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Order-parameter symmetry and mode-coupling effects at dirty superconducting quantum phase transitions

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We derive an order-parameter field theory for a quantum phase transition between a disordered metal and an exotic (non-*s*-wave) superconductor. Mode coupling effects between the order parameter and other fermionic soft modes lead to an effective long-range interaction between the anomalous density fluctuations which is reflected in singularities in the free energy functional. However, this long-range interaction is not strong enough to suppress disorder fluctuations. The asymptotic critical region is characterized by run-away flow to large disorder. For weak coupling, this asymptotic region is very narrow. It is preempted by a wide crossover regime with mean-field critical behavior and, in the *p*-wave case, logarithmic corrections to scaling in all dimensions.

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I. INTRODUCTION

Quantum phase transitions are one of the most intriguing problems in today's condensed matter physics.[1–3] In addition to being of fundamental interest, they are believed to underlie a number of interesting low-temperature phenomena, in particular, various forms of exotic superconductivity.[4–6]

In a seminal paper,[1] Hertz introduced a general scheme for the analysis of quantum phase transitions in itinerant electronic systems. This scheme is based on the Landau-Ginzburg-Wilson (LGW) approach of integrating out the fermionic degrees of freedom and deriving a free energy functional in terms of the order parameter fluctuations only. However, in recent years, it has become clear that for many transitions, there are problems with Hertz' scheme because in addition to the order parameter fluctuations, which are soft (gapless) at the critical point, there are additional fermionic soft modes in the system. These additional soft modes exist not only at the critical point but also in the bulk phases. They are related to conservation laws and/or broken symmetries and constitute examples of generic scale invariance.[7] If the coupling between the order parameter and these additional soft modes is sufficiently strong it generates an effective long-range interaction between the order parameter fluctuations. This is reflected in a nonanalytic wave-number dependence of the vertices of the LGW theory of the corresponding transition.[8–10] Generically, such nonlocalities in the LGW theory will lead to non-mean-field critical behavior of the quantum phase transition.

The precise influence of the mode-coupling effects on a quantum phase transition depends on the structure of the additional soft modes and their coupling to the order parameter. For the clean ferromagnetic transition, the mode-coupling effects can either lead to a fluctuation-induced first order transition or to non-mean-field critical behavior.[11,12] For dirty electrons, the transitions is generically of second order but with highly unusual exponents.[13,14] Even stronger effects were found for the transition between a dirty metal and a conventional (*s*-wave) superconductor.[15] Here, the mode-coupling effects lead to a critical point with exponential scaling, i.e., the correlation length behaves as $\xi \sim e^{1/|t|}$, where t is the distance from the quantum critical point. Based on general symmetries of itinerant electronic systems, it was recently shown[16] that homogeneous ($\mathbf{q}=0$)[17,18] order parameters in the particle-particle (Cooper) and spin-triplet particle-hole channels are generically affected by mode coupling effects while order parameters in the particle-hole spin-singlet channel do allow for a local LGW theory.

All of the above examples are quantum phase transitions with zero angular momentum order parameters. The effect of mode coupling on order parameters with finite angular momentum are much less understood. Herbut[19] studied the *d*-wave superconducting quantum phase transition in two dimensions within Hertz' scheme (which is equivalent to Gorkov's mean-field theory). He found that the typical Cooper channel (BCS) logarithmic singularities are demoted to irrelevant terms by the *d*-wave symmetry. This raises the important general question: How does a finite order parameter angular momentum influence the coupling between order parameter and additional fermionic soft modes?

In this paper, we study this question for quantum phase transitions between a metal and an exotic (non-*s*-wave) superconductor in the presence of nonmagnetic quenched disorder. These transitions are of experimental importance since various superconducting states with p,d and maybe higher symmetries have been observed recently, and their quantum phase transitions are experimentally accessible[4–6] in principle. Experiments performed on the weakly ferromagnetic compounds UGe_2 (Ref. 5) and $ZrZn_2$ (Ref. 6) revealed the existence of a superconducting phase within the ferromagnetic phase at temperatures below 1K. It is believed[5] that both superconductivity and ferromagnetism arise from the same electrons. One possible mechanism for this type of superconductivity is *p*-wave triplet pairing mediated by mag-

netic fluctuations in the vicinity of a magnetic quantum critical point,[20,21] although this is still not a settled issue. The onset of the phenomenon has proven to be very sensitive to the presence of nonmagnetic disorder, making it observable only in highly pure samples. This fact also points to a non-*s*-wave order parameter. In $ZrZn_2$ the superconducting quantum phase transition as a function of disorder has already been observed.[6]

Our results can be summarized as follows: Mode-coupling induced singularities exist for all order parameter angular momenta L. However, with increasing L, they are more and more suppressed (by a factor $|\mathbf{q}|^{2L}$.) As a result, the LGW theory of our superconducting transition is equivalent to that of the itinerant antiferromagnetic transition.[1,22–24] While the asymptotic critical behavior of this theory is not understood because of runaway flow to large disorder, we also show that for weak bare disorder the asymptotic region is exponentially narrow. It is preempted by a wide crossover regime with mean-field critical behavior and (in the *p*-wave case) logarithmic corrections to scaling. The paper is organized as follows. In Sec. II we derive the LGW free energy functional. In Sec. III, we study the LGW theory by means of the renormalization group and determine the critical behavior. In Sec. IV, we analyze our findings from a mode-coupling point of view, and discuss differences between paramagnetic and ferromagnetic as well as clean and dirty systems.

