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Gravitationally coupled Dirac equation for antimatter

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The coupling of antimatter to gravity is of general interest because of conceivable cosmological consequences ("surprises") related to dark energy and the cosmological constant. Here, we revisit the derivation of the gravitationally coupled Dirac equation and find that the prefactor of a result given previously by Brill and Wheeler [Rev. Mod. Phys. **29**, 465 (1957)] for the affine connection matrix is in need of a correction. We also discuss the conversion of the curved-space Dirac equation from the so-called "East-Coast" to the "West-Coast" convention, in order to bring the gravitationally coupled Dirac equation to a form where it can easily be unified with the electromagnetic coupling as it is commonly used in modern particle physics calculations. The Dirac equation describes antiparticles as negative-energy states. We find a symmetry of the gravitationally coupled Dirac equation, which connects particle and antiparticle solutions for a general space-time metric of the Schwarzschild type and implies that particles and antiparticles experience the same coupling to the gravitational field, including all relativistic quantum corrections of motion. Our results demonstrate the consistency of quantum mechanics with general relativity and imply that a conceivable difference of gravitational interaction of hydrogen and antihydrogen should directly be attributed to a a "fifth force" ("quintessence").

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I. INTRODUCTION

In view of the recent dramatic progress of antimatter gravity experiments [1,2], it seems indicated to reexamine the theoretical status of antimatter coupling to gravity. A number of experimental collaborations are actively pursuing related experiments [3–8]. A key factor in recent experimental progress [2] of the ALPHA collaboration has been their special Penning-Ioffé trap which simultaneously traps both positrons as well as antiprotons. Superimposed on the Penning trap fields (which trap the charged constituent particles), the antihydrogen atom Ioffé trap employed by ALPHA relies on a strong octupole magnetic-field configuration generated by eight superconducting current bars, which wind back on themselves in a sinuous pattern, glued to the inner chamber of the ALPHA experiment by a three-dimensional winding machine at Brookhaven National Laboratory (see Ref. [9]). This leads to an effective trapping of positrons and antiprotons, and antihydrogen atoms.

Antimatter gravity experiments aim to test the interaction of antihydrogen atoms with gravitational fields. According to general relativity [10], gravitational interactions can be described by the induced space-time curvature around massive objects. Furthermore, on the classical level, the motion of a particle in curved space-time is described by the following geodesic equation [10]:

$$\frac{d^2 x^\mu}{d^2 s} + \Gamma^\mu{}_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} = 0, \tag{1}$$

which implies that a particle of mass m experiences a "force" $F^\mu = m\, d^2 x^\mu / ds^2$ and moves along a zero geodesic in the gravitationally curved space-time (s is the proper time). Here, the $\Gamma^\mu{}_{\rho\sigma}$ are the Christoffel symbols [10], derived from the curved-space metric $\overline{g}_{\mu\nu}$ as follows:

$$\Gamma_{\alpha\rho\sigma} = \frac{1}{2}\left(\frac{\partial \overline{g}_{\alpha\sigma}}{\partial x^\rho} + \frac{\partial \overline{g}_{\alpha\rho}}{\partial x^\sigma} - \frac{\partial \overline{g}_{\rho\sigma}}{\partial x^\alpha}\right), \tag{2}$$

where the Einstein summation convention is used. The $\Gamma^\mu{}_{\rho\sigma}$ are derived from the $\Gamma_{\alpha\rho\sigma}$ by raising the first index with the help of the metric, i.e., $\Gamma^\mu{}_{\rho\sigma} = \overline{g}^{\mu\alpha}\,\Gamma_{\alpha\rho\sigma}$. We should clarify that in the current article, in a somewhat nonstandard notation, the symbol $\widetilde{g}_{\mu\nu}$ will be reserved for the flat-space metric in the following, whereas $\overline{g}^{\mu\alpha}$ denotes the metric of curved space. If, according to Einstein's equivalence principle, we assume that gravitational mass and inertial mass are proportional to each other, then classical geometrodynamics [10], on the basis of Eq. (2), makes the unique prediction that the force on a particle and antiparticle in a gravitational field are the same, provided the mass of particle and antiparticle are equal, i.e., both particle as well as antiparticle motion are described by Eq. (1). However, on the quantum level, the situation is less clear. It is often argued [7] that "general relativity is incompatible with quantum mechanics" and that, assuming rather peculiar couplings of antimatter to gravity [11], one can imagine that antimatter actually is repulsed by gravity. This observation provides part of the motivation for a number of antimatter gravity experiments currently under preparation [3–8].

Here, we reexamine the status of theoretical predictions regarding the coupling of Dirac particles and antiparticles to curved space-time. Indeed, closer inspection shows that considerable insight into the gravitational coupling of antiparticles can be gained based on rather straightforward generalizations of previous treatments which rely on a combination of relativistic quantum mechanics with general relativity. We note the works of Brill and Wheeler [12], Boulware [13], and Soffel, Müller, and Greiner [14]. The Dirac equation [15,16] describes both particles and antiparticles simultaneously, and symmetries of the solutions which connect particles and antiparticles are therefore relevant for antigravity experiments. We find that it is highly indicated to revisit a number of aspects of the derivation. We employ units with $\hbar = c = \epsilon_0 = 1$.

II. FORMALISM

Antihydrogen consists of two spin-1/2 particles, the electron and the proton. Spin-1/2 particles are described by the Dirac equation. In curved and flat space-time, respectively, the anticommutators $\{\cdot,\cdot\}$ of the Dirac γ matrices fulfill the algebraic relations

$$\{\overline{\gamma}^{\mu}(x),\overline{\gamma}^{\nu}(x)\} = 2\,\overline{g}^{\mu\nu}(x), \quad \{\widetilde{\gamma}^{\mu},\widetilde{\gamma}^{\nu}\} = 2\,\widetilde{g}^{\mu\nu}, \tag{3}$$

where the curved-space metric is $\overline{g}^{\mu\nu}$ with $\mu,\nu = 0,1,2,3$, while the local flat-space metric ("vierbein") in our conventions is $\widetilde{g}^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. The precise form of the $\overline{\gamma}^{\mu}(x)$ matrices depends on the space-time geometry and in particular, on the space-time coordinate x. We use the "West-Coast" signature diag$(1,-1,-1,-1)$ for the free-space metric $\widetilde{g}$ instead of the "East-Coast" conventions diag$(1,1,1,-1)$ or diag$(-1,1,1,1)$, in order to ensure compatibility with the sign convention usually adopted in the modern particle physics literature [17–20].

This is different from the sign conventions adopted in the traditional literature on general relativity [10], and also different from the sign conventions used in previous works on the gravitationally coupled Dirac equation [12–14]. E.g., the conventions of Misner, Thorne, and Wheeler [10] are given in Eq. (2.10) on page 53 of Ref. [10] and involve a metric diag$(-1,1,1,1)$ with $\mu,\nu = 0,1,2,3$. If we ever wish to study the combined "gravito-magnetic" effect of gravitational and electromagnetic fields on Dirac particles simultaneously, and conceivably use established results for the electromagnetic sector in a perturbative, then it is helpful to convert the gravitational Dirac equation into West-Coast conventions, because these are used in the particle physics and quantum electrodynamics (QED) literature.

One might ask why we are using the tilde in order to denote the flat-space metric, not just the plain symbol $g_{\mu\nu}$. The answer to that question is as follows. We would like to be as unique in our notation as possible, and avoid possible confusion upon comparison with the literature [12–14]. In Refs. [12–14], the curved-space Dirac matrices are denoted as γ^{μ}, but in the particle physics literature [17–20], one denotes the flat-space matrices as γ^{μ}. There is no way to unify the notations without introducing some ambiguity, and we have therefore decided to differentiate the matrices either by overlining or using the tilde, making their identification unique.

The flat-space action for the free Dirac particle in special relativity reads as

$$\begin{aligned} S_0 &= \int d^4x\,\overline{\psi}(x)\,(i\,\widetilde{\gamma}^{\mu}\partial_{\mu} - m)\psi(x) \\ &= \int d^4x\,\overline{\psi}(x)\left(\frac{i}{2}\,\widetilde{\gamma}^{\mu}\overleftrightarrow{\partial}_{\mu} - m\right)\psi(x), \end{aligned} \tag{4}$$

where $\overline{\psi}(x) = \psi(x)^{\dagger}\,\widetilde{a}$ is the Dirac adjoint, $\partial_{\mu} = \partial/\partial x^{\mu}$ is the derivative with respect to x^{μ}, and the symmetric derivative operator acts as

$$A(x)\,\overleftrightarrow{\partial}_{\mu}\,B(x) \equiv A(x)\,\partial_{\mu}\,B(x) - B(x)\,\partial_{\mu}\,A(x). \tag{5}$$

Furthermore, $\widetilde{a}$ is a Hermitizing matrix with the property

$$\widetilde{a}\,(\widetilde{\gamma}^{\mu})^{\dagger}\;\widetilde{a} = \widetilde{\gamma}^{\mu}. \tag{6}$$

Here, $b^{\dagger}$ denotes the Hermitian conjugate of a matrix b. An infinitesimal global Lorentz transformation Λ and the corresponding spinor Lorentz transformation $S(\Lambda)$ in flat space then read as

$$\Lambda^{\mu}{}_{\nu} = \widetilde{g}^{\mu}{}_{\nu} + \widetilde{\omega}^{\mu}{}_{\nu}, \tag{7a}$$

$$S(\Lambda) = 1 - \frac{i}{4}\,\widetilde{\sigma}^{\alpha\beta}\,\widetilde{\omega}_{\alpha\beta}, \tag{7b}$$

$$\widetilde{\sigma}^{\alpha\beta} = \frac{i}{2}\,[\widetilde{\gamma}^{\alpha},\widetilde{\gamma}^{\beta}]. \tag{7c}$$

Here, $\widetilde{\omega}^{\mu}{}_{\nu} + \widetilde{\omega}_{\nu}{}^{\mu} = 0$. In formulating the generators $\widetilde{\sigma}^{\alpha\beta}$ of spinor Lorentz transformations, we follow the conventions of Chap. 2 of Ref. [19]. Furthermore, in view of the relation

$$[\widetilde{\gamma}^{\mu},\widetilde{\sigma}^{\alpha\beta}] = 2\,i\,\widetilde{g}^{\mu\alpha}\,\widetilde{\gamma}^{\beta} - 2\,i\,\widetilde{g}^{\mu\beta}\,\widetilde{\gamma}^{\alpha}, \tag{8}$$

the $\widetilde{\gamma}$ matrices are shape-invariant under Lorentz transformations,

$$\widetilde{\gamma}'^{\mu} = \Lambda^{\mu}{}_{\nu}\,S(\Lambda)\,\widetilde{\gamma}^{\nu}\,S(\Lambda)^{-1} = \widetilde{\gamma}^{\mu}, \tag{9}$$

and $\overline{\psi}$ transforms with the inverse Lorentz transformation, $\overline{\psi}'(x') = \overline{\psi}(x)\,S(\Lambda)^{-1}$. This can be shown easily by observing that

$$\widetilde{a}\;[S(\Lambda)]^{+}\;\widetilde{a} = S(\Lambda)^{-1}. \tag{10}$$

Standard representations of the flat-space Dirac matrices $\widetilde{\gamma}$ include the Dirac and the Majorana representation [19,20].

The generalization of the Dirac action (4) to curved space-time involves two steps: (i) an obvious generalization of the anticommutator relations (3) to curved space, $\{\overline{\gamma}^{\mu}(x),\overline{\gamma}^{\mu}(x)\} = 2\,\overline{g}^{\mu\nu}(x)$, and (ii) a coupling of the derivative operator ∂_{μ} in the Dirac equation to the gravitational field on the basis of a covariant derivative, in the sense of the replacement $\partial_{\mu} \to \nabla_{\mu} = \partial_{\mu} - \Gamma_{\mu}$, where ∇_{μ} is the covariant derivative and $\Gamma_{\mu} \equiv \Gamma_{\mu}(x)$ is the affine connection matrix. The action for the Dirac particle in curved space-time then reads as

$$S = \int d^4x\,\sqrt{-\det\,\overline{g}}\;\;\overline{\psi}(x)\,[i\,\overline{\gamma}^{\mu}(x)\nabla_{\mu} - m]\;\psi(x), \tag{11}$$

where $\det\,\overline{g} = \det\overline{g}_{\mu\nu} < 0$ is the determinant of the space-time metric, and $\overline{\psi} = \psi^{\dagger}\,\overline{a}(x)$ is the curved-space Dirac adjoint, where $\overline{a}(x)$ is a Hermitizing matrix with the local properties

$$\overline{a}(x)\,[\overline{\gamma}^{\mu}(x)]^{\dagger}\;\overline{a}(x) = \overline{\gamma}^{\mu}(x), \tag{12}$$

$$\overline{a}(x)\;[S(L(x))]^{\dagger}\;\overline{a}(x) = S(L(x))^{-1}. \tag{13}$$

Here, $S(L(x))$ is the spinor transformation corresponding to an infinitesimal, local Lorentz transformation $L(x)$ and reads as

$$L(x)^{\mu}{}_{\nu} = \overline{g}^{\mu}{}_{\nu} + \overline{\omega}^{\mu}{}_{\nu}(x), \tag{14a}$$

$$S(L(x)) = 1 - \frac{i}{4}\,\overline{\sigma}^{\alpha\beta}(x)\,\overline{\omega}_{\alpha\beta}(x), \tag{14b}$$

$$\overline{\sigma}^{\alpha\beta}(x) = \frac{i}{2}\,[\overline{\gamma}^{\alpha}(x),\overline{\gamma}^{\beta}(x)]. \tag{14c}$$

These equations generalize Eq. (7) to curved space-time and ensure that the $\overline{\gamma}$ matrices are shape invariant under Lorentz transformations,

$$\overline{\gamma}'^{\mu}(x) = L(x)^{\mu}{}_{\nu}\,S(L(x))\,\overline{\gamma}^{\nu}(x)\,S(L(x))^{-1} = \overline{\gamma}^{\mu}(x). \tag{15}$$

From now on, we shall suppress the space-time coordinate argument x in the $\overline{\gamma}$ and $\overline{\sigma}$ matrices. The generalization of Eq. (8) to general relativity is given by

$$[\overline{\gamma}^{\mu},\overline{\sigma}^{\alpha\beta}] = 2\,i\,\overline{g}^{\mu\alpha}\,\overline{\gamma}^{\beta} - 2\,i\,\overline{g}^{\mu\beta}\,\overline{\gamma}^{\alpha}. \tag{16}$$

One can show this relation using Eq. (3) only. For tensors, the covariant derivative ∇_μ is well established [see Exercise 8.4 on page 211 of Ref. [10]], but for spinors, a nontrivial generalization is required. Let us assume the structure

$$\nabla_\nu \psi = (\partial_\nu - \Gamma_\nu)\psi, \tag{17}$$

where the affine connection matrix Γ_ν remains to be determined. We postulate that the covariant derivative operator commutes with the current matrix $\overline{\gamma}^{\mu}(x)$, i.e., $[\overline{\gamma}_{\mu}(x),\,\nabla_\nu] = 0$. Then,

$$\overline{\gamma}_{\mu}(x)\,\nabla_\nu\psi(x) = \nabla_\nu\,(\overline{\gamma}_{\mu}(x)\psi(x)), \tag{18}$$

and we can symmetrize Eq. (11) as follows:

$$S = \int d^4x\,\sqrt{-\det\overline{g}}\;\;\overline{\psi}(x)\left(\frac{i}{2}\overline{\gamma}^{\mu}(x)\;\overleftrightarrow{\nabla}_{\mu} - m\right)\psi(x). \tag{19}$$

By variation, the gravitationally coupled Dirac equation is obtained as

$$(i\,\overline{\gamma}^{\mu}\,\nabla_\mu - m)\psi(x) = 0. \tag{20}$$

An additional electromagnetic field could be incorporated by the replacement $\nabla_\mu \to \nabla_\mu + i\,q\,A_\mu$, where A_μ is the vector potential and q is the charge. However, the gravitational Dirac equation is primarily interesting when all electromagnetic interactions are compensated and the residual gravitational interaction dominates the kinematics.

The affine connection matrices Γ_μ remain to be determined. Using the ansatz (17), one can write the condition given in Eq. (18) as follows [12–14,21,22]:

$$\nabla_\nu\overline{\gamma}_{\mu} = \partial_\nu\overline{\gamma}_{\mu} - \Gamma^{\rho}{}_{\mu\nu}\overline{\gamma}_{\rho} - \Gamma_\nu\,\overline{\gamma}_{\mu} + \overline{\gamma}_{\mu}\,\Gamma_\nu = 0. \tag{21}$$

The $\Gamma^{\rho}{}_{\mu\nu}\overline{\gamma}_{\rho}$ are the Christoffel symbols defined in Eq. (2). For a Lorentz vector T^{μ}, we recall that [10]

$$\nabla_\mu T_\alpha = \partial_\mu T_\alpha - \Gamma^{\lambda}{}_{\alpha\mu}\,T_\lambda, \tag{22a}$$
$$\nabla_\mu T^\alpha = \partial_\mu T^\alpha + \Gamma^{\alpha}{}_{\mu\lambda}\,T^\lambda. \tag{22b}$$

The third and fourth term on the right-hand side in Eq. (21) represent the spinor structure contributions to the covariant derivative of the $\overline{\gamma}_{\mu}$ matrix.

The condition (21) defines the Γ_ν matrix up to a multiple of the unit matrix. In the vierbein formalism, we can represent the $\overline{\gamma}^{\nu}$ matrices in terms of the vierbein $\widetilde{\gamma}^{\mu}$ matrices as follows:

$$\overline{\gamma}_{\rho} = b_\rho{}^{\alpha}\,\widetilde{\gamma}_{\alpha}, \quad \widetilde{\gamma}_{\rho} = a^{\alpha}{}_{\rho}\,\overline{\gamma}_{\alpha}, \tag{23a}$$
$$\overline{\gamma}^{\alpha} = a^{\alpha}{}_{\rho}\,\widetilde{\gamma}^{\rho}, \quad \widetilde{\gamma}^{\alpha} = b_\rho{}^{\alpha}\,\overline{\gamma}^{\rho}. \tag{23b}$$

The metric is recovered as

$$\{\overline{\gamma}_{\rho},\overline{\gamma}_{\sigma}\} = b_\rho{}^{\alpha}\,b_\sigma{}^{\beta}\,\{\widetilde{\gamma}_{\alpha},\widetilde{\gamma}_{\beta}\} = 2\,\widetilde{g}_{\alpha\beta}\,b_\rho{}^{\alpha}\,b_\sigma{}^{\beta} = 2\,\overline{g}_{\rho\sigma}, \tag{24a}$$

$$\{\overline{\gamma}^{\rho},\overline{\gamma}^{\sigma}\} = a^{\rho}{}_{\alpha}\,a^{\sigma}{}_{\beta}\,\{\widetilde{\gamma}^{\alpha},\widetilde{\gamma}^{\beta}\} = 2\,\widetilde{g}^{\alpha\beta}\,a^{\rho}{}_{\alpha}\,a^{\sigma}{}_{\beta} = 2\,\overline{g}^{\rho\sigma}. \tag{24b}$$

We note that the matrix with components $\overline{g}^{\rho\sigma}$ is the inverse of the matrix with components $\overline{g}_{\rho\sigma}$, where the entries of $\widetilde{g}_{\alpha\beta}$ and $\widetilde{g}^{\alpha\beta}$ are identical. It is possible to show, using rather lengthy algebra, that the following affine connection matrix,

$$\Gamma_\rho = -\frac{i}{4}\,\overline{g}_{\mu\alpha}\left(\frac{\partial b_\nu{}^{\beta}}{\partial x^{\rho}}\,a^{\alpha}{}_{\beta} - \Gamma^{\alpha}{}_{\nu\rho}\right)\overline{\sigma}^{\mu\nu}, \tag{25}$$

with $\overline{\sigma}^{\mu\nu} = \frac{i}{2}\,[\overline{\gamma}^{\mu},\overline{\gamma}^{\nu}]$, fulfills Eq. (21) for a general metric $\overline{g}^{\mu\nu}$. Our result (25) differs from the result given in Eq. (8) of Ref. [12] in the correction of an obvious and in some sense trivial typographical error. Namely, the expression $\Gamma^{\alpha}{}_{\nu l}$ in Eq. (8) of Ref. [12] should be replaced by the expression $\Gamma^{\alpha}{}_{\nu k}$ (we have used ρ for the corresponding subscript of Γ_ρ, not Γ_k as in Ref. [12]). It is less trivial to see that a prefactor 1/4 is missing from Eq. (8) of Ref. [12] and needs to be supplemented as given in Eq. (25). In order to clarify the matter, we should also point out that the additional imaginary unit in the prefactor is entirely due to our different conventions for the γ matrices and the flat-space metric which follow modern West-Coast conventions [19,20].

It is rather lengthy but straightforward to show that Eq. (25) solves Eq. (21). One needs to use Eqs. (23) and (16) repeatedly, and one needs to observe that the b matrix is the inverse of the a matrix, i.e., $a^{k}{}_{\alpha}\,b_\rho{}^{\alpha} = \delta^{k}{}_{\rho}$, where δ is the Kronecker symbol. Furthermore, the relation $\Gamma^{\beta}{}_{\sigma\rho} + \Gamma_\sigma{}^{\beta}{}_{\rho} = \overline{g}^{\beta\alpha}\,\partial_\rho\overline{g}_{\alpha\sigma}$ is useful in intermediate steps of the calculation. Here, $\Gamma_\sigma{}^{\beta}{}_{\rho} = \overline{g}^{\beta\alpha}\,\Gamma_{\sigma\alpha\rho}$ with $\Gamma_{\sigma\alpha\rho}$ given in Eq. (2).

Using the identity $\overline{\sigma}^{\mu\nu} = i\,\overline{g}^{\mu\nu} - i\overline{\gamma}^{\nu}\,\overline{\gamma}^{\mu}$, it is possible to rewrite Eq. (25) in a simpler form,

$$\Gamma_\rho = -\frac{\overline{\gamma}^{\nu}}{4}\,(\partial_\rho\overline{\gamma}_{\nu} - \Gamma^{\mu}{}_{\nu\rho}\,\overline{\gamma}_{\mu}) + \mathcal{A}_\rho\,\mathbb{1}_{4\times4}, \tag{26a}$$

$$\mathcal{A}_\rho = \frac{1}{8}\,[2(\partial_\rho b_\alpha{}^{\beta})\,a^{\alpha}{}_{\beta} - (\partial_\rho\overline{g}_{\alpha\beta})\,\overline{g}^{\alpha\beta}]. \tag{26b}$$

For a diagonal structure of the metric tensor (the only nonvanishing elements are the $\overline{g}_{\alpha\beta}$ with $\alpha = \beta$), with $b_\alpha{}^{\beta} = \sqrt{|\overline{g}_{\alpha\beta}|}$ and $a^{\alpha}{}_{\beta} = \sqrt{|\overline{g}^{\alpha\beta}|} = 1/\sqrt{|\overline{g}_{\alpha\beta}|}$, the additional term $\mathcal{A}_\rho$ vanishes. This is the case for the (generalized) Schwarzschild metric to be discussed below. We have checked that, up to the term $\mathcal{A}_\rho$ and up to the conversion from East-Coast to West-Coast conventions, the result (26a) is formally identical to the result previously given in Eq. (9) of Ref. [14]. As a byproduct of our calculation of the $\mathcal{A}_\rho$, we thus show that the two different results for the affine connection matrix given in Refs. [12,14] are equivalent up to the term $\mathcal{A}_\rho\,\mathbb{1}_{4\times4}$, which is proportional to the unit matrix and not determined by the defining Eq. (21). For a diagonal metric, $\mathcal{A}_\rho$ vanishes. To the best of our knowledge, the precise form of the term $\mathcal{A}_\rho$ has not yet been indicated in the literature.

Our construction of the spinor Lorentz transformation in curved space [see Eq. (13)] follows ideas outlined in Ref. [13]. However, our result for the covariant derivative of a spinor manifestly contains additional terms as compared to the result given in Eq. (2.8) of Ref. [13]. In particular, in view of the condition (21), it is clear that the derivative terms [the first terms in round brackets on the right-hand sides of Eqs. (25) and (26a)] are an essential contribution to the affine connection matrix; these terms seem to be missing in the vierbein formalism formulated in later steps of the derivation leading to Eq. (2.8) of Ref. [13].

III. SCHWARZSCHILD-TYPE METRIC

A. Radially dependent metrics

In the following, we shall describe an important application of the above formalism. Namely, we shall discuss a generalization of the Schwarzschild metric, which describes, to good approximation, the gravitational field of a planet, e.g., the Earth. In West-Coast conventions (where the sign of the timelike component is positive), the Schwarzschild metric reads as follows [12–14,23]:

$$\overline{g}_{\mu\nu} = \mathrm{diag}\left(e^{\nu}, -e^{\lambda}, -r^2, -r^2\sin^2\theta\right) = \mathrm{diag}\left(u^2, -\frac{1}{u^2}, -r^2, -r^2\sin^2\theta\right), \tag{27}$$

$$u^2 = e^{\nu} = e^{-\lambda} = 1 - \frac{2M_G}{r}, \tag{28}$$

$$r_s = 2M_G = 2\,G\,M_P. \tag{29}$$

Here, the Schwarzschild radius is $r_s = 2M_G = 2G\,M_P/c^2 \approx 0.0089$ m, and G is Newton's gravitational constant (here, we supplement the factor c^{-2} although we use units with $c = 1$ in this article otherwise). Furthermore, M_P is the mass of the Earth (or, of the planet under consideration). The invariant line element ds^2 in the Schwarzschild geometry is given by

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2\,(d\theta^2 + \sin^{\theta}\, d\varphi^2). \tag{30}$$

The Schwarzschild metric is valid for a spherically symmetric geometry of space. However, it has a problem. Namely, as pointed out by Eddington [24], for the original Schwarzschild metric, the speed of light in the radial direction is not equal to the speed of light in the transverse directions; the prefactor in front of the "angular" term $r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$ is not the same as the one in front of the "radial term" proportional to dr^2. This structure implies that one has to resort to a highly nonstandard representation of the Dirac algebra [14] if one would like to separate the gravitationally coupled Dirac equation in the original form of the Schwarzschild metric (27).

For example, without explicit mention, a representation of the following form has apparently been used in Ref. [14]:

$$\widetilde{\gamma}^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}, \quad \widetilde{\gamma}^1 = \begin{pmatrix} 0 & -i\,\mathbb{1}_{2\times2} \\ -i\,\mathbb{1}_{2\times2} & 0 \end{pmatrix}, \tag{31a}$$

$$\widetilde{\gamma}^2 = \begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \quad \widetilde{\gamma}^3 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \tag{31b}$$

$$\widetilde{\gamma}^5 = i\,\widetilde{\gamma}^0\,\widetilde{\gamma}^1\,\widetilde{\gamma}^2\,\widetilde{\gamma}^3 = \begin{pmatrix} 0 & -i\sigma^1 \\ i\sigma^1 & 0 \end{pmatrix}. \tag{31c}$$

The authors of Ref. [14] use the matrix $\widetilde{\gamma}^1$ for the "radial" part of the Dirac equation. Specifically, near Eq. (21) of Ref. [14], it is stated without further explanation that a representation of the Dirac algebra is used where $\widetilde{\gamma}^1$ assumes a particularly simple form, proportional to the expression given in Eq. (31a). Indeed, such representations exist, as we show in Eq. (31), thus leading to a ramification of the somewhat *ad hoc* statement made in Ref. [14]. It is easy to verify that the relations $\{\widetilde{\gamma}^{\mu},\widetilde{\gamma}^{\nu}\} = 2\,\widetilde{g}^{\mu\nu} = 2\,\mathrm{diag}(1,-1,-1,-1)$ are fulfilled. Here, the $\vec{\sigma} = (\sigma^1,\sigma^2,\sigma^3)$ are the $(2\otimes2)$ Pauli spin matrices, and $\mathbb{1}_{2\times2}$ denotes the $(2\otimes2)$ unit matrix.

B. Eddington's reparametrization

In Sec. 43 of Chap. 3 of Ref. [24], Eddington has pointed out that a coordinate transformation exists which converts the Schwarzschild metric into spatially isotropic form. It reads as follows:

$$r = r_1\left(1 + \frac{r_s}{4r_1}\right)^2, \quad r_1 = \frac{r}{2} - \frac{r_s}{4} + \sqrt{\frac{r}{4}\,(r - r_s)}. \tag{32}$$

After this transformation, the invariant line element (30) becomes

$$ds^2 = \frac{(4r_1 - r_s)^2}{(4r_1 + r_s)^2}\,dt^2 - \left(1 + \frac{r_s}{4r_1}\right)^4 \left(dr_1^2 + r_1^2 d\theta^2 + r_1^2\,\sin^2\theta\, d\varphi^2\right). \tag{33}$$

Using this isotropic form of the metric, we can now transform the spatial part to Cartesian coordinates,

$$ds^2 = \frac{(4r_1 - r_s)^2}{(4r_1 + r_s)^2}dt^2 - \left(1 + \frac{r_s}{4r_1}\right)^4 \left(dx_1^2 + dy_1^2 + dz_1^2\right), \tag{34}$$

where $x_1 = r_1\,\sin\theta\,\cos\varphi$, $y_1 = r_1\,\sin\theta\,\sin\varphi$, and $z_1 = r_1\,\cos\theta$. We now redefine

$$r_1 \to r, \quad x_1 \to x, \quad y_1 \to y, \quad z_1 \to z, \tag{35a}$$

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{35b}$$

Furthermore, we define $w(r)$ and $v(r)$ as follows:

$$w(r) = \frac{4r - r_s}{4r + r_s}, \quad v(r) = \left(1 + \frac{r_s}{4r}\right)^2. \tag{36}$$

The transformed (according to Ref. [24]) Schwarzschild metric can now be written in the following form:

$$\overline{g}_{\mu\nu} = \mathrm{diag}(w^2(r), -v^2(r), -v^2(r), -v^2(r)). \tag{37}$$

The considerations below are valid for a general form (37) of the metric and not tied to the specific form given in Eq. (36). The vierbein coefficients are given as

$$b_0{}^{\beta} = b_{\beta}{}^0 = \delta_0{}^{\beta}\,w(r), \quad b_i{}^j = \delta_i{}^j\,v(r), \tag{38}$$

$$a^{\alpha}{}_0 = a^0{}_{\alpha} = \frac{\delta^0{}_{\alpha}}{w(r)}, \quad a_i{}^j = \frac{\delta_i{}^j}{v(r)}, \tag{39}$$

where $i,j = 1,2,3$ and $\delta_{\alpha}{}^{\beta}$ and $\delta^{\alpha}{}_{\beta}$ are Kronecker symbols (i.e., equal to 1 if the indices are equal, otherwise 0).

With these coefficients, using computer algebra [25], it is easy to evaluate all Christoffel symbols and to establish that

$$\overline{\gamma}^{\mu}\,\Gamma_{\mu} = -\frac{\vec{\gamma}\cdot\vec{r}}{r}\,w(r)\,G(r), \tag{40a}$$

$$G(r) = \frac{2\,v'(r) + w'(r)}{2\,v(r)\,w(r)}. \tag{40b}$$

This result has been verified by us both using the representation (25) as well as the representation given in (26a), for the metric (37).

C. Reduction to radial equation

In our further analysis, we use the Hamiltonian form of the gravitationally coupled Dirac equation,

$$i(\overline{\gamma}^0)^2\,\partial_t\psi = (\vec{\alpha}\cdot\vec{p} + i\,\overline{\gamma}^0\,\overline{\gamma}^\mu\,\Gamma_\mu + \overline{\gamma}^0\,m)\,\psi, \tag{41}$$

where $\vec{p} = -i\partial/\partial\vec{r}$ with $\vec{r} = (x,y,z)$, and $\alpha^j = \overline{\gamma}^0\,\overline{\gamma}^j$ for $j = 1,2,3$. The expression "Hamiltonian form" is used in analogy with flat-space. Namely, in flat space, the expression on the left-hand side simply represents the "noncovariant time-evolution operator" because $(\overline{\gamma}^0)^2 \to (\tilde{\gamma}^0)^2 = \mathbb{1}_{4\times4}$ and $H \to i\,\partial_t$ (see Refs. [26–29]). In curved space, with the metric given in Eq. (38), we have $(\overline{\gamma}^0)^2 = \frac{1}{w^2(r)}\,\mathbb{1}_{4\times4}$.

In the following, we use the Dirac matrices in the Dirac representation,

$$\tilde{\gamma}^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}, \quad \tilde{\gamma}^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \tag{42a}$$

$$\tilde{\gamma}^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad \tilde{\gamma}^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}, \tag{42b}$$

$$\tilde{\gamma}^5 = i\,\tilde{\gamma}^0\,\tilde{\gamma}^1\,\tilde{\gamma}^2\,\tilde{\gamma}^3 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}. \tag{42c}$$

In the form (41), the gravitationally coupled Dirac equation allows a solution of the standard form [26–29],

$$\psi = \begin{pmatrix} f(r)\,\chi_{\varkappa\mu}(\theta,\varphi) \\ i g(r)\,\chi_{-\varkappa\mu}(\theta,\varphi) \end{pmatrix} \exp(-i\,E\,t), \tag{43}$$

where the $\chi_{\varkappa\mu}(\theta,\varphi)$ [sometimes denoted as $\chi^{\mu}_{\varkappa}(\theta,\varphi)$] are the standard spin-angular functions [26–29]. They have the property

$$(\vec{\sigma}\cdot\vec{L} + 1)\,\chi_{\varkappa\mu}(\theta,\varphi) = -\varkappa\,\chi_{\varkappa\mu}(\theta,\varphi). \tag{44}$$

We recall that the eigenvalues of the operator $K = \vec{\sigma}\cdot\vec{L} + 1$ are $-\varkappa$ [see the text following Eq. (2.9) of Ref. [27]]. In the text following Eq. (18) of Ref. [14], the eigenvalues is assumed to be $+\varkappa$, an apparent typographical error. It is extremely instructive to write the Hamiltonian form (41), using the ansatz (43), in terms of $(2\otimes 2)$ spin matrices,

$$\begin{aligned}
\frac{1}{w^2(r)} i\,\partial_t\psi(\vec{r}) &= \begin{pmatrix} \dfrac{m}{w(r)} & \dfrac{\vec{\sigma}\cdot\hat{r}}{v(r)}\left(-i\,\dfrac{\partial}{\partial r} + i\,\dfrac{\vec{\sigma}\cdot\vec{L}}{r} - i\,G(r)\right) \\ \dfrac{\vec{\sigma}\cdot\hat{r}}{v(r)}\left(-i\,\dfrac{\partial}{\partial r} + i\,\dfrac{\vec{\sigma}\cdot\vec{L}}{r} - i\,G(r)\right) & -\dfrac{m}{w(r)} \end{pmatrix} \begin{pmatrix} f(r)\,\chi_{\varkappa\mu}(\hat{r}) \\ i\,g(r)\,\chi_{-\varkappa\mu}(\hat{r}) \end{pmatrix} \\
&= \begin{pmatrix} \left[-\dfrac{1}{v(r)}\left(\dfrac{\partial}{\partial r}g(r) - \dfrac{1}{r}\,g(r)\,(\varkappa-1) + G(r)\,g(r)\right) + \dfrac{m}{w(r)}\,f(r)\right]\chi_{\varkappa\mu}(\hat{r}) \\ \left[-(-i)\,\dfrac{1}{v(r)}\left(\dfrac{\partial}{\partial r}f(r) - \dfrac{1}{r}\,f(r)\,(-\varkappa-1) + G(r)\,f(r)\right) - \dfrac{im}{w(r)}\,g(r)\right]\chi_{-\varkappa\mu}(\hat{r}) \end{pmatrix} \\
&= \begin{pmatrix} \left[-\dfrac{1}{v(r)}\left(\dfrac{\partial}{\partial r} + \dfrac{1-\varkappa}{r} + G(r)\right)g(r) + \dfrac{m}{w(r)}\,f(r)\right]\chi_{\varkappa\mu}(\hat{r}) \\ i\left[\dfrac{1}{v(r)}\left(\dfrac{\partial}{\partial r} + \dfrac{\varkappa+1}{r} + G(r)\right)f(r) - \dfrac{m}{w(r)}\,g(r)\right]\chi_{-\varkappa\mu}(\hat{r}) \end{pmatrix} = \frac{E}{w^2(r)}\begin{pmatrix} f(r)\,\chi_{\varkappa\mu}(\hat{r}) \\ i\,g(r)\,\chi_{-\varkappa\mu}(\hat{r}) \end{pmatrix}.
\end{aligned} \tag{45}$$

Here, we have used the relation $\vec{\sigma}\cdot\hat{r}\,\chi_{\varkappa\mu}(\hat{r}) = -\chi_{-\varkappa\mu}(\hat{r})$, which can be found in Eq. (7.2.5.23) of Ref. [30], where $\hat{r} = \vec{r}/|\vec{r}|$ is the position unit vector. The radial equations are thus given as

$$\left(\frac{\partial}{\partial r} + \frac{1-\varkappa}{r} + G(r)\right)g(r) = \frac{v(r)}{w(r)}\left(m - \frac{E}{w(r)}\right)f(r), \tag{46a}$$

$$\left(\frac{\partial}{\partial r} + \frac{1+\varkappa}{r} + G(r)\right)f(r) = \frac{v(r)}{w(r)}\left(m + \frac{E}{w(r)}\right)g(r). \tag{46b}$$

An important symmetry property of Eq. (46) is given by its invariance under the simultaneous replacements

$$E \leftrightarrow -E, \quad f(r) \leftrightarrow g(r), \quad \varkappa \leftrightarrow -\varkappa. \tag{47}$$

So, if E is an eigenvalue of the gravitationally coupled Dirac equation, so is $-E$. Invoking the reinterpretation principle [31–33] and interpreting the negative energy $-E < 0$ as $+E > 0$ for antiparticles (which propagate "into the past"), we find that the spectrum of the gravitationally coupled Dirac Hamiltonian is exactly the same for particles and antiparticles. This important finding is true for any space-time metric of the form given in Eq. (37) and not necessarily tied to the Schwarzschild geometry.

Let us now establish the connection to the flat-space result given in Ref. [27]. Specifically, Eqs. (2.12a) and Eq. (2.12b) of Ref. [27], using the identity $r^{-1}\partial_r[r\,h(r)] = \partial_r h(r) + r^{-1}\,h(r)$, for $\hbar = c = 1$ and $Z\alpha \to 0$, can be written as

$$\left(\frac{\partial}{\partial r} + \frac{1-\varkappa}{r}\right)g(r) = (m - E)\,f(r), \tag{48a}$$

$$\left(\frac{\partial}{\partial r} + \frac{1+\varkappa}{r}\right) f(r)) = (m+E)\, g(r), \qquad (48b)$$

where we have used the form (43) for the wave function. These equations therefore become identical to our Eq. (46) in the limit $v(r) \to 1$, $w(r) \to 1$.

The symmetry property (47) is physically tied to the reinterpretation principle which is very well known in the particle physics community [17–20] but less well known in the general relativity community. Some remarks are therefore in order. We consider a space-time interval $\Delta x = (\Delta t, \Delta\vec{r})$, and a scalar product $k \cdot \Delta x = |E|\, \Delta t - \vec{k} \cdot \Delta\vec{r}$ (with $\Delta t > 0$). The antiparticle amplitude $\exp(i\, k \cdot \Delta x)$ then is proportional to

$$e^{i|E|\,\Delta t - i\vec{k}\cdot\Delta\vec{r}} \to e^{-i|E|\,(-\Delta t) + i\vec{k}\cdot(-\Delta\vec{r})}, \qquad (49)$$

where $-\Delta t > 0$ and one can thus reinterpret the antiparticle trajectory, initially propagating "into the past" (advanced contribution to the Feynman propagator) and along the distance interval $\Delta\vec{r}$ ("from point a to point b"), as a positive-energy trajectory with energy $+E$, covering the inverse space-time interval $(-\Delta t > 0, -\Delta\vec{r})$ i.e., propagating into the future with four-momentum $(|E|, \vec{k})$, i.e., "from point b to point a." Applied to gravitational interactions, the currently available accepted interpretation based on particle physics principles [17–20] therefore dictates that "an antiparticle falls upward in the gravitational field, but backward in time, and with the same modulus of the kinetic energy as the corresponding particle." Therefore, after reinterpretation, the formalism of the gravitationally coupled Dirac equation predicts that antiparticles and particles receive exactly the same energy perturbations in a gravitational field, at least within space-time geometries that have the general form (37). This important result generalizes Eq. (1) to the relativistic quantum domain.

IV. CONCLUSIONS

In this article, we reexamine the gravitationally coupled Dirac equation in Sec. II, explaining a number of aspects of the derivation in greater detail. In particular, we show that the condition (21) follows naturally as a consequence of the fundamental anticommutator property of the curved-space Dirac matrices (3), together with the known fact that the covariant derivative of the metric tensor has to vanish [10]. Under these assumptions, the covariant derivative of the curved-space Dirac matrix $\overline{\gamma}^{\mu}(x)$ also has to vanish, and the condition (21) follows as a consequence of the ansatz (17) for the covariant derivative of a spinor, together with the fundamental commutator property (18). The symmetrization of the covariant action of the Dirac field given in Eq. (19) then becomes possible, in analogy to the flat-space action Eq. (4). Furthermore, under a proper definition of the local spinor Lorentz transformation (13), expressed in space-time coordinates, the local Dirac matrices $\overline{\gamma}^{\mu} = \overline{\gamma}^{\mu}(x)$ are shape invariant, as shown in Eq. (15). For a general metric $\overline{g}^{\mu\nu} = \overline{g}^{\mu\nu}(x)$, we find the vierbein representation (25) of the affine connection matrices $\Gamma_{\rho} = \Gamma_{\rho}(x)$ which differs from the result given previously in Eq. (8) of Ref. [12] by a factor 1/4. With the additional prefactor, the result given in Eq. (25) then is in agreement with the result for the affine connection matrices given in Eq. (9) of Ref. [14]. In West-Coast conventions for the metric, the gravitationally Dirac equation reads as $(i\,\overline{\gamma}^{\mu}\,\nabla_{\mu} - m)\psi(x) = 0$ [see Eq. (20)], as opposed to the East-Coast form $(\overline{\gamma}^{\mu}\,\nabla_{\mu} + m)\psi(x) = 0$ [see Refs. [12,14]].

The gravitationally coupled radial Dirac equation given in (46) for a Schwarzschild-type metric (37) describes the coupling of a particle (and corresponding antiparticle) to the gravitational field of a planet. Our Eq. (46), in appropriate limits, is in agreement with the fundamental properties of upper and lower components describing particles and antiparticles at rest ($E \to m$ and $E \to -m$), if we additionally consider the limit of flat space-time [$v(r) \to 1$, $w(r) \to 1$]. This limit is explored easily, starting from Eq. (2.12) of Ref. [27].

The symmetry $E \leftrightarrow -E$, $f \leftrightarrow g$, $\varkappa \leftrightarrow -\varkappa$ given in Eq. (47) implies that the quantum states of spin-1/2 antiparticles, in the gravitational field of the Earth, have exactly the same spectrum as those of the corresponding particles, including all relativistic corrections of motion. Therefore, this statement also holds for superpositions of quantum states, including those which describe a wave packet evolving along a classical trajectory (these states would other be known as "coherent" or "Glauber" states). Our quantum-theory based findings go beyond the simple statement that particle and antiparticle trajectories in curved space-time are the same on the classical level [see Eq. (1)].

Finally, let us include a remark regarding the validity of the gravitationally coupled Dirac equation for antiparticles. One might contemplate if antiparticles should be described by a different equation in the context of gravity than particles, but by the same equation in the context of electromagnetism (electromagnetically coupled Dirac equation). In this case, the gravitationally coupled Dirac equation (20) would only describe particles, not antiparticles, even if it admits negative-energy solutions. However, in this case one gets into trouble in the limit of a vanishing gravitational interaction, in which case space becomes flat. This is because the Dirac equation is known to describe antiparticles very well in this limit [17–20]. At least, this concept is used in all perturbative QED calculations, including the notoriously difficult bound-state problems [34]. If we conjecture that the flat-space limit is smooth, then the gravitationally coupled Dirac equation (20) must remain valid for both particles and antiparticles. Brill and Wheeler [12], Boulware [13], and Soffel, Müller, and Greiner [14] all used the same methods for deriving the coupled Dirac equation (we here attempt to resolve some discrepancies found in the literature regarding the final steps in the derivation). The gravitationally coupled Dirac equation (20) involves $(4 \otimes 4)$ matrices and allows for two positive-energy solutions, which are naturally interpreted as particles, and two negative-energy solutions, which are naturally interpreted as antiparticles, according to usual practice in particle physics [17–20]. We may thus assume that the gravitationally coupled Dirac equation given in Eq. (20) should be valid for particles and antiparticles, simultaneously. We know for sure that the corresponding variant of the equation in flat space describes particles and antiparticles simultaneously, as described in detail, e.g., in Chap. 2 of Ref. [19].

Despite our theoretical considerations, it is still of utmost value to the scientific community to carry out the planned antimatter gravity experiments [3–8]. We conclude with two remarks: (i) The "inertial mass term" in the sense of the equivalence principle enters the Schrödinger equation for free and bound electrons, e.g., for a bound electron in a hydrogen atom. According to experimental evidence, inertial and gravitational mass are the the same for atoms (such as atomic hydrogen), which is composed of spin-$1/2$ particles (electrons and protons), therefore, the action principle $\delta S = 0$ [see Eq. (11)] provides for a solid basis of the discussion of relativistic quantum effects in gravitational coupling, with the m term entering the equation being equal to the inertial (gravitational) mass. (ii) Our investigations suggest that any conceivable differences of the gravitational coupling of particles and antiparticles should be assigned to a "fifth force," not to any conceivable "modifications of the gravitational mass" of antiparticles versus particles.

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[1] G. Gabrielse *et al.* (ATRAP Collaboration), Phys. Rev. Lett. **100**, 113001 (2008).
[2] C. Amole *et al.* (ALPHA Collaboration), Nature (London) **483**, 439 (2012).
[3] (ALPHA Collaboration) *(Antihydrogen Laser Physics Apparatus)*, see the URL http://alpha-new.web.cern.ch.
[4] (ATHENA Collaboration) *(ATHENA Antihydrogen Apparatus)*, see the URL http://athena.web.cern.ch/.
[5] (ASACUSA Collaboration) *(Atomic Spectroscopy And Collisions Using Slow Antiprotons)*, see the URL http://asacusa.web.cern.ch/.
[6] (ATRAP Collaboration) *(Antihydrogen trap)*, see the URL http://gabrielse.physics.harvard.edu/.
[7] A. D. Cronin *et al.* (AGE Collaboration), Letter of Intent: Antimatter Gravity Experiment (AGE) at Fermilab (2009), available at the URL http://www.fnal.gov/directorate/program_planning/Mar2009PACPublic/AGELOIFeb2009.pdf; see also the URL http://www.phy.duke.edu/~phillips/gravity/frameIndex.html.
[8] A. Kellerbauer *et al.* (AEGIS Proto-Collaboration), Nucl. Instrum. Methods Phys. Res. B **266**, 351 (2008).
[9] See the URL http://www.quantumdiaries.org/2011/06/06/trapping-antimatter-with-magnets/.
[10] C. W. Misner, K. S. Thorne, and J. Wheeler, *Gravitation* (W. H. Freeman, New York, 1973).
[11] M. Kowitt, Int. J. Theor. Phys. **35**, 605 (1996).
[12] D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. **29**, 465 (1957).
[13] D. G. Boulware, Phys. Rev. D **12**, 350 (1975).
[14] M. Soffel, B. Müller, and W. Greiner, J. Phys. A **10**, 551 (1977).
[15] P. A. M. Dirac, Proc. R. Soc. London, Ser. A **117**, 610 (1928).
[16] P. A. M. Dirac, Proc. R. Soc. London, Ser. A **118**, 351 (1928).
[17] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[18] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).
[19] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
[20] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Perseus, Cambridge, MA, 1995).
[21] G. V. Shishkin and W. D. Cabos, J. Math. Phys. **33**, 914 (1991).
[22] G. V. Shishkin and V. M. Villalba, J. Math. Phys. **33**, 4037 (1992).
[23] K. Schwarzschild, Sitz. K. Preuss. Akad. Wiss. **7**, 189 (1916).
[24] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, England, 1924).
[25] S. Wolfram, *Mathematica—A System for Doing Mathematics by Computer* (Addison-Wesley, