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Recommended Citation

K. L. Kosbar and A. D. Polydoros, "A Lower-Bound for the Error-Variance of Maximum-Likelihood Delay Estimates of Discontinuous Pulse Waveforms," *IEEE Transactions on Information Theory*, vol. 38, no. 2, pp. 451-457, Institute of Electrical and Electronics Engineers (IEEE), Mar 1992.

The definitive version is available at <https://doi.org/10.1109/18.119703>

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A Lower Bound for the Error-Variance of Maximum-Likelihood Delay Estimates of Discontinuous Pulse Waveforms

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Abstract—A new lower bound is developed for the error variance of maximum-likelihood time-delay estimation when the received signal is a square pulse, corrupted by additive white Gaussian noise. The bound is generated by combining concepts previously developed for a special class of stochastic processes, induced by the signal model. For moderate signal to noise ratios, the new bound is significantly tighter than previously known ones.

Index Terms—Maximum-likelihood, time-delay estimation, discontinuous signals.

I. INTRODUCTION

In numerous applications it is necessary to use wide-bandwidth signals to achieve system goals. Two wideband signals that are relatively easy to generate are periodic pulse trains and pseudonoise (PN) sequences, both modulated by square pulses. These signalling formats are used in navigation systems [1], Direct-sequence spread-spectrum systems [2], and numerous radar applications [3]. To operate properly, the receivers in these systems must derive an estimate of the time epoch of such a deterministic, discontinuous signal after it has been corrupted by a noisy channel. The purpose of this correspondence is to develop a new lower bound that limits the error variance of the unbiased maximum-likelihood (ML) time-delay estimate for systems where a square-pulse signal is a useful modeling abstraction. The result is also a valid lower bound on the mean square estimation error of biased estimates.

Numerous lower bounds on the error variance of time-delay estimates have previously been developed [4]–[13]. These bounds can be divided into two classes, based on the continuity of the transmitted waveform. Since the physical signals encountered in practice are continuous, it is tempting to employ the continuous-signal theory and disregard the discontinuous theory as a mathematical curiosity that, in the strict sense, is inapplicable. The shortcoming with this approach is that the continuous-signal bounds (such as the Cramér–Rao bound) can be very loose for wide-bandwidth signals. This was demonstrated by Barton and Ward [14], when they examined a sequence of bandlimited square pulses. The Cramér–Rao bound [4, 5] became trivial as the root mean-square bandwidth increased to infinity. This implies that either the Cramér–Rao bound is extremely loose in this case, or else the delay-estimation problem is singular for discontinuous signals. Previous work has shown [4]–[13] that this is not a singular problem by developing nontrivial bounds for the discontinuous case. So, even though the discontinuous-signal bounds do not apply in the strict sense, they can provide

a much more reasonable approximation to the true behavior of wideband systems than their continuous counterparts.

In this correspondence, we develop a procedure for identifying a tight lower bound on the error variance of the unbiased ML time-delay estimate of a square pulse that has passed undistorted through an additive white Gaussian noise (AWGN) channel. This is done by constructing an appropriate function, whose variance is warranted to be less than that of the aforementioned estimator for all signal-to-noise ratios (SNR). The procedure is fairly general, in that it can incorporate various partial statistics that are available for the underlying stochastic process. Here, it is specifically applied to available knowledge pertaining to Gaussian processes whose mean and autocorrelation functions are triangular.

Section II discusses previously developed bounds that can be applied to this problem. It also reviews a related (and popular) procedure that approximates, but does not bound, the performance of ML estimators. The new bound is developed in Section III. Since it was not possible to express the new bound in closed form, examples of numerical evaluations are presented in Section IV, where we also confirm the tightness of the bound by Monte Carlo simulations. As a particular application, we look at the optimization of the pulse-width for a given SNR. A brief appendix summarizes the necessary probability density functions (pdf) used in the calculations.

II. PREVIOUS BOUNDS AND APPROXIMATIONS

Since there has been a considerable amount of research in the field of delay estimation, it is useful to briefly survey this work before proceeding with the development of the new bound. For example, it is well known [14], [15] that when pulse-type signals are transmitted, there is an SNR threshold where the performance characteristics of a ML delay-estimator change abruptly. This is because at high SNR the estimator will seldom make errors larger than the width of the transmitted pulse. Under these conditions, the performance of the estimator is strongly influenced by the shape of the autocorrelation function, $\rho(\tau)$, of the transmitted waveform in the neighborhood of the origin, $\tau = 0$. However, at low SNR, the ML estimator will frequently make errors which are larger than the pulse width. Then, the shape of the autocorrelation function about the origin is not as critical as its width with respect to the total uncertainty range. A tight bound or accurate approximation of the error variance of the ML estimate must account for both types of errors if it is to be useful over a wide range of SNR. One such approximation procedure was described by Van Trees [15, pt. I, p. 282]. The practical importance of such approximations and bounds was demonstrated in a recent application to the ranging problem for receivers that are subject to significant acceleration and “jerk” [16].

The approximation described in [15] applies to continuous, pulse-type waveforms. The autocorrelation function of these waveforms must be continuous, and possess a continuous first derivative. This method cannot be applied to discontinuous signals, such as a square pulse or PN sequence, since their autocorrelation functions are triangular, and lack a continuous first derivative. Also, from a mathematical standpoint, this procedure results in an *approximation* of the error variance of the ML delay-estimate. In contrast, the work of Section III provides a strict lower bound on the error variance of the unbiased ML delay-estimate for signals with triangular autocorrelation functions. Despite these differences, a close examination of the approximating procedure of [15] can help clarify the development of the new bound.

This work was supported in part by NSF/PYI Grant #NCR-8552527, and the Air Force Engineering Foundation Grant #RI-B-90-25. This work was presented in part at the IEEE International Symposium on Information Theory, San Diego, CA, January 14–19, 1990.

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IEEE Log Number 9103962.

As previously mentioned, there are two types of estimation errors; small errors, those less than the pulse width, and interval errors, those greater than the pulse width. Van Trees expressed the total error variance, $\text{var}(\bar{\tau})$, as

$$\text{var}(\bar{\tau}) = \Pr(\text{small error}) \text{var}(\bar{\tau}|\text{small error}) + \Pr(\text{interval error}) \text{var}(\bar{\tau}|\text{interval error}), \quad (1)$$

where $\bar{\tau}$ is the difference between the true time epoch of the received signal and the estimate generated by the receiver. For continuous signals, the first conditional variance in (1) was approximated by the Cramér–Rao bound and the second by the variance of a uniform distribution. However, as pointed out in the Introduction, the Cramér–Rao bound is trivial for the discontinuous signal used in Section III. A second difficulty with (1) concerns the determination of $\Pr(\text{interval error})$. In [15], this quantity was approximated by formulating the delay estimation problem as an M -ary hypothesis-testing problem, where M is the number of orthogonal pulses that can be placed in the uncertainty region. This procedure will produce an estimate of the probability of an interval error, but is not guaranteed to be a lower or upper bound on the true value. This is not critical at very low or very high SNR, where the quantity is essentially 1 or 0, respectively. However, it is unclear how accurate this approximation will be at moderate SNR. As will be shown later, this is a particularly critical region when one desires to optimize the pulse width for minimum variance at a fixed SNR. The bound developed, in Section III, is valid and reasonably tight at all SNR, including moderate values where both types of errors are common.

Numerous other lower bounds have been developed for the error variance of the (assumed unbiased) ML delay-estimate. Some of these bounds [4]–[13] can be applied when the transmitted signal is a square pulse of width Δ :

$$s_{sq}(t) = \begin{cases} \sqrt{E/\Delta}, & |t| \leq \Delta/2, \\ 0, & |t| > \Delta/2. \end{cases} \quad (2)$$

At a sufficiently high SNR all of these bounds have the same form, namely,

$$\frac{E(\bar{\tau}^2)}{\Delta^2} \geq C_{\text{norm}}(E/N_0)^{-2} \quad (3)$$

where N_0 is the single-sided power spectral density of the AWGN and

$$E = \int_{-\Delta/2}^{\Delta/2} s_{sq}^2(t) dt$$

is the energy received. The value of the constant C_{norm} varies as a function of the underlying assumptions made in the development of the particular bound. A summary of these values is presented in Table I. As shown in Fig. 1, these bounds diverge at low SNR. This graph is for the particular case where there is one pulse of width Δ in the observation interval T , with $\Delta = T/33$. The Terent'yev bound is tight at high SNR; however it becomes extremely loose at low SNR. At moderate SNR there is a substantial difference between the greatest lower bound and the simulation results. A facet of these bounds that is not evident in the figure, is that all bounds monotonically decrease with decreasing pulse width. In contrast, the new bound, derived in Section III, has the following attributes: a) it is considerably tighter than present bounds at moderate SNR, b) it

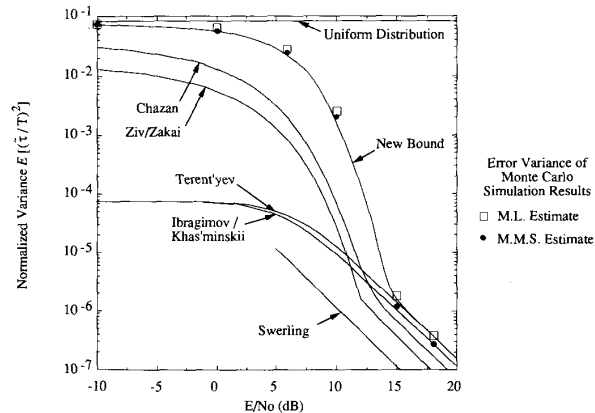


Fig. 1. Known bounds for a square pulse of width $T/33$ and simulation results.

TABLE I
BOUND COEFFICIENTS FOR SQUARE PULSES

Bound	C_D
Swerling [7]	0.125
Ziv/Zakai [8]	0.405
Manasse [10]	0.500
Chazan/Zakai/Ziv [11]	0.750
Ibragimov/Khas'minskii [12]	1.250
Terent'yev [13]	1.625

converges to the tightest lower bound at high SNR, and c) it suggests that there exists a nonzero pulse width that will minimize the variance at any fixed SNR. I; fig 1

As an aid in determining the tightness of the various bounds, lengthy Monte Carlo simulations were performed on a digital computer. Both ML and minimum mean-square (MMS) estimates, with the associated error variances, were calculated. The results of these simulations are shown in Fig. 1. As one would expect, the MMS estimate is superior at all SNR; however the ML estimate is within 0.5 dB at all measured points. In the strict sense, the bound derived in the next section applies only to the error variance of the ML estimate. However the simulation results suggest that the bound is also useful as an approximation of the MMS estimator performance.

III. THE NEW BOUND

Maximum-likelihood estimation is performed by locating the maximum value of the likelihood function $\Lambda(\tau)$. When an AWGN channel model is used, it is easily shown, (see, for instance [15, section 4.2 (101)]) that for *open-loop* (or *batch*) estimation

$$\ln \Lambda[\tau] = \frac{2}{N_0} \int_{-T/2}^{T/2} r(t)s(t-\tau)dt - \frac{1}{N_0} \int_{-T/2}^{T/2} s^2(t-\tau) dt, \quad (4)$$

where $r(t)$ is the received waveform, τ is the unknown delay, $s(t-\tau)$ is the transmitted pulse delayed by τ seconds, and T is the observation interval $-T/2 \leq t \leq T/2$. In practical applications, it is often reasonable to assume that the second integral on the right-hand side of (4) is constant for all τ . By exploiting the

monotonicity of the logarithmic operator, it is possible to find the ML estimate, τ_{ML} , from the property

$$y(\tau_{ML}) \geq y(\tau) \quad \forall \tau, \quad (5)$$

where

$$y(\tau) = \int_{-T/2}^{T/2} r(t)s(t-\tau) dt. \quad (6)$$

The function $y(\tau)$, a linear transformation of the Gaussian random process $r(t)$, is itself Gaussian. The mean and covariance of $y(\tau)$ are

$$m_y(\tau) = E\rho(\tau), \quad (7)$$

$$K_y(\tau) = \frac{E\rho(\tau)N_0}{2}, \quad (8)$$

where

$$E = \int_{-T/2}^{T/2} s^2(t) dt$$

is the signal energy and

$$\rho(\tau) = \frac{1}{E} \int_{-T/2}^{T/2} s(t)s(t-\tau) dt$$

is the signal autocorrelation function. So, the ML delay-estimation problem is equivalent to finding the location of the maximum of a Gaussian stochastic process. Unfortunately, the process is colored, nonzero-mean and nonstationary, so it is difficult to determine the pdf of the random variable (r.v.) τ_{ML} .

The principal idea presented here is to integrate bounds originally developed for similar or other problems [12], [13], [17]–[20] by a certain procedure, the first step of which is to carefully partition the uncertainty region as follows: For all partitions A_1, A_2, \dots, A_N such that

$$\bigcup_{i=1}^N A_i = [0, T]$$

it is possible to rewrite (5) as

$$y(\tau_{ML}) = \max_{i=1, \dots, N} \left\{ \max_{\tau \in A_i} y(\tau) \right\}.$$

While this is true for any N , in this analysis we deliberately choose $N = 3$. To simplify the notation in the subsequent discussion, the variables A_1, A_2 , and A_3 will be replaced by A, B , and C , respectively. Thus, the ML estimate must satisfy

$$y(\tau_{ML}) = \max_{W=A, B, C} \left\{ \max_{\tau \in W} y(\tau) \right\}. \quad (9)$$

The specific partition chosen here is

$$A = \left\{ \tau: |\tau| \leq \frac{\Delta}{2} \right\},$$

$$B = \left\{ \tau: \frac{\Delta}{2} < |\tau| \leq \Delta \right\},$$

$$C = \left\{ \tau: \Delta < |\tau| \leq \frac{T}{2} \right\},$$

depicted in Fig. 2. Let us now define the r.v. x_W as the peak value of $y(\tau)$ in region W , $x_W \triangleq \max_{\tau \in W} y(\tau)$, and τ_W as the location of this maximum: $y(\tau_W) = x_W$. Then, (9) is equivalent to

$$\tau_{ML} = \tau_W \quad \text{iff } x_W \geq x_V \quad \forall V \in \{A, B, C\}. \quad (10)$$

Some of the known joint statistics for the r.v.s $x_A, x_B, x_C, \tau_A, \tau_B, \tau_C$ are summarized, in reference form, in Table II.

In principle, knowledge of the 6-dimensional joint pdf of these r.v.'s, along with equations (9) and (10), would lead to the determination of the pdf of the ML estimation error, $f_{ML}(\tau)$. Unfortunately, such complete statistical information is not currently available. However, it is possible to bound the variance of $f_{ML}(\tau)$ by the following procedure: Let us consider a collection of conditions that $f_{ML}(\tau)$ should necessarily satisfy. For example, two obvious conditions are $f_{ML}(\tau) \geq 0$ and

$$\int_{-\infty}^{\infty} f_{ML}(\tau) d\tau = 1,$$

since $f_{ML}(\tau)$ is a pdf. We will assume that the true delay is zero; hence, the symmetry of the problem makes it possible to construct an unbiased ML estimator with a symmetric error pdf in $|\tau| \leq T/2$. In that case, a third condition is

$$\int_{-T/2}^{T/2} \tau f_{ML}(\tau) d\tau = 0.$$

Additional conditions will be identified next. Regarding the assumption on unbiasedness, it is well known [8] that, for batch estimation, there exists a bias that is a function of the true value of the delay. This bias will increase the second moment of the estimation error of the true ML estimator. This will effect the tightness, but not the validity of the lower bound generated below.

The aforementioned collection of conditions define a space G of real functions $G(\tau)$ that satisfy them. Let $g_m(\tau)$ denote that member of G which possesses the minimum second moment (identical to its variance)

$$\sigma^2 \triangleq \int_{-T/2}^{T/2} \tau^2 g(\tau) d\tau$$

over all functions in G . By the way G is identified, $f_{ML}(\tau) \in G$; thus, the second moment of $g_m(\tau)$ will lower bound the error variance of the unbiased ML estimate. Clearly, the more restrictions that can be identified regarding $f_{ML}(\tau)$, the smaller the size of G , and, hence, the tighter the bound. Such restrictions come about from a combination of the established facts of Table II, along with equation (10), as we indicate next.

To start with, $g(\tau) \in G$ should be pdf-like and zero-mean, exactly as the three obvious conditions dictate. The first nontrivial necessary conditions placed on the set G are created as follows: The probability that τ_{ML} is in some infinitesimally small interval $\alpha \subset A$ is bounded by

$$\Pr[\tau_{ML} \in \alpha] = \Pr[(\tau_A \in \alpha) \cap (x_A > x_B) \cap (x_A > x_C)]$$

$$\leq \Pr[\tau_A \in \alpha]. \quad (11)$$

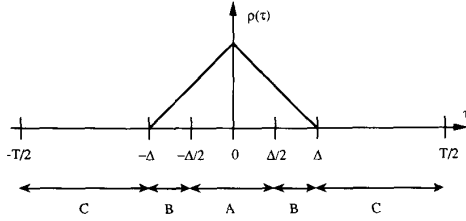


Fig. 2. Partitioning of the observation interval.

TABLE II
PARTIAL STATISTICS OF THE PROCESS $y(\tau)$ FOR PARTITION OF FIG. 2*

Known Probability Density Function	Reference
$f_{x_A}(x)$	Bar-David [18]
$f_{x_{A \cup B}}(x)$	Bar-David [17]
$f_{\tau_A}(\tau)$	Terent'yev [13]
$f_{x_{B^c}}(x, \tau)$	Shepp [20]

* Some of the references given in this table have typographical errors, or miscalculations. This analysis uses the form of these results presented in the Appendix.

This leads to the condition

$$0 \leq f_{ML}(\tau) \leq f_{\tau_A}(\tau) \quad \text{in the region } \tau \in A. \quad (12)$$

Similar reasoning for region B will produce the condition

$$0 \leq f_{ML}(\tau) \leq f_{\tau_B}(\tau) \quad \text{in the region } \tau \in B. \quad (13)$$

Note that the pdf's $f_{\tau_A}(\tau)$ and $f_{\tau_B}(\tau)$ can be evaluated using the results of [13] and [20] respectively. Thus, the conditions (12) and (13) must also be satisfied by all $g(\tau) \in G$.

The subsequent conditions that we create apply to the distribution of the probability mass between regions. We show shortly that it is possible to find a lower bound, P_C^L , on the value of the quantity P_C , defined as

$$P_C \triangleq \int_C f_{ML}(\tau) d\tau = \Pr[x_C > \max(x_A, x_B)]. \quad (14)$$

In addition, we note that $y(\tau)$ is a zero-mean stationary stochastic process in region C , hence, $f_{\tau_C}(\tau)$ will be uniformly distributed. The symmetry of the problem in this region also guarantees that x_C and τ_C are independent r.v.'s. Thus, we can restrict G to the set of functions that have at least P_C^L of their mass in region C , and also have this mass uniformly distributed over the region $\Delta/2 \leq |\tau| < T/2$. It follows that the minimum-variance function $g_m(\tau)$ must satisfy

$$g_m(\tau) = \begin{cases} \frac{P_C^L}{T - 2\Delta}, & \text{for } \Delta \leq |\tau| < \frac{T}{2}. \end{cases} \quad (15)$$