II. LANDAU-GINZBURG-WILSON THEORY

A. *p*-wave pairing case

In this section, we derive an effective LGW theory for the disorder-driven quantum phase transition between a paramagnetic metal and a *p*-wave triplet superconductor. Our starting point is a microscopic action for interacting electrons in $d>2$ dimensions and subject to a static, single-particle random potential $V(\mathbf{x})$. We assume a Gaussian distributed potential with $[V(\mathbf{x}_1)V(\mathbf{x}_2)]_{\text{dis}}=W\delta(\mathbf{x}_1-\mathbf{x}_2)$, with W being measure of disorder strength. The partition function Z can be written as a functional integral over Grassmann variables $\psi, \overline{\psi}$:

$$Z=\int D[\overline{\psi},\psi]e^{S[\overline{\psi},\psi]}. \tag{1}$$

We decompose the action $S=S_p+S_0$ into the *p*-wave interaction part S_p and a reference system S_0 which comprises the single-particle part, the random potential and S_{int} (the interaction in all channels other than the *p*-wave channel):

$$S_0=\int dx\sum_\sigma \overline{\psi}_\sigma(x)\left[\partial_\tau+\frac{\nabla^2}{2m}+\mu\right]\psi_\sigma(x) \tag{2}$$

$$+\int dx\sum_\sigma \overline{\psi}_\sigma(x)V(\mathbf{x})\psi_\sigma(x)+S_{\text{int}}, \tag{3}$$

$$S_p=\sum_{\{\sigma\}}\frac{\Gamma_t^{\{\sigma\}}}{2}\int dx\overline{\mathbf{n}}_{\sigma\sigma'}(x)\cdot\mathbf{n}_{\sigma_1\sigma_1'}(x). \tag{4}$$

We use a $(d+1)$-vector notation, with $x=(\mathbf{x},\tau)$, $k=(\mathbf{k},\Omega)$, $\int dx=\int_V d^dx\int_0^\beta d\tau$ and $\Sigma_k=\Sigma_{\mathbf{k}}\,T\Sigma_\Omega$, $\mathbf{x}$ is a real space coordinate, τ imaginary time, $\mathbf{k}$ momentum vector, and Ω Matsubara frequency. $\mathbf{n}_{\sigma\sigma'}(x)$ is the *p*-wave anomalous density whose Fourier transform in terms of the fermion fields is given by

$$\mathbf{n}_{\sigma\sigma'}(q)=\sum_k \hat{\mathbf{e}}_{\mathbf{k}}\,\psi_\sigma(k+q/2)\psi_{\sigma'}(k-q/2), \tag{5}$$

with $\hat{\mathbf{e}}_{\mathbf{k}}=\mathbf{k}/|\mathbf{k}|$, σ, σ' being spin indices and where a centered dot denotes a scalar product in the orbital space. Due to the Pauli principle the spin state of the Cooper pair has to be a triplet, i.e., $\sigma\sigma'\in\{(\uparrow\uparrow),(\downarrow\downarrow),1/\sqrt{2}(\uparrow\downarrow+\downarrow\uparrow)\}$. Which combination of the three possible triplet components is actually realized depends on the system under consideration. The reference ensemble S_0 describes interacting electrons in the presence of nonmagnetic quenched disorder and no bare interaction in the *p*-wave Cooper channel. (A nonvanishing interaction in this channel will be generated in perturbation theory.) S_0 thus describes a general system of disordered interacting electrons with the only restriction being that it must not undergo a phase transition in the parameter region we are interested in.

A standard procedure[1] is used to derive a LGW order-parameter field theory. We decouple the interaction term using a Hubbard-Stratonovich transformation[25,26] introducing a complex field $\Delta_{\sigma\sigma'}(x)$ which plays the role of an order parameter. Quenched disorder treated using the replica trick,[27] and fermionic degrees are integrated out, leading to an expression for the critical part of the partition function in terms of the order parameter only

$$Z=\int \mathrm{D}[\Delta]\,e^{-\Phi[\Delta]}. \tag{6}$$

Since our emphasis is on the mode-coupling effects, and in order to avoid unnecessary complications in notation, we restrict our analysis to a certain spin component $[\sigma\sigma'=(\uparrow\uparrow)]$ of the order parameter. The LGW free energy $\Phi[\Delta]$ is expanded in powers of the order parameter $\Delta\equiv\Delta_{\uparrow\uparrow}$. Up to quartic order it reads

$$\begin{aligned}\Phi[\Delta]=&\sum_{q,\alpha}\overline{\Delta}^\alpha(q)[1-\Gamma_t\chi^{(2)}(q)]\Delta^\alpha(q)\\&-\sum_{\substack{q_1\cdots q_3\\ \alpha,\beta}}\Gamma_t^2\chi^{(4)}_{\alpha\beta}(q_1,q_2,q_3)\overline{\Delta}^\alpha(q_1)\Delta^\alpha(q_2)\\&\times\overline{\Delta}^\beta(q_3)\Delta^\beta(q_1+q_3-q_2)+O(\Delta^6),\end{aligned} \tag{7}$$

where $\Gamma_t^{\uparrow\uparrow}\equiv\Gamma_t$. Here α,β are replica indexes. The coefficients of the LGW functional are determined by the two-point and four-point anomalous density correlation functions of the reference system S_0 which can be written as $\chi^{(2)}=\langle\overline{n}^\alpha n^\alpha\rangle_0$ and $\chi^{(4)}_{\alpha\beta}=\langle\overline{n}^\alpha\overline{n}^\beta n^\beta n^\alpha\rangle_0$ (with the spin and component indices suppressed).

