=\pm} \big[b_\sigma^+(k) \overline{\mathcal{U}}_\sigma(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}k\cdot x} + d_\sigma(k) \overline{\mathcal{V}}_\sigma(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}k\cdot x} \big]. \tag{44}$$

In momentum space, according to (30), the amplitude for creation of the particle at (t_1, x_1) and annihilation of the particle at (t_2, x_2) is proportional to the matrix element (vacuum expectation value)

$$\begin{aligned}\mathcal{M} &= \langle 0 | b_\sigma(k_1) b_\sigma^+(k_2) | 0 \rangle = \big\{ b_\sigma(k_1), b_\sigma^+(k_2) \big\} \\ &= (-\sigma)(2\pi)^3 \frac{E}{m} \delta^3(\boldsymbol{k}_1 - \boldsymbol{k}_2).\end{aligned} \tag{45}$$

The Lorentz transform of this expression is

$$\begin{aligned}\mathcal{M}' &= L\mathcal{M}L^{-1} = \big\{ L b_\sigma(k_1) L^{-1}, L b_\sigma^+(k_2) L^{-1} \big\} \\ &= (-\sigma)(2\pi)^3 \frac{E'}{m} \delta^3(\boldsymbol{k}'_1 - \boldsymbol{k}'_2),\end{aligned} \tag{46}$$

where L is the representation of the Lorentz transformation in the space of the operators, and $k'_1 = (E'_1, \boldsymbol{k}'_1)$ is the Lorentz transform of $k_1 = (E_1, \boldsymbol{k}_1)$. According to (5), E'_1

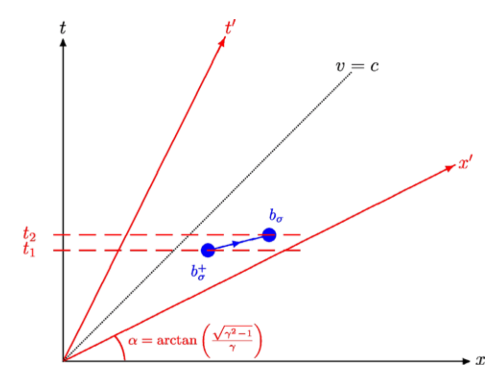

Fig. 3 (Color online) Events 1 and 2 seen in the laboratory frame. A particle is created at space-time point (t_1, x_1) and propagates to (t_2, x_2) where it is annihilated. The creation and annihilation operators are b_σ^+ and b_σ

can be negative if E_1 is positive, and we consider this situation in the Minkowski diagrams given in Figs. 3–5. According to Fig. 4, the time ordering and the sign of the energy is reversed in the primed (moving frame). The Lorentz transform of the corresponding particle annihilation operator is thus calculated, according to (4.7) of Ref. [5], as follows:

$$Lb_\sigma(k_1)L^{-1} = b_\sigma\left(-k_1'\right) = d_\sigma^+\left(k_1'\right), \tag{47}$$

i.e. the particle annihilation operator turns into an antiparticle creation operator. We can easily show that the anticommutator is Lorentz covariant,

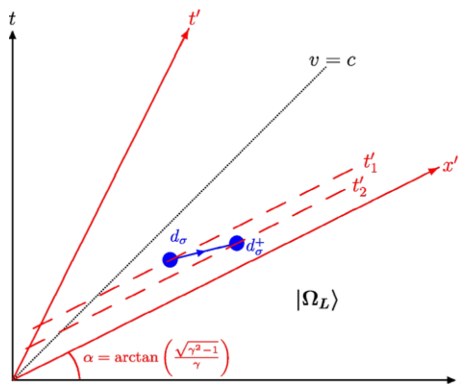

Fig. 4 (Color online) Events 1 and 2 seen in the moving frame, with a Lorentz-transformed vacuum state $|\Omega_L\rangle$. The time ordering is reversed, and the operator b_σ^+ transforms into an antiparticle annihilation operator d_σ, whereas b_σ transforms into d_σ^+. In order for the matrix element to be Lorentz covariant, we must transform the vacuum state $|0\rangle \to |\Omega_L\rangle$

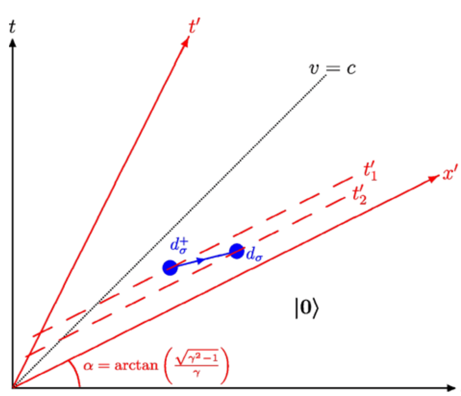

Fig. 5 (Color online) The matrix element describing events 1 and 2 could also be evaluated in a different way by the moving observer. The trajectory is backward in time. Therefore, it is an antiparticle trajectory in the moving frame. Using the untransformed vacuum, we evaluate, in the moving frame, the transition amplitude by the appropriate antiparticle creation and annihilation operators. This reverses the ordering of the annihilation and creation operators with respect to Fig. 4 and leads to the result given in (52)

$$\begin{aligned}\mathcal{M}' &= \left\{d_\sigma^+\left(k_1'\right), d_\sigma\left(k_2'\right)\right\}\\ &= (-\sigma)(2\pi)^3\frac{E'}{m}\delta^3\left(\boldsymbol{k}_1' - \boldsymbol{k}_2'\right),\end{aligned} \tag{48}$$

but we cannot use the vacuum expectation value because

$$\langle 0|Lb_\sigma(k_1)L^{-1}Lb_\sigma^+(k_2)L^{-1}|0\rangle = \langle 0|d_\sigma^+(k_1)d_\sigma(k_2)|0\rangle = 0. \tag{49}$$

The vacuum cannot be Lorentz invariant. We have to transform it according to $|\Omega_L\rangle = L|0\rangle$, so that

$$\begin{aligned}\mathcal{M}' &= \langle\Omega_L|d_\sigma^+\left(k_1'\right)d_\sigma\left(k_2'\right)|\Omega_L\rangle\\ &= (-\sigma)(2\pi)^3\frac{E'}{m}\delta^3\left(\boldsymbol{k}_1' - \boldsymbol{k}_2'\right)\end{aligned} \tag{50}$$

holds. This can be achieved as follows. Namely, according to (5.7) of Ref. [5], the Lorentz-transformed vacuum is actually filled with all particle and antiparticle states whose energy changes sign under the Lorentz transformation. Therefore,

$$\begin{aligned}\mathcal{M}' &= \int\frac{d^3k'}{(2\pi)^3}\frac{m}{E'}(-\rho)\langle 0|d_\rho\left(k'\right)d_\sigma^+\left(k_1'\right)d_\sigma\left(k_2'\right)d_\rho^+\left(k'\right)|0\rangle\\ &= \int\frac{d^3k'}{(2\pi)^3}\frac{m}{E'}(-\rho)\\ &\quad\times\langle 0|\left\{d_\rho\left(k'\right), d_\sigma^+\left(k_1'\right)\right\}\left\{d_\sigma\left(k_2'\right), d_\rho^+\left(k'\right)\right\}|0\rangle\\ &= \int\frac{d^3k'}{(2\pi)^3}\frac{m}{E'}(-\rho)(-\sigma)\delta_{\sigma\rho}(2\pi)^3\frac{E'}{m}\delta^3\left(\boldsymbol{k}' - \boldsymbol{k}_1'\right)\\ &\quad\times(-\sigma)\delta_{\sigma\rho}(2\pi)^3\frac{E'}{m}\delta^3\left(\boldsymbol{k}' - \boldsymbol{k}_2'\right)\\ &= (-\sigma)(2\pi)^3\frac{E'}{m}\delta^3\left(\boldsymbol{k}_1' - \boldsymbol{k}_2'\right)\end{aligned} \tag{51}$$