The value of P_C^L can be determined by considering a subset C' of C . Obviously the peak value in region C' must equal or exceed the

peak value in region C' , $x_C \geq x_{C'}$, so

$$\begin{aligned} P_C &= 1 - \Pr[x_C < \max(x_A, x_B)] \\ &\geq 1 - \Pr[x_{C'} < \max(x_A, x_B)]. \end{aligned} \quad (16)$$

We now choose $C' = \cup_{i=1}^M D_i$, where D_i are segments of length 2Δ , which are separated by Δ from each other and from the combined region $A \cup B$. Fig. 3 illustrates this for $\tau > 0$. In this case, M is the largest integer greater than, or equal to, $(T - 2\Delta)/3\Delta$. This construction divides $y(\tau)$ into M statistically independent segments in C , which are also independent from the segment $A \cup B$. Thus, $\{x_{D_i}\}$ are independent identically distributed (i.i.d.) r.v.'s, as are $\{\tau_{D_i}\}$. We can then proceed and determine P_C^L from (16) as follows:

$$\begin{aligned} P_C &\geq 1 - \int_{-T/2}^{T/2} F_{x_C}(x) f_{x_{A \cup B}}(x) dx \\ &= 1 - \int_{-T/2}^{T/2} \left(\prod_{i=1}^M F_{x_{D_i}}(x) \right) f_{x_{A \cup B}}(x) dx \\ &= 1 - \int_{-T/2}^{T/2} [F_{x_D}(x)]^M f_{x_{A \cup B}}(x) dx \triangleq P_C^L, \end{aligned} \quad (17)$$

where $F_{x_w}(x)$ is the cumulative probability distribution function of x_w . It is possible to numerically evaluate P_C^L by using Bar-David's [17] results¹. The remaining $(1 - P_C^L)$ of the probability mass of $g_m(\tau)$ will be in region $A \cup B$.

Let us now investigate conditions that apply to the distribution of the probability mass in region $A \cup B$. We would like to restrict G by finding an upper limit on the amount of probability mass that can be in region A . It is possible to identify an upper bound $P_{A,B}$ such that

$$\begin{aligned} \Pr[\tau_{ML} \in A] &= \Pr[(x_A > x_B) \cap (x_A > x_C)] \\ &\leq \Pr[x_A > x_B] \leq P_{A,B}. \end{aligned} \quad (18)$$

This is done by the following argument: for all α , it is true that

$$\begin{aligned} \Pr[x_A > x_B] &\leq \Pr[x_A > \alpha] + \Pr[x_B < \alpha] \\ &= 1 - (\Pr[x_B > \alpha] - \Pr[x_A > \alpha]) \\ &\leq 1 - \max_{\alpha} (\Pr[x_B > \alpha] - \Pr[x_A > \alpha]) \\ &\triangleq P_{A,B}. \end{aligned} \quad (19)$$

Again, the probabilities $\Pr[x_B > \alpha] = 1 - F_{x_B}(\alpha)$ and $\Pr[x_A > \alpha] = 1 - F_{x_A}(\alpha)$ can be evaluated numerically by using the results of Bar-David [18] and Shepp [20]. Thus, G can be restricted to the space of functions that have no more than

$$P_A^U = \min[P_{A,B}, (1 - P_C^L)] \quad (20)$$

of their mass in region A . It follows after some thought that the

¹ The rationale for choosing the length of the D_i 's to be 2Δ is to conform with the available results in [11].

² Any function $g(\tau) \in G$ with mass in region A less than P_A^U will necessarily have a larger variance than $g_m(\tau)$.

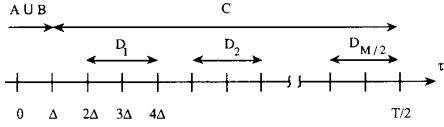


Fig. 3. Segmentation of region C.

minimum-variance function $g_m(\tau) \in G$ must satisfy²

$$\int_{\tau=0}^{\Delta/2} g_m(\tau) d\tau = \frac{1}{2} P_A^U. \quad (21)$$

We have now arrived at the final point: We wish to determine the function $g_m(\tau)$ with the smallest second moment in the region $A \cup B$ (it has already been determined in region C by (17)), subject to the inequality constraints (12) and (13), and the equality (21). This can be solved using linear-programming techniques, and is equivalent to the relaxation of integer constraints on the knapsack problem [21]. Looking first at region A , it is relatively straightforward to find the solution as

$$g_m(\tau) = \begin{cases} f_{\tau_A}(\tau), & \text{for } 0 \leq |\tau| < a, \\ 0, & \text{for } a \leq |\tau| < \frac{\Delta}{2}, \end{cases} \quad (22)$$