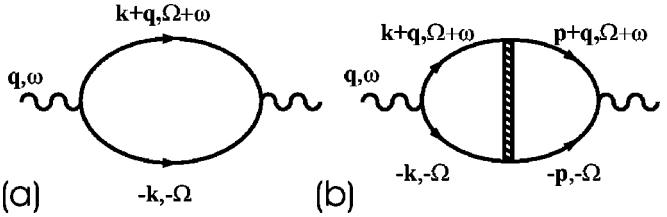

FIG. 1. Contributions to the leading terms of the Gaussian part of the LGW functional. (a) Provides a constant ($\sim N_F$) and the frequency dependence $|\omega|\tau$ while (b) gives the leading momentum dependence, $|\mathbf{q}|^2 \ln(1/|\mathbf{q}|)$.

B. Anomalous density correlation functions in the *p*-wave channel

In this subsection we use diagrammatic perturbation theory in disorder and interactions to calculate $\chi^{(2)}$ and $\chi^{(4)}$ of the reference ensemble S_0, focusing on the behavior for weak disorder. Thus, we neglect all diagrams with crossed impurity lines, i.e., all weak localization effects, which become important only at higher impurity concentrations. This is justified for the ferromagnetic superconductors[5,6] where the superconducting quantum phase transition occurs at very small disorder.

We start our analysis by examining the two-point correlation function $\chi^{(2)}=\langle \bar{n}^{\bar{\alpha}} n^{\alpha}\rangle_0$. The leading contributions are obtained from the diagrams shown in Fig. 1 (details of the calculation of these diagrams are given in Appendix A). Here, the external vertices represent anomalous *p*-wave densities, the solid lines are fermionic propagators in Born approximation

$$G_{\sigma}^{-1}(\mathbf{k},\omega) = i\omega - \epsilon_{\mathbf{k},\sigma} + \mu + (i/2\tau)\mathrm{sgn}(\omega), \tag{8}$$

where $\epsilon_{\mathbf{k},\sigma}$ is the dispersion relation and τ is the scattering time. The double line represents a particle-particle impurity ladder

$$W_R(\mathbf{q},\Omega,\omega) = W\begin{cases} 1 & \text{if } \Omega(\Omega+\omega)<0, \\ \dfrac{1}{|2\Omega+\omega|\tau + \ell^2|\mathbf{q}|^2/d} & \text{if } \Omega(\Omega+\omega)>0. \end{cases} \tag{9}$$

$\ell = k_F\tau/m$ is the elastic mean free path and $W=1/(2\pi N_F\tau)$ with N_F being the density of states at the Fermi level. The calculation of the diagrams in Fig. 1 for $|\mathbf{q}|=\omega=0$ leads to $\chi^{(2)}=(N_F/3)\ln(2\epsilon_F\tau)$. The well-known logarithmic Cooper channel (BCS) singularities are cut off by the disorder, reflecting the suppression of exotic superconductivity by nonmagnetic scatterers in analogy to the suppression of *s*-wave superconductivity by magnetic impurities.[28,29] [We note that, in contrast, *s*-wave superconductivity is not influenced by weak nonmagnetic scatterers, as is signified by Anderson's theorem.[30]] However, a closer investigation of Fig. 1(b) for finite $|\mathbf{q}|$ reveals that a nonanalyticity of the form $|\mathbf{q}|^2\ln(1/|\mathbf{q}|)$ survives. Thus, the *p*-wave symmetry has demoted the BCS singularity to quadratic order in $|\mathbf{q}|$ because each of the renormalized external vertices picks up an extra power of $|\mathbf{q}|$.[31]

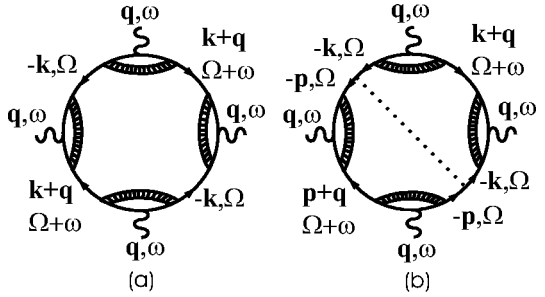

FIG. 2. Leading singular contributions to $\chi^{(4)}_{\mathrm{diag}}$. After expansion in small q, the leading order terms of (a) and (b) cancel (see Appendix C).