where the integral over d^3k' is over those states whose energy changes sign and the factor $(-\rho)$ in the Lorentz-transformed vacuum ensures that it has positive norm. (For the sum over the chiralities ρ of the occupied states, we appeal to the summation convention.) This confirms that upon acting on the field operators and on the vacuum with the proper Lorentz transformation, the matrix $\mathcal{M}$ is covariant, but the necessity for a Lorentz transformation of the vacuum is unsettling. Furthermore, according to (50), in the Lorentz-transformed expression, the creation operator stands before the annihilation operator, which is somewhat unnatural.

Suppose, though, that the moving observer had used a Lorentz-invariant vacuum $|0\rangle$ and had evaluated the process according to form-invariant field operators which differ from those of the laboratory frame only in the space-time arguments. According to Fig. 5, the moving observer identifies antiparticle creation with momentum k_1' and antiparticle annihilation with momentum k_2'. So, the amplitude evaluated by the moving observer using shape-invariant field operators and a Lorentz-invariant vacuum is

$$\mathcal{M}' = \langle 0|d_\sigma(k_2')d_\sigma^+(k_1')|0\rangle = (-\sigma)(2\pi)^3 \frac{E'}{m} \delta^3(\boldsymbol{k}_1' - \boldsymbol{k}_2'), \tag{52}$$

which is exactly equal to the result obtained using the Lorentz-transformed vacuum, and the Lorentz-transformed field operators.

We here conjecture that a generalization of this statement should hold, namely, that the Lorentz-transformed probability amplitude for any process, calculated using shape-invariant field operators for the tachyonic field (where creators and annihilators retain their entity and only the space-time arguments are transformed) is equal to the amplitude calculated using Lorentz-transformed field operators (which mix creators and annihilators) and the Lorentz-transformed vacuum. In this context, it is instructive to observe that the form of the tachyonic propagator given in (38) involves only Lorentz-invariant quantities.

5 Neutrinoless double beta decay

Finally, let us briefly comment on charge conjugation. It is known and relatively easy to show that the (ordinary, tardyonic) Dirac equation is invariant under charge conjugation, provided one changes the sign of the charge $e \to -e$ when describing the antiparticle. For neutral particles, the Dirac equation thus is fully charge conjugation invariant. Therefore, it is possible to construct Majorana solutions which are invariant under charge conjugation. For the massless Dirac equation, we have

$$\Psi_+(x) = u_{(+)}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}k\cdot x} + \big(-v_-(\boldsymbol{k})\big)\mathrm{e}^{\mathrm{i}k\cdot x}, \tag{53}$$

which is invariant because

$$\begin{aligned}\Psi_+^{\mathcal{C}}(x) &= C\bar{\Psi}_+(x)^{\mathrm{T}} \\ &= C\overline{u}_+(\boldsymbol{k})^{\mathrm{T}}\mathrm{e}^{\mathrm{i}k\cdot x} + C\big(-\overline{v}_-(\boldsymbol{k})^{\mathrm{T}}\big)\mathrm{e}^{-\mathrm{i}k\cdot x} \\ &= \big(-v_-(\boldsymbol{k})\big)\mathrm{e}^{\mathrm{i}k\cdot x} + u_+(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}k\cdot x} = \Psi_+(x).\end{aligned} \tag{54}$$

However, the tachyonic Dirac equation is not charge conjugation invariant but changes [22] under charge conjugation to

$$\big(\mathrm{i}\gamma^\mu \partial_\mu + \gamma^5 m\big)\psi^{\mathcal{C}}(x) = 0, \tag{55}$$

i.e., the sign of the γ^5 term reverses its sign. This is because the tachyonic Dirac equation is $\mathcal{CP}$, and $\mathcal{T}$ invariant, but not $\mathcal{C}$ invariant [22]. It thus describes a tachyonic fermionic field with particle and antiparticle states which are manifestly different. It is rather straightforward to calculate the properties

$$\mathcal{U}_+^{\mathcal{C}}(\boldsymbol{k}) = C\overline{\mathcal{U}}_+(\boldsymbol{k})^{\mathrm{T}} = -\mathcal{U}_-(\boldsymbol{k}), \tag{56a}$$

$$\mathcal{U}_-^{\mathcal{C}}(\boldsymbol{k}) = C\overline{\mathcal{U}}_-(\boldsymbol{k})^{\mathrm{T}} = -\mathcal{U}_+(\boldsymbol{k}), \tag{56b}$$

$$\mathcal{V}_+^{\mathcal{C}}(\boldsymbol{k}) = C\overline{\mathcal{V}}_+(\boldsymbol{k})^{\mathrm{T}} = -\mathcal{V}_-(\boldsymbol{k}), \tag{56c}$$

$$\mathcal{V}_-^{\mathcal{C}}(\boldsymbol{k}) = C\overline{\mathcal{V}}_-(\boldsymbol{k})^{\mathrm{T}} = -\mathcal{V}_+(\boldsymbol{k}), \tag{56d}$$

which hold for the solutions of the tachyonic Dirac equation. Particle solutions are not transferred to antiparticle solutions by charge conjugation, but rather, to solutions with the opposite chirality, in full agreement with the relation of (2) to (55) [$\gamma^5 \to -\gamma^5$]. It thus seems unlikely that one could modify the tachyonic Dirac equation such as to allow for charge conjugation invariant solutions, or, to construct a Majorana field from the solutions of the tachyonic Dirac equation. Neutrinoless double decay thus is not allowed if we assume that the neutrino is described by the tachyonic Dirac equation.

6 Conclusions

Let us summarize a few observations made in our investigations. Feinberg [5] noted that if one writes the field operator with creation and annihilation operators, then it is impossible to impose the canonical commutation relations for the field operators, because under a Lorentz transformation, creation operators may transform into annihilation operators and vice versa, which reverses the sign of the quantization condition if commutators are used [see (4.8) of Ref. [5]]. He therefore suggested to quantize a scalar, tachyonic theory using anticommutators, effectively imposing Fermi–Dirac statistics onto scalar particles. Alternatively, we may interpret the necessity to invoke anticommutators instead of commutators for the quantization of a tachyonic theory as suggesting that only fermions are suitable candidates for tachyonic particles. *This observation applies to the neutrinos which are spin-$\frac{1}{2}$ particles.*

We here introduce the tachyonic Dirac equation [see (2)] which is obtained from the ordinary (tardyonic) Dirac equation by a matrix-valued representation of the imaginary unit $\mathrm{i} \to \beta\gamma^5$. The spinor solutions to the equation have some peculiar properties and normalizations and imply special anticommutation relations (30) which imply that the right-handed neutrino states have negative norm [see (31)]. Such states, for the indefinite-metric photon field, are excluded from the physical states by a Gupta–Bleuler type condition while being present in the propagator. *This observation applies to the neutrinos which have never been observed in right-handed helicity states.*

The tachyonic Dirac equation applies to particles which cannot be stopped; they remain superluminal in any subluminal reference frame. The rest frame is that of infinite velocity [see (22) and (23)]. The charge conjugate of the tachyonic Dirac equation implies that the antiparticles of particles described by the tachyonic Dirac equation differ only in the sign of the hirality, or helicity [cf. (2) and (55)]. Furthermore, the tachyonic Dirac equation is $\mathcal{CP}$ invariant, where we observe that the parity transformation again restores the original sign of the helicity. *This observation applies to neutrinos and antineutrinos which differ in the sign of the helicity and have never been observed at rest.*

We have therefore carried out a more detailed analysis of the spinor solutions of the tachyonic Dirac equation (Sect. 2), before quantizing the theory in Sect. 3, using anticommutators. We find that helicity-dependent anticommutators have to be used in order to quantize the theory. The tachyonic Dirac equation naturally implies a physically different behavior of the chirality components of the neutrino field. Related observations, which avoid an explicit quantization of the fields and use the Lagrangian formalism, have been described previously in Refs. [19, 46]. Further considerations regarding the problematic Lorentz covariance of the vacuum state and possible solutions of this problem are described in Sect. 4. Finally, in Sect. 5 we conclude that if the neutrino is a tachyonic particle described by the tachyonic Dirac equations, then neutrinoless double beta decay is forbidden.

