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Traversal Times for Random Walks on Small-World Networks

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We study the mean traversal time τ for a class of random walks on Newman-Watts small-world networks, in which steps around the edge of the network occur with a transition rate F that is different from the rate f for steps across small-world connections. When $f \gg F$, the mean time τ to traverse the network exhibits a transition associated with percolation of the random graph (i.e., small-world) part of the network, and a collapse of the data onto a universal curve. This transition was not observed in earlier studies in which equal transition rates were assumed for all allowed steps. We develop a simple self-consistent effective-medium theory and show that it gives a quantitatively correct description of the traversal time in all parameter regimes except the immediate neighborhood of the transition, as is characteristic of most effective-medium theories.

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I. INTRODUCTION

There has been intense recent interest in statistical and dynamical properties associated with random networks possessing so-called "small-world" properties, i.e., networks whose bonds possess a large degree of local clustering, but with a relatively small minimal path length connecting nodes of the system. At least some of this interest has arisen from the realization that algorithmically constructed small-world networks (SWNs) introduced by Watts and Strogatz [1], and by Newman and Watts [2], appear to share many statistical features with social networks, and thus form a useful topological substrate on which to model dynamical processes relevant to the medical and social sciences [3]. Power grids and information networks may also possess properties associated with small-world or scale-free networks [4], and even certain aspects of polymers may be understood in terms of an underlying small-world structure [5].

Independent of their possible applications to a wide range of problems in the social, physical, and information sciences, small-world networks and their extensions are inherently interesting because they provide a class of disordered structures similar to those that have been studied in the condensed matter literature for many years [6–10]. There is an extensive literature, for example, in which the dynamics of random walkers diffusing through various kinds of disordered media have been studied through a variety of methods, including numerical simulations [6], percolation ideas [7], effective-medium theories (EMTs) [7–10], and others.

In the present paper we study traversal times associated with random walks on the Newman-Watts small-world networks (NWSWN's). These structures are defined on a ring of N sites, each of which has bonds that connect it to its $2K$ nearest neighbors. On this translationally invariant lattice, a (generally) disordered small-world structure is imposed by associating with probability q a SW connection along each of the remaining $N(N-2K-1)/2$ bonds in the system. For a given value of q, there are on average $n_{\mathrm{SW}}=q(N-2K-1)/2 \sim qN/2$ small-world bonds per site in the system, a parameter that allows for a useful comparison between SWNs of different size. Except for the end points at $q=1$ and $q=0$, where the system is perfectly ordered, the network forms a disordered system on which various forms of dynamics can be studied.

The additional feature of our analysis, which is based upon a master equation for the site occupation probabilities $P_n(t)$, is that we take the jump rate f associated with SW connections to be independent of the jump rate F associated with steps around the edge of the network. Models of this sort might arise, for example, in attempts to design, modify, or optimize existing information networks in order to reduce the mean access time, by incorporating a small number of fast connections into an existing random network already possessing a large number of other (perhaps slower) connections. By introducing this simple extension to the situation in which all allowed steps occur with the same probability (and hence same transition rate), our theory is able to reveal interesting universal behavior that occurs for $f \gtrsim F$, including a percolation transition that occurs at a critical fraction of SW bonds. Such a transition, which is clearly a consequence of a well-known result by Erdös and Rényi [11] for random graphs, was not observed in earlier random walk studies of the NWSWN [12,13], in which only equal jump probabilities ($f=F$) were considered. The transition that we observe numerically and analyze using a simple self-consistent effective-medium theory, is critically dependent on the fact that the topological disorder associated with the small-world connections is "quenched." We show, for example, that an exactly solvable model with "annealed disorder," in which the walker decides at each step whether to move along the edge or to a randomly chosen site elsewhere in the network, shows no such transition as a function of the branching ratio between steps around the edge of the ring and steps associated with small-world type "shortcuts."

The rest of the paper is laid out as follows. In the next section we introduce a master equation description of continuous time random walks on Newman-Watts small-world networks, and review the relation between the Green's func-

tions associated with those equations to the mean time τ for a random walker on such a network to traverse the system. We then present numerical calculations of the mean traversal time and show that for $f \gtrsim F$, there is a transition in the underlying transport mode between motion that is dominated (and limited) by edge diffusion and motion that takes place predominantly among small-world connections in the system. In that section we also consider a model with annealed disorder and show that it does *not* display the transition exhibited by the model with quenched disorder. In Sec. III we analytically consider the traversal time for NWSWN's using a simple self-consistent effective-medium theory similar to that which has been used to understand other disordered systems. As in previous work, we find that EMT provides an accurate and computationally efficient means for calculating transport properties as a function of the underlying parameters of the system, except in the immediate vicinity of the transition, where the underlying transport mechanism changes abruptly. We are thus able to use it to reliably investigate, in regimes away from such points, the manner with which various transport properties scale with system size and connectivity.

II. MASTER EQUATIONS, TRAVERSAL TIMES, AND PERCOLATION

We consider a continuous time random walk with symmetric, translationally invariant jump rates $F_{n,m}=F_{n-m}$ governing transitions between sites n and m on a one-dimensional ring of N sites. Although it is not crucial in the analysis that follows, for specificity we assume that $F_{n,m}=F$ is constant and nonzero only between a given site and its $2K$ closest neighbors on the ring, which we will refer to collectively as "neighbor sites." The small-world structure of the system is imposed by associating with probability q a nonzero SW hopping rate f along each of the $N(N-2K-1)/2$ non-neighbor bonds in the system. For $q=1$ and $q=0$, the system is perfectly ordered; in the former, all sites are connected, whereas in the latter, connections occur only between sites on the periphery of the ring. For intermediate values of q, a particle moving along the ring encounters sites from which it can move across the system without taking steps to its $2K$ nearest neighbors. The small-world connections thus superimpose a "random graph" structure on the otherwise translationally invariant lattice.

Random walks on any single NWSWN constructed in this manner can be described by the master equation

$$\frac{dP_m}{dt} = \sum_{n=-K}^{K} F(P_{m+n} - P_m) + \sum_{n\neq -K}^{K} f_{mn}(P_{m+n} - P_m) \qquad (1)$$

for the probability $P_m(t)$ for the particle to be at site m at time t, where we assume periodic boundary conditions so that $P_{m+N}=P_m$. In (1), the first sum on the right describes steps around the edge of the ring, while the last term describes SW transitions. In this last term, the f_{mn} are symmetric random rates that in any given realization equal f with probability q and equal zero with probability $1-q$.

A full solution to the problem involves obtaining the set of propagators, or Green's functions $g_{m,n}(t)$, the solutions to Eq. (1) for each initial site n at which the particle may start. These quantities, or their Laplace transforms $\tilde{g}_{m,n}(\varepsilon)=\int_0^\infty g_{m,n}(t)e^{-\varepsilon t}dt$, are readily computed for moderately sized systems $N \lesssim 10^3$ using standard numerical techniques. For example, rewriting the equations of motion (1) in the form

$$\frac{dP_m}{dt} + \sum_n A_{mn}P_n(t) = 0, \qquad (2)$$

which, along with Eq. (1), implicitly defines a transition matrix $\mathbf{A}$ with elements A_{mn}, we can write the Green's function in the time domain as the m,n element of an exponential

$$g_{m,n}(t) = [e^{-\mathbf{A}t}]_{m,n} \qquad (3)$$

of the matrix $\mathbf{A}$. Similarly, we can express its Laplace transform as the corresponding element

$$\tilde{g}_{m,n}(\varepsilon) = [(\varepsilon + \mathbf{A})^{-1}]_{m,n} \qquad (4)$$

of the resolvent matrix $(\varepsilon+\mathbf{A})^{-1}$, which is easily computed using matrix inversion routines.

Numerical solutions of this sort can be used to find information about the evolution of the probabilities $P_m(t)$, their moments, or other quantities that characterize transport in the system. In the condensed matter literature, for example, much attention has been placed on calculating the diffusion constant D which in an infinite Euclidean network characterizes the asymptotic linear growth of the mean-square displacement $\langle n^2(t)\rangle \sim Dt$ of an ensemble of random walkers. Insofar as we are interested in the properties of finite networks, in which the mean-square displacement always saturates at long times, we focus here on calculating properties that may be more relevant to applications for which small-world concepts are currently employed. Specifically, we focus here on calculating the mean traversal time τ, which we define to be the earliest time, on average, that a random walker visits the point on the ring the farthest from where it started. For a walker starting at site n at $t=0$, the mean time $\tau_{m,n}$ to arrive at an arbitrary site m for the first time is the first moment of the probability density $F_{m,n}(t)$ for a walker to first arrive at site m at time t for these initial conditions, i.e.,

$$\tau_{m,n} = \int_0^\infty F_{m,n}(t)t\,dt = -\lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}\int_0^\infty e^{-\varepsilon t}F_{m,n}(t)dt \qquad (5)$$

$$= -\lim_{\varepsilon\to 0}\frac{d\tilde{F}_{m,n}(\varepsilon)}{d\varepsilon}, \qquad (6)$$

where in the second line we have expressed the result in terms of the Laplace transform $\tilde{F}_{m,n}(\varepsilon)$ of $F_{m,n}(t)$. The first passage probability density $F_{m,n}(t)$ is related to elements of the Green's functions described above through the relation

$$g_{m,n}(t) = \int_0^t g_{m,m}(t-t')F_{m,n}(t')dt' \qquad (7)$$

which physically expresses the probability for the walker to be found at m at the current time t, in terms of its probability to have arrived at that site for the first time at some earlier

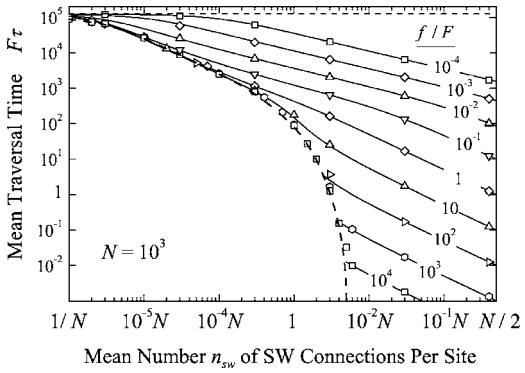

FIG. 1. Mean traversal time τ as a function of the mean number n_{SW} of small-world connections per site on a SWN of 10^3 sites, with values of f/F as indicated.

moment t', and the probability that it is now at that site given that it did so. From the convolution theorem for Laplace transforms, it then follows that

$$\tilde{F}_{m,n}(\varepsilon) = \frac{\tilde{g}_{m,n}(\varepsilon)}{\tilde{g}_{m,m}(\varepsilon)}, \tag{8}$$

so that we can write Eq. (5) as

$$\tau_{m,n} = -\lim_{\varepsilon \to 0} \frac{d}{d\varepsilon}\left[\frac{\tilde{g}_{m,n}(\varepsilon)}{\tilde{g}_{m,m}(\varepsilon)}\right]. \tag{9}$$

The mean first passage time is thus readily computed from a numerical solution to the Laplace transformed Green's function of the system (4), or equivalently, through the resolvent of the matrix **A**.

In Fig. 1 we display the results of a numerical calculation of the mean traversal time

$$\tau = N^{-1}\sum_{m} \langle \tau_{m+N/2,m} \rangle \tag{10}$$

scaled by the average nearest-neighbor hopping time F^{-1}, averaged over an ensemble of 100 $K=1$ NWSWN's with $N=10^3$ as a function of the average number n_{SW} of small-world bonds per site, for different values of the ratio f/F. As expected, the mean traversal time τ decreases monotonically as the number n_{SW} of small-world connections increases. In the limit in which $n_{\mathrm{SW}} < N^{-1}$, i.e., in the absence of any small-world shortcuts, the mean traversal time reduces to the mean time τ_0 that it takes for the walker to diffuse around the ring, for which $2F\tau_0 \sim (N/2)^2$. In addition, as we might expect, for very small values of f/F, only modest decreases in the traversal time occur with the addition of small-world network connections.

For values of $f \geqslant F$, however, the situation appears very different, there generally being a very strong decrease in the traversal time near $n_{\mathrm{SW}} \sim 1-5$, and a strong collapse of the numerical data onto a single curve for values of $n_{\mathrm{SW}} < 1$. In this region, SW connections are sparse, but fast. For $n_{\mathrm{SW}} < 1$ and $f \gg F$, the SW connections act as short-circuits, but transport across the system is still limited by diffusion within segments of the ring that are free of shortcuts, because there is not in this regime a percolating path of small-world connections spanning the system. However, as shown by Erdös and Rényi [11], in a completely random graph a percolating path develops (as $N \to \infty$) as n_{SW} approaches 1 from below. Thus, as $f/F \to \infty$, and $N \to \infty$, percolation of the random-graph part of the NWSWN network leads to a critical change in the traversal time near $n_{\mathrm{SW}} \sim 1$.

It is interesting to note that other transport properties of the NWSWN's, including scaling properties of the diagonal element $\tilde{g}_{00}(\varepsilon)$ of the average Green's function [12], and the distinct number of sites visited $S(t)$ by a walker on the network [12] have been studied previously, but only for the case in which $f=F$; i.e., in which hops associated with small-world connections occur with the same transition rate as steps along the periphery. As we see from the current study, however, the case $F=f$ lies at the edge of a parameter regime in which the behavior of the system, as a function of the number of small-world connections, changes drastically.

It is natural to ask whether the quenched disorder occurring in the small-world networks is an essential element for the observed transport threshold to occur, or whether a simpler *annealed disorder* model with uniform transnetwork connections could exhibit a similar transport transition as a function of their strength. We show explicitly below that a simple annealed disorder model does not exhibit such a transition. Specifically, we study an exactly soluble model in which a walker at any site takes steps to its neighbors on the edge of the ring with rate F, as before, and takes steps to one of n randomly chosen non-neighbor sites on the ring with rate f, but in such a way that the n randomly chosen sites to which it may move are not fixed in time, but are selected anew *each time the particle visits the site*. Thus, in this model, if we take $n=2n_{\mathrm{SW}}$ the branching ratio between steps associated with short cuts across the system and those around the edge is, on average, the same as in the NWSWN model with quenched disorder already considered. Since the *a priori* probability of the particle moving to any other site on the ring other than its neighbors is the same, the dynamics on this annealed disorder model is equivalent to one in which the shortcuts allow the walker to move to any one of its non-neighbors with a uniform rate

$$\omega = 2n_{\mathrm{SW}} f/N_n, \tag{11}$$

where $N_n = N-2K-1$ is the number of non-neighbor sites to which it can move. Thus, in any given site, the walker moves to one of its neighbors with probability $2KF/(2KF+2n_{\mathrm{SW}}f)$ and to a non-neighbor with probability $n_{\mathrm{SW}}f/(2KF+2n_{\mathrm{SW}}f)$. The resulting master equation for this model,

$$\frac{dP_m}{dt} = \sum_{n=-K}^{K} F(P_{m+n} - P_m) + \sum_{n \neq -K}^{K} \omega(P_{m+n} - P_m), \tag{12}$$

is translationally invariant and is readily solved by introducing Fourier transformed probabilities $P^k(t) = \Sigma_n P_n(t) e^{-ikn}$. We find, for example, that for this annealed disorder model the Laplace transformed Green's functions, $\tilde{g}^A_m(\varepsilon) = \tilde{g}^A_{m,0}(\varepsilon)$, are given by the relation

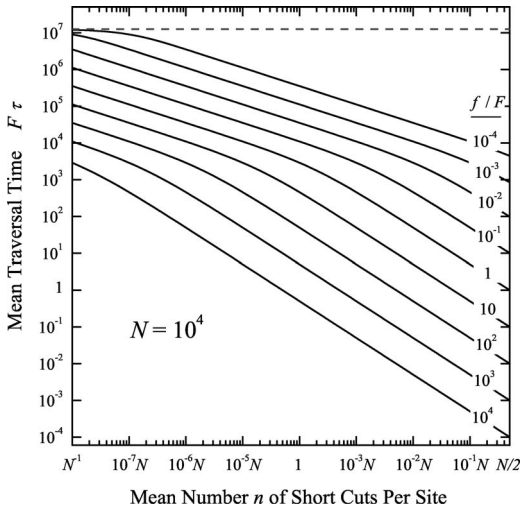

FIG. 2. Mean traversal time τ as a function of the mean number n of transnetwork connections per site for the annnealed disorder model on a system with 10^4 sites, with values of f/F as indicated, showing the failure of the annealed disorder to capture the transport transition observed to occur on real small-world networks.

$$\tilde{g}_m^A(\varepsilon) = \frac{N\omega}{N\varepsilon(\varepsilon + N\omega)} + \tilde{G}_m(\varepsilon + N\omega), \tag{13}$$

in which

$$\tilde{G}_m(\varepsilon) = N^{-1}\sum_k \frac{e^{ikm}}{\varepsilon + (F-\omega)A_k}, \tag{14}$$

$$A_k = \sum_{n=1}^{K} 2(1-\cos kn), \tag{15}$$

and the sum in (14) is over all wave vectors $k = 2\pi n/N$ in the Brillouin zone.

In Fig. 2 we show how the mean traversal time τ scales in the annealed disorder model as function of n (now considered a continuous parameter), for various values of f for a system with $N = 10^4$. Clearly, although the limiting diffusive behavior seen in the quenched disorder model of Fig. 1 as $n_{SW} \to 0$ and $f \to 0$ is the same as that observed in Fig. 2, the previously observed behavior for $f \gg F$, and in particular the transition near $n_{SW} \sim 1$ is completely lacking in the model with annealed disorder. Thus, as we might expect, the underlying percolative behavior is clearly a property associated with a system with quenched disorder. We note in passing that the annealed disorder model described above can be viewed as an *approximation* to the quenched disorder model in which the random matrix $\mathbf{A}$ is replaced by its ensemble average $\langle \mathbf{A} \rangle$; i.e., in which

$$\langle (\varepsilon + \mathbf{A})^{-1} \rangle \approx (\varepsilon + \langle \mathbf{A} \rangle)^{-1}. \tag{16}$$

For disordered systems, approximations of this type are known to give results that are often qualitatively very bad, particularly in percolative systems, where they typically wash out any transition that occurs. It is not surprising, therefore, that the two models have very different behaviors.

It is clear that the annealed disorder model fails to capture the essential features of the transport transition that occurs in the NW small-world networks. However, the idea of replacing a disordered system with a translationally invariant one that captures, on large time and length scales, the macroscopic transport properties of the ensemble of systems that it replaces is venerable and has a long history in the condensed matter literature. Indeed, the goal behind a significant body of theoretical work on various kinds of disordered systems has been to construct an appropriate, self-consistently determined effective medium, whose properties capture qualitative features of the actual system under consideration. In the next section, we explore this idea, and introduce a simple effective-medium theory that does, in fact, provide an excellent quantitative prediction of the traversal time for the NWSWN in all parameter regimes except the immediate neighborhood of the critical point.

III. SELF-CONSISTENT EFFECTIVE-MEDIUM THEORY OF THE TRAVERSAL TIME

Theoretical justification for the search for a translationally invariant effective medium whose properties capture the essential features of the ensemble of disordered systems that they replace (and which capture the large time and length scale properties, typically, for any member of the ensemble) lies with the fact that *average* transport properties of the ensemble are, in fact, translationally invariant. For example, the ensemble-averaged probabilities $p_m(t) = \langle P_m(t) \rangle$ associated with any fixed initial condition evolve in a translationally invariant way, and therefore obey translationally invariant equations of motion which, if we knew what they were, would serve to define the effective medium that we seek. Indeed, the goal of effective-medium theory is to self-consistently determine properties associated with this average evolution. To this end, we consider the simplest set of linear, homogeneous, and translationally invariant equations of motion that reflect the structure of the original equations, and the symmetry properties of the underlying ensemble. In particular, the Laplace transforms $\tilde{p}_m(\varepsilon)$ of the average probabilities $p_m(t)$ we take to obey the equations

$$\varepsilon\tilde{p}_m(\varepsilon) - p_m(0) = \sum_{n=-K}^{K} F(\tilde{p}_{m+n} - \tilde{p}_m) + \sum_{n\neq -K}^{K} \tilde{w}(\varepsilon)(\tilde{p}_{m+n} - \tilde{p}_m). \tag{17}$$

Here, $\tilde{w}(\varepsilon)$ is a frequency-dependent rate—equivalently, a memory function in the Laplace domain [14]—connecting pairs of sites on the network capable of being connected, in any realization, by a small-world rate f. In these effective-medium equations of motion, transport *around* the ring edge is characterized by the same rate F that obtains throughout the ensemble, but $\tilde{w}(\varepsilon)$ must be determined from self-consistent considerations. Note that these EMT equations of motion have exactly the same form as the Laplace transform of those describing the annealed disorder model of the last

section, except that in that model the rate $\tilde{w}(\varepsilon)=\omega$ was simply set equal to its ensemble average, while in the current treatment $\tilde{w}(\varepsilon)$ is to be determined self-consistently through other considerations.