In addition to the BCS logarithms $\chi^{(2)}$ contains nonanalyticities similar to that in the itinerant ferromagnet. They are caused by the leading corrections to scaling at the dirty Fermi liquid fixed point[32] and can be viewed as particle-particle analogs of the well known Altshuler-Aronov corrections to density of states and conductivity.[33] For *s*-wave order parameters these singularities (which only arise for interacting electrons) are of the form $|\mathbf{q}|^{d-2}$.[32] For *p*-wave order parameters, an inspection of the corresponding contributions (for details see Appendix B) reveals that they are suppressed by a factor $|\mathbf{q}|^2$ by the same mechanism as the BCS logarithms. (Note that an analogous suppression occurs in the particle-hole channel, as can be seen from a power counting analysis of the Altshuler-Aronov correction to the conductivity: $\delta\sigma=\delta\langle jj\rangle/\omega\sim\omega^{(d-2)/2}\sim|\mathbf{q}|^{d-2}$. Thus, the correction to the current-current correlation function $\langle jj\rangle$, which is proportional to the *p*-wave density, behaves as $\delta\langle jj\rangle\sim|\mathbf{q}|^d$. This means it has picked up an additional factor $|\mathbf{q}|^2$ compared to the zero angular momentum channel.)

As a result, we find that the leading singularities in $\chi^{(2)}$ are the Cooper channel logarithms, and the leading behavior of $\chi^{(2)}$ for $(|\mathbf{q}|,\omega)\to 0$ is given by

$$\chi^{(2)}_{ij}(\mathbf{q},\omega) = \frac{N_F}{3}\left[\ln(2\epsilon_F\tau) - |\omega|\tau - \frac{\ell^2|\mathbf{q}|^2}{10}\right]\delta_{ij} + \frac{N_F}{3}q_iq_j\ell^2\left[-\frac{1}{5}+\frac{1}{3}\ln\left(\frac{\ell^2|\mathbf{q}|^2}{3}\right)\right], \tag{10}$$

where i,j are the order parameter component indices. The anisotropic $\mathbf{q}$ dependence in the last term reflects the spatial anisotropy of the *p*-wave order parameter.

We now turn our attention to the four-point correlation function $\chi^{(4)}$which can be split into a replica-diagonal part and a replica-off-diagonal part $\chi^{(4)}_{\alpha\beta}=\delta_{\alpha\beta}\chi^{(4)}_{\mathrm{diag}}+\chi^{(4)}_{\mathrm{off}}$. A detailed discussion of our calculation is given in Appendix C. We find that the leading contributions to $\chi^{(4)}_{\mathrm{diag}}$ in the long-wavelength, low-frequency limit are coming from the diagrams shown in Fig. 2. While each of the two diagrams individually diverges for $(\mathbf{q},\omega)\to 0$, their leading singularities cancel, and the remaining contribution is finite and proportional to $N_F\tau^2$. The leading contribution to the replica-off-diagonal part of $\chi^{(4)}$ is produced by the diagram shown on the Fig. 3. Thus, we finally obtain

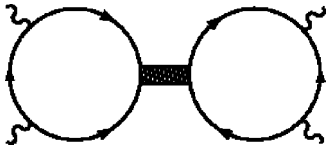

FIG. 3. Leading contribution to the $\chi^{(4)}_{\rm off}$ (see Appendix C).

$$\chi^{(4)}_{\rm diag} = A_d\, N_F \tau^2\, F_d(\mathbf{q}_i, \omega_i), \tag{11}$$

$$\chi^{(4)}_{\rm off} = A_0 (N_F^2/k_F^3) F_0(\mathbf{q}_i, \omega_i), \tag{12}$$

where F_d and F_0 are dimensionless functions with values between 0 and 1 and A_d and A_0 are dimensionless prefactors of order 1.

The results (11) and (12) have been obtained from low order perturbation theory. Within perturbation theory, it is nontrivial to prove the leading nonanalyticity to all orders. Therefore we follow the guidance of the corresponding results for the *s*-wave case which have been rigorously established using *Q* field theory and renormalization group arguments.[32] (Corresponding rigorous results for finite angular momentum modes do not yet exist.) Indeed, simply correcting the *s*-wave results from Ref. 15 for the *q* dependence of the renormalized vertex leads to $\chi^{2n} \sim |\mathbf{q}|^{4-2n}$ in agreement with Eqs. (11) and (12). Note that the singularity becomes stronger in the higher order anomalous density correlation functions, in agreement with general mode-coupling arguments.[16,34]

Inserting Eqs. (10)–(12) into Eq. (7) we obtain the LGW functional up to quartic order in Δ,

$$\begin{aligned}\Phi = & \sum_{\substack{q,\alpha \\ i}} \Delta^{*\alpha i}(q)(t + |\omega| + c_2|\mathbf{q}|^2)\Delta^{\alpha i}(q) \\ & + \sum_{\substack{q,\alpha \\ ij}} \Delta^{*\alpha i}(q) c_l q_i q_j \left[-\frac{1}{5} + \frac{1}{3}\ln\left(\frac{1}{\ell^2|\mathbf{q}|^2}\right)\right]\Delta^{\alpha j}(q) \\ & + U\sum_{\alpha} \int d\mathbf{r} d\tau\, |\Delta^{\alpha}(\mathbf{r},\tau)|^4 \\ & - V\sum_{\alpha\beta} \int d\mathbf{r} d\tau d\tau'\, |\Delta^{\alpha}(\mathbf{r},\tau)|^2 |\Delta^{\beta}(\mathbf{r},\tau')|^2. \end{aligned} \tag{13}$$

Here we have scaled the order parameter with $(\Gamma_t N_F \tau)^{1/2}$ and replaced the quartic coefficients by numbers which is sufficient for power counting purposes. The coefficients are $t \sim 1/\tau[1/\tilde{\Gamma} - \ln(2\epsilon_F \tau)]$, $c_2 \sim c_l \sim \ell^2/\tau$, $U \sim 1/N_F$, and $V \sim 1/(k_F^3 \tau^2)$, with $\tilde{\Gamma} = \Gamma_t N_F$ being a dimensionless measure of the interaction strength. The parameter t represents the distance from the quantum critical point. Generically, $U > 0$, which leads to a second order transition.[35] This completes the derivation of the LGW theory.

C. Higher angular momentum channels

In this subsection we generalize the findings to pairings in higher angular momentum (L) channels. For angular momentum $L > 0$, the renormalized anomalous density vertex is proportional to $|\mathbf{q}|^L$. The leading non-analyticity in the static anomalous density susceptibility $\chi^{(2)}_L(\mathbf{q})$ is given by the BCS logarithms in Fig. 1(b). They take the form

$$\delta\chi^{(2)}_L(\mathbf{q}) = \delta\langle \overline{n}^M_L(q) n^M_L(q)\rangle \sim |\mathbf{q}|^{2L} \ln(1/|\mathbf{q}|), \tag{14}$$

i.e., they are suppressed by a factor $|\mathbf{q}|^{2L}$ compared to the *s*-wave case. Here $n^M_L(q) = \Sigma_k\, Y^M_L(\hat{\mathbf{e}}_\mathbf{k})\psi_\sigma(k+q/2)\psi_{\sigma'}(k-q/2)$ is a component of the anomalous density for angular momentum L. Note that for $L > 1$ the BCS logarithm is subleading compared to the regular $|\mathbf{q}|^2$ term coming from Fig. 1(a) while in the *p*-wave case the BCS logarithm provides the leading wave-number dependence in the LGW functional. The same mechanism also suppresses the interaction induced mode-coupling singularities related to corrections to scaling at the dirty Fermi liquid fixed point. An explicit calculation outlined in Appendix B shows that these mode coupling singularities behave at most as $|\mathbf{q}|^{2L}|\mathbf{q}|^{d-2}$ (d is the spatial dimensionality). Therefore they are subleading compared to the BCS logarithms for all $d > 2$. We now turn to the four-point anomalous density correlation function $\chi^{(4)}_L$. Because of the momentum dependence of the renormalized anomalous density vertex, $\chi^{(4)}_L$ picks up an extra power of $|\mathbf{q}|^{4L}$ compared to the *s*-wave case. More generally, any $2n$-point anomalous density correlation function $\chi^{(2n)}$ picks up an extra power of $|\mathbf{q}|^{2nL}$ compared to the *s*-wave case, i.e., $\chi^{(2n)} \sim |\mathbf{q}|^{4+(2L-4)n}$. Therefore their singular contributions are demoted to subleading order and do not play a role for the critical behavior. As a result, the leading terms in the LGW functional for *d*-wave and higher order parameter angular momentum take the same form (13) as in the *p*-wave case except for the missing logarithmic wave number dependence in the Gaussian part.[19]

III. RENORMALIZATION GROUP ANALYSIS

In this section, we analyze the LGW theory (13) by means of the renormalization group. There is a Gaussian fixed point with mean-field static exponents $\nu = 1/2$, $\gamma = 1$, $\eta = 0$, and a dynamical exponent of $z = 2$. In the *p*-wave case there are logarithmic corrections to the mean-field behavior in all dimensions. In order to check the stability of this Gaussian fixed point we study the importance of quantum and disorder fluctuations. The scale dimensions of U and V at the Gaussian fixed point can be calculated by power counting. We obtain $[U] = 2 - d$ and $[V] = 4 - d$. Thus, the conventional fluctuation term (the U term) is renormalization group irrelevant for $d > 2$, but the disorder term (the replica-off-diagonal quartic V term) is relevant for $d < 4$. In three dimensions the Gaussian fixed point is unstable with respect to the disorder term, and thus the calculation of loops is necessary to determine the asymptotic critical behavior. This includes the possibility of replica-symmetry breaking in the replica-off-diagonal quartic term.

Rather than carrying out this program explicitly, we use the analogy between our transition and the disordered

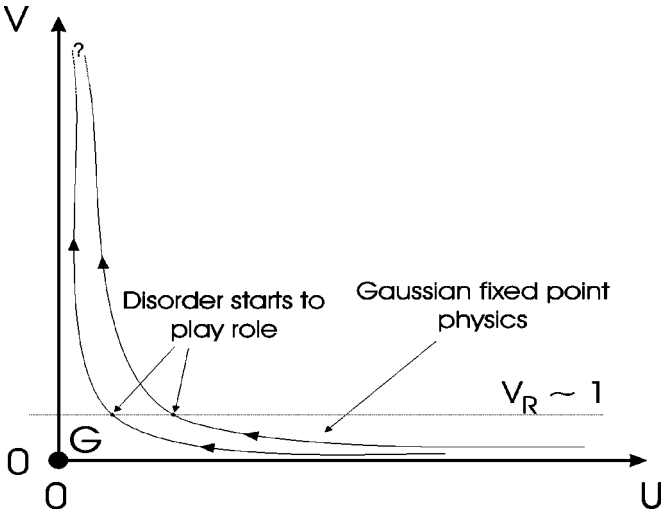

FIG. 4. Schematic flow diagram on the critical surface. The Gaussian (G) fixed point is unstable; the flow goes toward large disorder V. For weak bare disorder, the flow stays close to the U axis until it almost reaches the Gaussian fixed point before crossing over (black dots) to the asymptotic destination. The dashed line separates the region described by the Gaussian fixed point from the strong disorder region.

itinerant antiferromagnetic quantum phase transition to discuss the asymptotic critical behavior: Except for the logarithmic corrections, the LGW theory, Eq. (13) is identical to that of a disordered itinerant antiferromagnet. This transition has been investigated in great detail in recent years.[22–24] By taking into account rare disorder fluctuations it was found that there is no critical fixed point in the perturbative region of parameter space, and the asymptotic critical behavior is characterized by run-away flow toward large disorder (Fig. 4) rendering the perturbation expansion unjustified. The physical implications of this runaway flow are not fully understood so far. Possible scenarios include a nonperturbative fixed point with conventional (power-law) scaling, an infinite randomness fixed point (relative magnitude of inhomogeneities increases without limit under coarse graining), resulting in activated scaling, or a complete destruction of a sharp phase transition. Thus, from the analogy with the quantum phase transition in itinerant antiferromagnets we conclude that the asymptotic critical behavior of our theory is unconventional and, at present, unknown.

However, in many relevant experimental systems the bare disorder is actually very small. Thus, one may ask at what length scale disorder effects start to play a role. The crossover scale between the Gaussian and the asymptotic critical behavior can be determined from the condition that the renormalized dimensionless disorder coupling constant $V_R = V/\sqrt{tc_2^3} \approx 1$. Now, the quantum phase transition occurs at $t = 0$ which leads to $1/\tilde{\Gamma}_c = \ln(2\epsilon_F \tau)$, with $\tilde{\Gamma}_c$ being the dimensionless critical coupling. Thus, the quantum phase transition occurs at an exponentially small bare disorder strength which implies that $V_R \sim 1$ requires an exponentially large length scale. Using the results above we find a Ginzburg-type criterion

$$\frac{|\tilde{\Gamma} - \tilde{\Gamma}_c|}{\tilde{\Gamma}_c} < \tilde{\Gamma}_c^3 \exp\left[-\frac{1}{\tilde{\Gamma}_c}\right]. \tag{15}$$

Therefore, disorder effects become important only inside an exponentially narrow region surrounding $\tilde{\Gamma}_c$. This asymptotic critical region is preempted by a wide Gaussian crossover region (region below dashed line on Fig. 4) with mean-field critical behavior. For p-wave symmetry there are logarithmic correction to the power-law scaling whose the $|\mathbf{q}|$ dependence reflects the underlying p-wave symmetry of the order parameter.

IV. CONCLUSIONS

In this paper, we have studied the quantum phase transition from a dirty metal to an exotic superconductor. Starting from a microscopic action of disordered interacting electrons, we have derived the LGW theory for this quantum phase transition which proved to be equivalent (up to logarithmic corrections in the Gaussian part in the case of p-wave pairing) to the extensively studied LGW theory of the dirty itinerant antiferromagnetic transition. A renormalization group analysis yielded runaway flow toward large disorder. As a result, the asymptotic fate of the quantum phase transition is presently unknown. However, we could derive a Ginzburg-type criterion for the importance of the disorder fluctuations. For weak bare disorder, as is realized in many experimental systems, the true asymptotic behavior is observed only exponentially close to the quantum critical point. It is preempted by a wide region with mean-field behavior (and logarithmic corrections for p-wave pairing). In this last section we analyze our results from a general mode-coupling point of view, and we also discuss experiments.

In deriving the LGW functional, we have paid particular attention to the coupling between the order parameter and additional soft modes. Mode-coupling-induced singularities are indeed present in all angular momentum channels, but they are increasingly suppressed for higher angular momentum: In the static order parameter susceptibility the singular terms pick up an extra power of $|\mathbf{q}|^{2L}$. This suppression can be understood as follows. In the presence of nonmagnetic quenched disorder, the dominant electronic soft modes are those that involve fluctuations of the number density, spin density, or anomalous density in the zero angular momentum channel[32] while the corresponding densities in higher angular momentum channels are not soft. Since the different angular momentum modes are orthogonal at zero wave number, the coupling between a finite angular momentum order parameter and the zero angular momentum soft modes must involve powers of the wave number $|\mathbf{q}|$. These arguments suggest a very general difference between the mode-coupling effects in clean and dirty electronic systems. While the only soft modes in the dirty case are in the zero angular momentum channel, in clean systems, the charge, spin, and anomalous density fluctuations in all angular momentum channels are soft (corresponding to an infinite number of Fermi liquid parameters). Therefore, one expects the mode coupling singularities in a clean system *not* to be suppressed by a higher

order parameter angular momentum. This is known to be true for the Cooper channel logarithmic singularities which do not pick up an extra $|\mathbf{q}|^{2L}$ in clean electronic systems. A systematic investigation of this question will be published elsewhere.[36]

The explicit calculations in this paper were for a superconducting quantum phase transition in a paramagnetic system. We now discuss to what extent the results change if the transition happens in a ferromagnetic system. Let us assume the magnetization is in the z direction. Obviously, not all possible order parameter components are equivalent. Specifically, the symmetric triplet $1/\sqrt{2}(\uparrow\downarrow+\downarrow\uparrow)$ (for p- or f-wave pairing) as well as the singlet $1/\sqrt{2}(\uparrow\downarrow-\downarrow\uparrow)$ (for s- and d-wave pairing) are suppressed because the exchange gap cuts off the Cooper-channel singularities. In contrast, for equal spin pairing ($\uparrow\uparrow$ and $\downarrow\downarrow$), the leading behavior is the same as discussed in Secs. II and III of this paper.

Possible candidates for an experimental verification of our predictions are the ferromagnetic superconductors UGe_2 (Ref. 5) or $ZrZn_2$.[6] For these systems, a likely mechanism for superconductivity is p-wave triplet pairing mediated by magnetic fluctuations due to the vicinity to a magnetic quantum critical point,[21] although this has not yet been established beyond doubt. In $ZrZn_2$ the vanishing of superconductivity as a function of disorder has actually already been observed.[6] A systematic study of this transition would therefore be very interesting.[37]

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APPENDIX A: TWO-POINT LGW VERTEX

In this appendix we sketch the derivation of expression (10) for the anomalous density susceptibility $\chi^{(2)}$ in $3D$. In a suitable parametrization and for p-wave pairing, Fig. 1(a) can be written as

$$D^a_{ij}(\mathbf{q},\omega) = T\sum_{\mathbf{k},\Omega} Y^i_1(\hat{\mathbf{e}}_\mathbf{k}) Y^j_1(\hat{\mathbf{e}}_\mathbf{k}) G(\mathbf{k}+\mathbf{q},\Omega+\omega) G(-\mathbf{k},-\Omega), \tag{A1}$$

with i, j being the order parameter component indices and $\mathbf{q},\omega$ the external momentum and frequency. $Y^M_L(\hat{\mathbf{e}}_\mathbf{k}) = Y^M_L(\theta,\phi)$ is a spherical harmonic and Green's function G is given by Eq. (8). A straightforward calculation leads to

$$D^a_{ij} = \frac{N_F}{3}\left[\ln(2\epsilon_F\tau) - |\omega|\tau - \frac{1}{10}\ell^2|\mathbf{q}|^2\right]\delta_{ij} - \frac{N_F}{15}\ell^2 q_i q_j. \tag{A2}$$

$\ell = k_F\tau/m$ is elastic mean free path and N_F the density of states at the Fermi level. Similarly, the diagram in Fig. 1(b) is

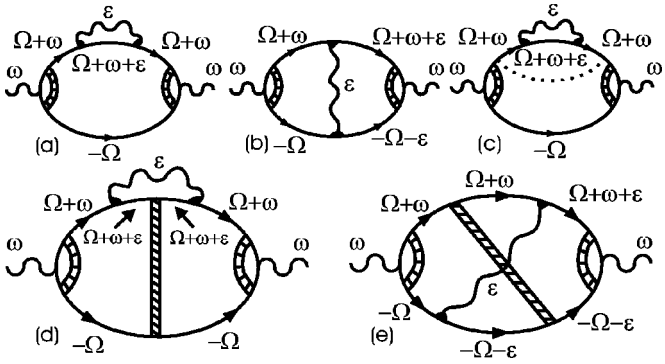

FIG. 5. Diagrams arising in the first order perturbation theory in interaction of the reference ensemble S_0, and produce a non analytic $|\mathbf{q}|^{2L}|\mathbf{q}|^{d-2}$ term, which is a consequence of the mode-coupling effects.

$$D^b_{ij} = \frac{N_F}{9}\ell^2 q_i q_j \ln\left(\omega\tau + \frac{\ell^2|\mathbf{q}|^2}{3}\right). \tag{A3}$$

Adding Eqs. (A2) and (A3) completes derivation of Eq. (10). For general angular momentum L, the analogous calculation shows that the BCS logarithm in D^b picks up an extra factor $|\mathbf{q}|^{2L}$ compared to the s-wave case.

APPENDIX B: INTERACTION EFFECTS

In this appendix we analyze the leading corrections to $\chi^{(2)}$ due to the interactions S_{int} in the reference ensemble S_0. They can be understood as corrections to scaling at the dirty Fermi liquid fixed point[32] and are particle-particle analogs of the well known Altshuler-Aronov corrections to density of states and conductivity.[33] We first consider a paramagnetic reference ensemble. To first order in the interactions, the relevant diagrams are those in Fig. 5 and their counterparts with bare external vertices. The wiggly line represents the interaction which is assumed to be short ranged and can thus be approximated by a number $\Gamma_{\sigma\sigma'}$ (where σ,σ' denote the spin at the two ends of the interaction line).

Particular attention must be paid to the diagrams a, b and c with bare vertices. In these diagrams the spherical harmonics in the two external vertices are not independent. Therefore, their contributions can potentially produce stronger terms than $|\mathbf{q}|^{2L}|\mathbf{q}|^{d-2}$. However, it turns out that these contributions do not produce any nonanalytic terms and only contribute to the regular terms. In the remaining contributions the angular variables of the external vertices are independent. They can be analyzed along the lines presented in Ref. 38. After a straightforward calculation one finds that the interaction corrections produce singularities of at most of the order $|\mathbf{q}|^{2L}|\mathbf{q}|^{d-2}$ which means they are suppressed by a factor $|\mathbf{q}|^{2L}$ compared to the zero angular momentum case.[13] The above conclusion is easily generalized to ferromagnetic reference ensembles: If the magnetization is in the z direction, the $\uparrow\uparrow$ and $\downarrow\downarrow$ components of the order parameter have the same type of singularity as discussed above, while the leading singularities in the $\uparrow\downarrow$ components are cutoff by the exchange gap.

APPENDIX C: 4-POINT VERTICES

In this appendix we present details of calculation of four-point susceptibility $\chi^{(4)}$ in 3*D*, Eqs. (11) and (12). We start with replica diagonal part $\chi^{(4)}_{\rm diag}$ which is calculated from the diagrams shown on Fig. 2. The most singular contributions are produced if the frequencies structure permits all four external vertices to be renormalized by an active (retarded-advanced) ladder (four-ladder diagrams). A direct calculation of the first diagram [Fig. 2(a)] leads to

$$D_a = \frac{2\ell^5\tau^2|\mathbf{q}|^4\cos^4(\alpha)}{81\pi}\sum_{\Omega}\frac{\Theta[\Omega(\Omega+\omega)]}{\left(|2\Omega+\omega|\tau+\frac{\ell^2|\mathbf{q}|^2}{3}\right)^4}$$

$$\times(1-9|2\Omega+\omega|\tau-\ell^2|\mathbf{q}|^2). \qquad \text{(C1)}$$

An analogous calculation for Fig. 2(b) gives (the frequency constraint requires the extra impurity line to act as a single impurity line rather than an active ladder):

$$D_b = -\frac{\ell^5\tau^2|\mathbf{q}|^4\cos^4(\alpha)}{81\pi}\sum_{\Omega}\frac{\Theta[\Omega(\Omega+\omega)]}{\left(|2\Omega+\omega|\tau+\frac{\ell^2|\mathbf{q}|^2}{3}\right)^4}$$

$$\times(1-12|2\Omega+\omega|\tau-2\ell^2|\mathbf{q}|^2). \qquad \text{(C2)}$$

Each of these two diagrams individually has an divergence $\sim 1/|\mathbf{q}|^2$. However, because the relative combinational factor of D_b is 2, the divergent contributions cancel, rendering the final value for $\chi^{(4)}_{\rm diag}$ finite [Eq. (11)].

Similar cancellations among individually diverging diagrams take place in the replica off-diagonal case, with the strongest singularities coming from diagrams with the largest number of active ladders. A set of such diagrams is shown on Fig. 6. Here, at most six ladders can be active, leading to an infrared singularity in each of the diagrams of the form $1/|\mathbf{q}|$. Similar calculations to the ones carried out above reveal that the singular contributions of diagrams (a), (b), and (c) canceled each other. The remaining contribution is finite and can be estimated from the simple diagram in Fig. 3.

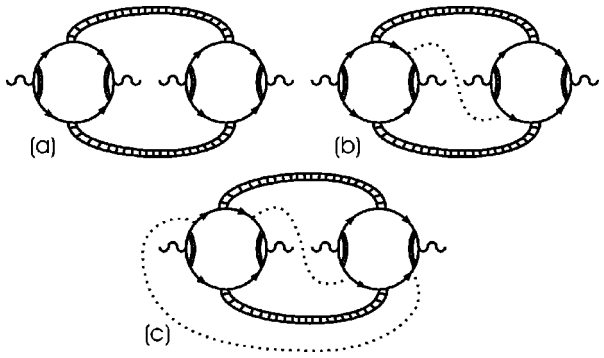

FIG. 6. Diagrams contributing to the $\chi^{(4)}_{\rm off}$. All four vertices are renormalized, with two extra active particle-particle ladders connecting two fermionic loops. The zeroth order terms in small $\mathbf{q}$ and ω of these diagrams cancel.

We emphasize once more that all the results for the singularities in the anomalous density correlation functions in the Appendixes A–C have been obtained in low-order perturbation theory. Within perturbation theory one cannot prove that the terms obtained indeed represent the leading singularities to all orders. We are nonetheless confident that we indeed identified the leading terms, because in the *s*-wave case we reproduce the known rigorous results from *Q*-field theory.[15,32]

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