A few remarks on the experimental status are in order. *All of the following remarks are somewhat speculative at the current time and should thus be taken with a grain of salt.* The general trend in measurements of the neutrino mass square points to negative (tachyonic) values. By inspection of neutrino data [34–42, 47] (for a summary overview see [48]), one may conclude that neutrinos at higher energy (in the GeV range) exhibit a larger tachyonic mass square than those observed at low energies (in the keV range). The trend in all data is toward negative values of the mass square. If the trend in the data is confirmed, it implies a "running" of the effective neutrino mass with the energy, which we shall not discuss in any further detail here. (The neutrino mass running might otherwise suppress conceivable decay processes of the Cerenkov type, because the neutrino mass in the final state of the decay process may substantially differ from the effective neutrino mass in the initial state of the decay process.)

The trend in the observed neutrino masses [34–42, 47] implies that the neutrino approaches the light cone closer and closer as its energy decreases, because its effective tachyonic mass tends to small values (in the eV range) for a total neutrino energy in the keV range [36–39]. Even in the GeV range, the deviation toward superluminal velocities is "only" in the range of a few parts in 10^5 (see Refs. [34, 35]). The conceivable violation of the causality principle implied by the small deviation of the propagation velocity toward superluminal velocities is thus restricted to small kinematic regions, but still causes a number of fundamental concerns. Tachyonic theory, including the reinterpretation principle discussed in the current paper, may solve a number of these issues. One may speculate about possible restrictions on "allowed" violations of the causality principle observed in subluminal reference frames. For instance, one might conceive that violations of the causality principle could be subject to a generalized "uncertainty principle" which describes the allowed deviations from the velocity of light for particles in a specific energy, or frequency range.

Sudarshan is being quoted in [15] with reference to an imaginary demographer who studies population patterns on the Indian subcontinent: "Suppose a demographer calmly asserts that there are no people North of the Himalayas, since none could climb over the mountain ranges! That would be an absurd conclusion. People of central Asia are born there and live there: they did not have to be born in India and cross the mountain range. So with faster-than-light particles." If neutrinos are faster-than-light particles, then, since the deviations from c are small [34, 35], it might well be possible that, figuratively speaking, we are allowed to "summit" and to take a glance over the top of the mountain range, but not much more. Furthermore, if we try to do so over longer time intervals, i.e., at smaller energies, then the decrease of the effective neutrino mass with the energy implies that we are less and less allowed to do. Further considerations on these issues are beyond the scope of the current article. *Again, the statements in the last three paragraphs above are somewhat speculative and should be taken with a grain of salt; they still seem to be in order in view of a topical experimental result* [35].

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Appendix A: Dirac equation with imaginary mass

Let us consider the free Dirac equation with an imaginary mass,

$$\big(\mathrm{i}\gamma^{\mu}\partial_{\mu} - \mathrm{i}m\big)\psi'(x) = 0. \tag{A.1}$$

The corresponding imaginary-mass Dirac Hamiltonian reads

$$H' = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \mathrm{i}\beta m. \tag{A.2}$$