Thus, the theoretical tasks are two: (i) the self-consistent determination of $\tilde{w}(\varepsilon)$ as a function of N and q (or n_{SW}), and (ii) the determination of transport properties arising through the solutions to (17). Both tasks require calculation of the effective-medium propagators $\tilde{g}_m^e(\varepsilon)$, i.e., the solutions to (17) for a walker initially at the origin,: $p_m(0)=\delta_{m,0}$. Since the structure of the equations of motion (17) are the same as those of the annealed disorder model, the form of the effective-medium Green's function associated with (17) is the same as (13), with ω replaced by $\tilde{w}(\varepsilon)$:

$$\tilde{g}_m^e(\varepsilon)=\frac{N\tilde{w}}{N\varepsilon(\varepsilon+N\tilde{w})}+\tilde{G}_m(\varepsilon+N\tilde{w}), \tag{18}$$

where

$$\tilde{G}_m(\varepsilon)=N^{-1}\sum_k \frac{e^{ikm}}{\varepsilon+2(F-\tilde{w})A_k}. \tag{19}$$

Using (18), we now develop a self-consistent expression for $\tilde{w}(\varepsilon)$. We proceed as in other EMTs [7–10], by embedding in the effective medium a typical "fluctuation from the average" associated with the disorder, and then requiring that upon averaging over the distribution associated with this fluctuation, we recover the effective medium in which it was embedded. Thus, we consider the bond between the origin and some other non-neighbor site n on the ring, and replace the effective-medium bond $\tilde{w}(\varepsilon)$ between those two sites with an actual rate $f_{0,n}=f_n$ (=0 or f), drawn from the ensemble. For a particle placed at the origin, the resulting $\tilde{p}_m(\varepsilon)$ obey

$$\varepsilon\tilde{p}_m(\varepsilon)-\delta_{m,0}=\sum_{n=-K}^{K}F(\tilde{p}_{n+m}-\tilde{p}_m)+\sum_{n\neq -K}^{K}\tilde{w}(\varepsilon)(\tilde{p}_{m+n}-\tilde{p}_m)-\tilde{\Delta}_m(\varepsilon), \tag{20}$$

where $\tilde{\Delta}_m(\varepsilon)=(\delta_{m,0}-\delta_{m,n})(f_n-\tilde{w})(\tilde{p}_0-\tilde{p}_n)$. The solution to the set of Eqs. (20) can be written in terms of (18) as

$$\tilde{p}_0=\tilde{g}_0^e-\frac{\tilde{\alpha}_n\tilde{\gamma}_n^2}{1+2\tilde{\alpha}_n\tilde{\gamma}_n}, \tag{21}$$

where $\tilde{\alpha}_n(\varepsilon)=f_n-\tilde{w}(\varepsilon)$, and $\tilde{\gamma}_n(\varepsilon)=\tilde{g}_0^e(\varepsilon)-\tilde{g}_n^e(\varepsilon)$.

We now impose self-consistency, and require that, upon averaging (21) over the dichotomous distribution

$$\rho(f_n)=(1-q)\delta(f_n)+q\delta(f_n-f) \tag{22}$$

of the rates f_n, on the one hand, *and* the location n of the small-world connection, on the other, we recover the propagator for the effective medium. For this to occur, the second term on the right-hand side of (21) must average out to zero. This leads to the self-consistent condition

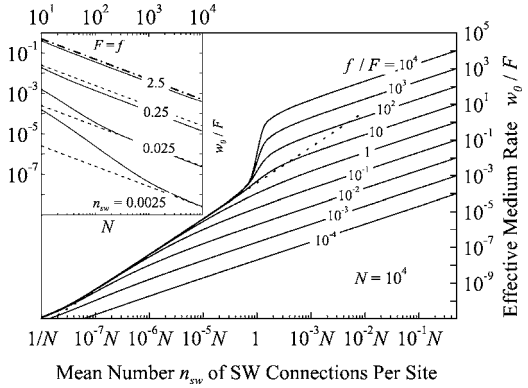

FIG. 3. Effective-medium parameter w_0 as a function of the mean number n_{SW} of small-world connections per site on a SWN of 10^4 sites, with values of f/F as indicated. Inset: Effective-medium parameter w_0 as a function of the number N of sites in the network, for the case $f=F$.

$$\sum_{n=K+1}^{N/2}\frac{q(f-\tilde{w})\tilde{\gamma}_n^2}{1+2(f-\tilde{w})\tilde{\gamma}_n}=\sum_{n=K+1}^{N/2}\frac{(1-q)\tilde{w}\tilde{\gamma}_n^2}{1-2\tilde{w}\tilde{\gamma}_n}, \tag{23}$$

from which $\tilde{w}(\varepsilon)$ can be determined. For $q=1$ and $q=0$, respectively, the solutions reduce to the exact results. For $q\neq 0$ or 1, Eq. (23) can be solved numerically without having to perform large matrix inverses.