where the limit a is found from

$$\int_0^a f_{\tau_A}(\tau) d\tau = \frac{1}{2} P_A^U. \quad (23)$$

If $P_A^U < (1 - P_C^L)$, then there will be P_A^U of the probability mass in region A , P_C^L of the mass in region C , and the remaining $(1 - P_A^U - P_C^L)$ of the mass will be in region B . Using reasoning similar to that used in region A , it is possible to show that

$$g_m(\tau) = \begin{cases} f_{\tau_B}(\tau), & \text{for } \frac{\Delta}{2} \leq |\tau| < b, \\ 0, & \text{for } b \leq |\tau| < \Delta, \end{cases} \quad (24)$$

where the limit b is determined from

$$\int_{\Delta/2}^b f_{\tau_B}(\tau) d\tau = \max \left[0, \frac{1}{2} (1 - P_A^U - P_C^L) \right]. \quad (25)$$

Note that, depending on the sign of $(1 - P_A^U - P_C^L)$, the integral in (25) may have to be set equal to zero; this can be satisfied by setting $b = \Delta/2$. In conclusion, the sought function $g_m(\tau)$ will have the form

$$g_m(\tau) = \begin{cases} f_{\tau_A}(\tau), & \text{for } 0 \leq |\tau| < a, \\ 0, & \text{for } a \leq |\tau| < \frac{\Delta}{2}, \\ f_{\tau_B}(\tau), & \text{for } \frac{\Delta}{2} \leq |\tau| < b, \\ 0, & \text{for } b \leq |\tau| < \Delta, \\ \frac{P_C^L}{T - 2\Delta}, & \text{for } \Delta \leq |\tau| < \frac{T}{2}, \end{cases} \quad (26)$$

with the constants and functions defined in equations (17), (19), (20), (22)–(25) and Table II.

IV. NUMERICAL RESULTS AND DISCUSSION

Unfortunately, it was not possible to find a simpler form for the new bound. However, it is possible to evaluate it through the use of numerical integration. Note that some of the references used in the development of the new bound contained minor errors. Corrected versions of the critical equations appear in the Appendix. The analysis that leads to these results is presented in greater detail in [22]. A sampling of numerical results is presented in Fig. 1, Fig. 4, and Fig. 5. The tightness of the new bound was verified through the use of Monte Carlo simulations, as demonstrated in Fig. 1. Note that the new bound is considerably tighter than previous bounds at moderate SNR. It also has the advantage of converging to the greatest lower bound at high SNR. Although no assumptions were made concerning low or high SNR regions, note that there is clearly a threshold where the estimator performance changes abruptly.

Fig. 4 shows the new bound for a wide range of pulse widths Δ . The minimum variance obtainable (for any pulse width) is also included in this graph. Note that the error variance has been normalized with respect to the observation time. If a highly accurate estimate is required, the observation time may be very long with respect to the pulse width. Although this analysis is based on a batch-estimation approach, it can also be used to approximate the performance of closed-loop systems with a commensurately narrow equivalent bandwidth. In this case, the observation time is related to the time constant of the loop, which again may be many orders of magnitude larger than the pulse width. At a fixed SNR, there exists a nonzero pulse width that produces the minimum lower bound. This is consistent with previous observations [15, p. 280]. However, some of the previous work in this area has required the SNR to be either very high or very low and did not attempt to predict the performance in the threshold region. Other work has produced bounds that are valid at all SNR [8], [11]–[13]; however, these bounds are all monotonically decreasing with decreasing pulse width, thus failing to indicate that there may exist an optimal pulse-width which will minimize the error variance. The bound developed here has the advantage that it is valid at all SNR's and suggests that at every SNR there exists such an optimal, nonzero pulse width. This is shown more clearly in Fig. 5. Here, curves are parameterized by the channel SNR. For any fixed value of SNR, the ratio T/Δ can be varied to obtain a minimum value of the lower bound. As long as the bound is reasonably tight, this optimal ratio will produce the lowest normalized error variance $E[(\hat{\tau}/T)^2]$ for a ML receiver. For example, if the system has an SNR of 16 dB, then the optimal pulse will be approximately 0.003 of the observation time. This can also find practical use in choosing pulse widths or chip rates in spread-spectrum navigation and communication systems.

V. APPENDIX

Some of the references used in this work have typographical errors or miscalculations. The version of these results that were used in this analysis of Section III are summarized next. A more extensive discussion of these changes appears in [22].

In [17], (24) has two typographical errors. The corrected version is

$$\begin{aligned} \hat{P}_a(-t_1, t_1; b) \\ = \Phi^2(b_n) \Phi(b_n - A_n) - 2 A_n^{-1} \phi(b_n) \Phi(b_n) \end{aligned}$$

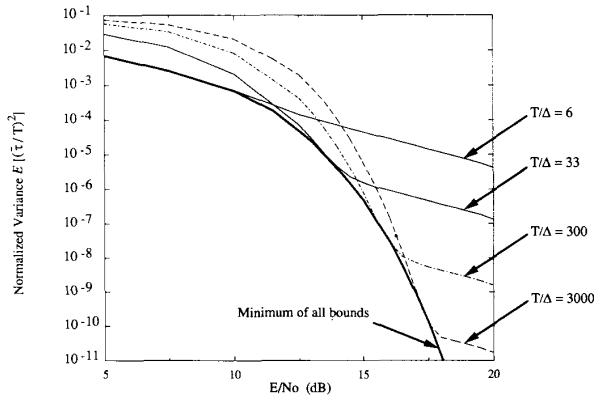


Fig. 4. Evaluation of the new bound.

$$\begin{aligned} & \cdot \left[\Phi(b_n) \exp\left(-\frac{1}{2}A_n^2 + A_n b_n\right) - \Phi(b_n - A_n) \right] \\ & + \phi^2(b_n) \left[b_n A_n^{-1} \Phi(b_n) \exp\left(\left(A_n b_n - \frac{1}{2}A_n^2\right) \right. \right. \\ & \quad \left. \left. - A_n^{-2} \exp\left(A_n b_n - \frac{1}{2}A_n^2\right) \Phi(b_n) \right. \right. \\ & \quad \left. \left. + A_n^{-2} \Phi(b_n - A_n) + A_n^{-1} \phi(b_n - A_n) \right] \right. \\ & \quad \left. - \frac{1}{2\pi} \sqrt{\frac{1}{3}} \Phi\left((b_n - A_n) \sqrt{\frac{1}{3}}\right) \right. \\ & \quad \left. \cdot \exp\left(-b_n^2 - \frac{(b_n - A_n)^2}{3}\right) + I_F + I_G. \right. \end{aligned}$$

This analysis also used (3.13) of [18]. The right-hand side of (3.13) used in Section III is

$$\begin{aligned} & \frac{\arctan 2^{-1/2}}{\pi} \left[1 + \exp(-2b(b-c)) - 2 \exp\left(-\frac{1}{2}(b^2 - c^2)\right) \right] \\ & + \operatorname{sgn}(2b-c) \left[\Phi\left(\sqrt{\frac{1}{3}}|2b-c|\right) - \frac{1}{2} \right] \\ & + 2V\left(\sqrt{\frac{1}{3}}|2b-c|, \sqrt{\frac{2}{3}}|2b-c|\right) \\ & + \exp(-2b(b-c)) \left\{ \operatorname{sgn}(c) \left[\Phi\left(\sqrt{\frac{1}{3}}|c|\right) - \frac{1}{2} \right] \right. \\ & \quad \left. + 2V\left(\sqrt{\frac{1}{3}}|c|, \sqrt{\frac{2}{3}}|c|\right) \right\} \\ & - \exp\left(-\frac{1}{2}(b^2 - c^2)\right) \\ & \cdot \left\{ -\operatorname{sgn}(b-2c) \left[\Phi\left(\sqrt{\frac{1}{3}}|b-2c|\right) - \frac{1}{2} \right] \right. \\ & \quad \left. + \operatorname{sgn}(3b-2c) \left[\Phi\left(\sqrt{\frac{1}{3}}|3b-2c|\right) - \frac{1}{2} \right] \right. \\ & \quad \left. - 2 \operatorname{sgn}((b-2c)(2b-c)) \right\} \end{aligned}$$

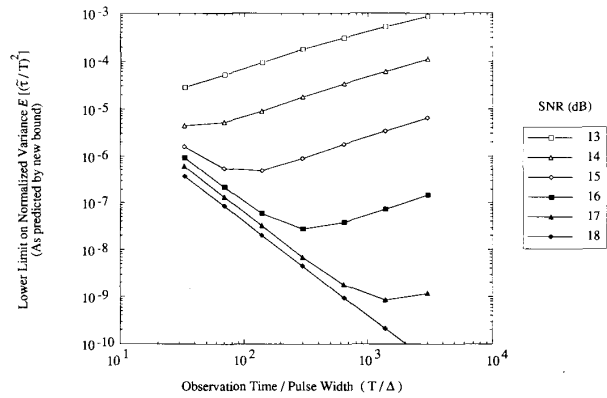


Fig. 5. Optimization of pulse width at fixed SNR.

$$\begin{aligned} & \cdot V\left(\sqrt{\frac{1}{3}}|b-2c|, \sqrt{\frac{2}{3}}|2b-c|\right) \\ & + 2 \operatorname{sgn}(c(3b-2c)) V\left(\sqrt{\frac{1}{3}}|3b-2c|, \sqrt{\frac{2}{3}}|c|\right) \}. \end{aligned}$$

A minor typographical error in [13] caused its (20) to be in error. The corrected version of this equation is

$$\begin{aligned} P_{2,\tau}(y) = & \frac{1}{\sqrt{2\pi}} \int_{[-\sqrt{R-4\tau}/2-2y/\sqrt{R-4\tau}] }^{\infty} e^{-z^2/2} dz \\ & - \frac{e^{-2y}}{\sqrt{2\pi}} \int_{[-\sqrt{R-4\tau}/2+2y/\sqrt{R-4\tau}] }^{\infty} e^{-z^2/2} dz \end{aligned}$$

for $\tau > 0$. The term for $\tau < 0$ is correct as originally published.

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Tobit Maximum-Likelihood Estimation for Stochastic Time Series Affected by Receiver Saturation

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Abstract—In many fields of engineering and scientific research, random processes with infinite dynamic range must be measured by systems with finite dynamic range. This paradox leads to receiver saturation, which has a profound biasing effect on first- (and higher) order statistical estimates derived from time series sampled by the receiver. General expressions are derived for the bias and mean-squared error of the n th noncentral moment estimator for a random variable (RV) with an arbitrary probability density function (PDF), obtained by sampling a stationary random process with a saturating receiver. Research results from the field of econometrics lead to the development of general expressions for a maximum-likelihood parameter estimator that remains efficient under conditions of receiver saturation. The so-called "Tobit model" is derived in detail for the Rayleigh PDF. Results of the Tobit model's performance under simulated conditions of receiver saturation are presented for the Rayleigh, Rice–Nakagami, Lognormal, and Nakagami- M PDF's (all associated with the two-dimensional random walk implicit in the scattering of acoustic, ultrasonic, radio-frequency (RF),

and optical transmissions). It is shown that when the (standard) maximum-likelihood estimator is efficient for the RV with unlimited dynamic range, the Tobit estimator is asymptotically efficient for the RV with dynamic range that has been limited by a saturating receiver. Tobit estimators are shown to be robust for the RF scattering PDF's under both standard and nonstandard assumptions. Under identical conditions, method-of-moment estimators and standard maximum-likelihood estimators employing sample moment estimates display high mean-squared error.

Index Terms—Censoring, efficiency, maximum-likelihood estimation, receiver saturation, Tobit model, truncation.

I. INTRODUCTION

Numerous scientific and engineering disciplines deal with the sampling of random processes that can be modeled as stationary processes with probability density. Often the sample functions obtained from these random processes have substantial dynamic range, which exceeds the dynamic range of the data acquisition instrumentation (hereafter referred to as the receiver). Stochastic processes stemming from RF, microwave, laser, acoustic, and ultrasonic transmission/scattering phenomena exemplify such random signals. Receivers with finite dynamic range used to measure these processes distort the measured signal. Receivers employing logarithmic analog-to-digital (A/D) converters substantially reduce—but do not eliminate—the distorting effects of saturation; furthermore, many acquisition systems are restricted to the use of linear converters for overriding economic or system integration considerations.

Receiver saturation can take the form of *truncation* (whereby all data exceeding the receiver's dynamic range are immeasurable, and "lost") or *censoring* (whereby all data exceeding the receiver's dynamic range are "counted" but not measured specifically). Sample moment estimates of received data are commonly used as the basis for both Maximum-Likelihood and Method-of-Moments parameter estimation techniques for hypothetical probability density functions (PDF's) against which sample data are compared. Receiver saturation affects these moment estimates and all conclusions drawn from them. The general expressions for the biasing effects of censoring and truncation on the sample moment estimators for a random variable (RV) with arbitrary PDF are substantial—even for relatively small amounts of saturation.

The Tobit (Tobin Probit) model is adapted from the field of econometrics as a maximum-likelihood estimator of PDF parameters for data that have been censored or truncated. A general expression for the Tobit estimator is presented. It is shown that when the (standard) maximum-likelihood estimator is efficient for the RV with unlimited dynamic range, the unbiased Tobit estimator is efficient for the censored/truncated RV. The model is proffered in detail for the Rayleigh PDF; its efficiency is confirmed, independent of the degree of truncation/censoring. Results from the application of Tobit estimation to simulated data with Rayleigh, Lognormal, Rice–Nakagami, and Nakagami- M PDF's are shown to exhibit very low mean-squared error as well. Finally, the limitations and computational complexities of the Tobit estimator are discussed. The reader who seeks detailed derivations of the following formulae should refer to Appendix 1 of [1] for the general Tobit model, and Appendixes 2–5 of [1] for PDF-specific Tobit models.

II. MOTIVATION

The motivation for this work stems from the authors' investigation of the effects of acute heart disease on the ultrasonic scattering properties of canine and human myocardium (heart muscle tissue). Establishing a link between the scattering properties of myocardium

Manuscript received December 19, 1988; revised March 26, 1991. This work was supported by a grant from the Mathers Foundation, and supported by technical equipment from the Medical Imaging Division of Hewlett-Packard Corporation, Andover, MA. The views and conclusions contained in this work are the authors', and should not be interpreted as representing the official policies (either expressed or implied) of the Mathers Foundation or Hewlett-Packard. This work was presented in part at the International Conference on Ultrasonic Imaging and Tissue Characterization, Washington, DC, June 1988.

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IEEE Log Number 9103965.