It has the property

$$H' = \eta' H'^{+} \eta'^{-1}, \quad \eta' = \gamma^5. \tag{A.3}$$

Let ψ' be an eigenfunction of H' with eigenvalue E'. Then, because the spectrum of the Hermitian adjoint of an operator consists of the complex-conjugate eigenvalues, there exists a wavefunction ϕ' with the property

$$H'^{+}\phi' = E^{*}\phi' \tag{A.4}$$

from which we infer that

$$\left(\eta' H'^{+} \eta'^{-1}\right)\eta\phi' = E^{*}\eta\phi' \tag{A.5}$$

and so $H'(\eta\phi') = E^{*}(\eta\phi')$. So, if E' is an eigenvalue of H', so is E'^{*}. Indeed, the eigenstates of the imaginary-mass Dirac Hamiltonian (A.2) can easily be found, simply by replacing $m \to \mathrm{i}m$ in the well-known solutions of the ordinary (tardyonic) Dirac equation. The latter are discussed in detail in Chap. 2 of Ref. [43]. The energy levels of the imaginary-mass Dirac equation are thus given by $E' = \sqrt{\boldsymbol{k}^2 + (\mathrm{i}m)^2} = \sqrt{\boldsymbol{k}^2 - m^2}$ and thus real (for $|\boldsymbol{k}| > m$). In view of (4), the Hamiltonian H' is pseudo-Hermitian [23–32]. Both the positive-energy as well as the negative-energy eigenvalues of H' are thus real rather than complex (for $|\boldsymbol{k}| > m$). Under time reversal, the imaginary-mass Dirac Hamiltonian changes to ($\boldsymbol{\alpha} \to -\boldsymbol{\alpha}$, $\boldsymbol{p} \to -\boldsymbol{p}$, $\mathrm{i}m \to -\mathrm{i}m$),

$$\mathcal{T} : H' = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \mathrm{i}\beta m \to \boldsymbol{\alpha} \cdot \boldsymbol{p} - \mathrm{i}\beta m, \tag{A.6}$$

i.e., the mass term changes its sign. By a straightforward generalization of the considerations described in Sect. II of Ref. [22], it can easily be shown that the imaginary-mass Dirac equation is $\mathcal{P}$ invariant, but the mass term changes sign under charge conjugation $\mathcal{C}$ and time reversal $\mathcal{T}$ (restoring $\mathcal{CPT}$ invariance).

A priori, one could implement the $\mathrm{i}\epsilon$ prescription in the form $m \to m(1 + \mathrm{i}\epsilon)$, and change E' to $E' = \sqrt{\boldsymbol{k}^2 - m^2 - \mathrm{i}\epsilon}$. However, this creates problems with regard to the reinterpretation principle, by which negative-energy solutions are being reinterpreted as antiparticle solutions propagating into the past. The inverse of the Hamiltonian, endowed with the $\mathrm{i}\epsilon$ prescription, should thus describe the propagation of a wave packet both into the future (particles) as well as into the past (antiparticles). Let us assume that the propagator again is given by the inverse of the Hamiltonian, with the $\mathrm{i}\epsilon$ prescription,

$$S'_T = \gamma^0 \frac{1}{E - H'} \to \gamma^0 \frac{1}{E - \boldsymbol{\alpha} \cdot \boldsymbol{p} - \mathrm{i}\beta(m + \mathrm{i}\epsilon)}, \tag{A.7a}$$

$$S'_T(k) = \frac{1}{\not{k} - \mathrm{i}(m + \mathrm{i}\epsilon)} = \frac{1}{\not{k} - \mathrm{i}m + \epsilon}. \tag{A.7b}$$

Combined with the reinterpretation principle, these formulas imply that propagation into the past is governed by the same Hamiltonian as propagation into the future. By contrast, as implied by (A.6), the mass term in the Hamiltonian changes sign under $\mathcal{T}$. It might still be possible to use the imaginary-mass Dirac equation for the description of tachyonic particles, but due to the lack of desired symmetry properties, such an investigation seems to be rather unattractive and may lead to an unnecessarily complex mathematical form of the field-theoretical propagator. Earlier attempts at a calculation of the field-theoretical propagator of spin-$\frac{1}{2}$ particles based on the imaginary-mass Dirac equation were not successful [12]. Fortunately, the tachyonic Dirac equation is $\mathcal{T}$ invariant [22], which facilitates the analysis presented in the current work.

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