Because we are interested primarily in properties of the network associated with large length scales, and thus with times much longer than that required for a single hop, we are interested in the Laplace domain on the behavior of the system for small values of the Laplace variable ε as in Eq. (9). Accordingly, we present in Fig. 3 a plot of the zero-frequency effective-medium parameter $w_0=\lim_{\varepsilon\to 0}\tilde{w}(\varepsilon)$ as a function of the number $n_{\mathrm{SW}}=qN/2$ of small-world connections per site, for different values of the SW hopping rate f, on a NWSWN with $N=10^4$ sites. On the right-hand side of the figure, at sufficiently large values of n_{SW} for any value of f, the effective-medium parameter w_0 appears well-described by the formula $w_0\sim qf\sim 2n_{\mathrm{SW}}f/N$. As we move to values of $n_{\mathrm{SW}}\sim 1$ and less, there is a convergence of the curves with $f>F$, similar to that which appeared in Fig. 1 for the traversal time of the NWSWN, thus giving hope that the self-consistent treatment introduced here provides a qualitatively correct description of transport in this disordered system. After some analysis of the results, we find that in the region $1>n_{\mathrm{SW}}$, where the different curves with $f>F$ converge, the parameter w_0 is independent of f and is well-characterized by the relation $w_0\sim Nq^2F\sim 4n_{\mathrm{SW}}^2F/N$, depicted as a dashed line in the figure. As $N\to\infty$, the region of the main part of Fig. 3 associated with this universal behavior extends downward and to the left, so that for any f there will be a value $n_{\mathrm{SW}}(f)$ below which w_0 will scale as $4n_{\mathrm{SW}}^2F/N$. In the inset of Fig. 3 we show how this scaling relation is obeyed as a function of system size for large N and small n_{SW}, for the case in which $f=F$. Dotted lines in the inset indicate the relation $w_0\sim 4n_{\mathrm{SW}}^2F/N$, which the curves with small n_{SW} approach for large N. At moderate values of $n_{\mathrm{SW}}>1$, scaling crosses over to the form $w_0\sim 2n_{\mathrm{SW}}f/N$, represented by the dot-

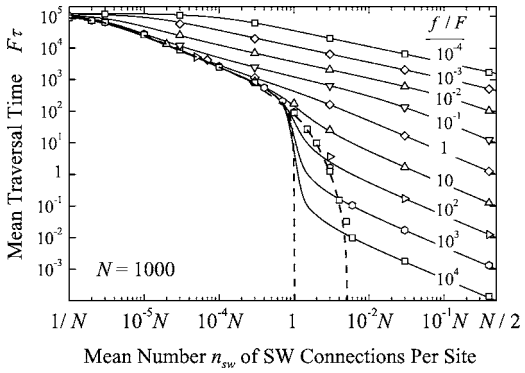

FIG. 4. Average time to first traverse the network as a function of the mean number n_{SW} of small-world connections per site, on a SWN of 10^3 sites, with values of f/F as indicated. Solid lines are from EMT; data points are the same as in Fig. 1.

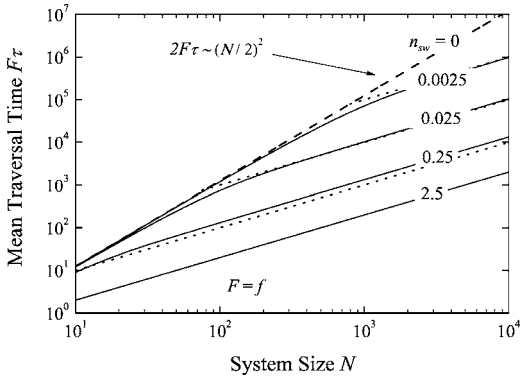

FIG. 5. Mean traversal time as a function of the number N of sites in the network, for $f=F$. Dashed line is the limiting case for pure diffusion, solid lines are the results of EMT. Dotted lines indicate scaling relationships discussed in the text.

dashed line, shown for $n_{\mathrm{SW}}=2.5$. For very small values of $n_{\mathrm{SW}} \lesssim N^{-1}$, uninteresting deviations from SW scaling occur in the regime where most members of the ensemble have no SW connections.

This interesting behavior of the effective medium parameter w_0 leads to two important questions. Does the behavior seen in the parameter w_0 lead to a corresponding effect on actual transport observables? Do the predictions of EMT for those observables describe the actual system of interest? At least in terms of the mean traversal time, the answer to both of these questions is affirmative. The derivatives and small-ε limit in (9) can be explicitly taken for the effective-medium propagators (18), leading to the simplification

$$\tau_m = N[\tilde{G}_0(Nw_0) - \tilde{G}_m(Nw_0)]. \tag{24}$$

In Fig. 4, we plot the mean traversal time $\tau=\tau_{N/2}$ as a function of n_{SW} for a system with $N=10^3$ sites. Solid lines in this figure are predictions of EMT, while the data points are the same as those appearing in Fig. 1. In Fig. 4 we see that the effective-medium theory predicts a transition in the vicinity of $n_{\mathrm{SW}}=1$, as well as a collapse of the numerical data for large f/F onto a single curve for $n_{\mathrm{SW}}<1$. For all values of f, traversal times saturate for very small n_{SW} to a value such that $2F\tau \sim (N/2)^2$, consistent with pure diffusion around the edge of the ring. In the region $1 \gtrsim n_{\mathrm{SW}} \gtrsim N^{-1}$ where the data collapse occurs for $f>F$, the traversal time of the numerical data and the EFT are both described by the functional relation $\tau \sim (2qF)^{-1} = N/4n_{\mathrm{SW}}F$, which is a reflection of the mean number of steps $F\tau \sim (2q)^{-1}$ the particle must take along the edge of the ring before it encounters a SW shortcut across the system.

Comparison of the data and the solid curves in Fig. 4 shows that, as in other EMTs, the theory derived above appears to be numerically accurate for all parameters except those near the underlying percolation transition, the effect of which only becomes apparent for large values of f/F in the neighborhood of $n_{\mathrm{SW}} \sim 1$. Provided that we avoid this particular regime (i.e., the region bounded by the dashed curves in Fig. 4), we can reliably use our EMT to calculate accurately other properties of the system. The two dashed curves in that figure represent limiting cases for very large $f/F > 10^8$ for the EMT (on the left) and the numerical results (on the right). The discrepancy between EMT and the exact numerical results does not decrease significantly with increasing N. Thus, the EMT of the present paper is typical of those for other disordered systems, which often capture the essential behavior of the system, but fail to accurately reproduce critical properties. Of course, the main advantage of the EMT we have derived is that we can now extend its calculations to larger networks (i.e., larger values of N) for which numerical solutions to the individual equations of motion (1) become prohibitively difficult. As an example, we present in Fig. 5 a plot of traversal time τ as a function of system size N. This plot has been computed using EMT for fixed values of n_{SW}, for the case $f=F$, which is accurately described by EMT for all n_{SW}, as is evident from Fig. 4. Clearly, the predicted scaling $\tau \sim N$, shown as dotted lines in Fig. 5, obtains for any value of n_{SW} for sufficiently large values of N. Indeed, for small n_{SW}, the time to traverse the system follows closely the diffusive result $\tau \sim N^2$ until the system size is sufficiently large that there is a significant probability to encounter a shortcut before reaching the other side of the ring via diffusion around the periphery.

IV. SUMMARY

In this paper we have considered the mean traversal time τ for a class of random walks on Newman-Watts small-world networks, in which steps around the edge of the network occur with a transition rate F that is different than the rate f for steps across small-world connections. Using numerical calculations of the Green's function we obtained present numerical data for the mean traversal time and showed that for $f \gtrsim F$, there is a transition in the underlying transport mode between diffusion limited and connection limited motion. We showed that a model with annealed disorder does not display a transition reflective of the underlying percolation transition occurring on the random-graph part of the network. We then

developed a self-consistent effective-medium theory (EMT) for the traversal time that provides an accurate and computationally efficient means for calculating this quantity except in the immediate vicinity of the transition. As a result, we have been able to use EMT to determine how various transport properties scale with system size and connectivity on the Newman-Watts small-world networks.

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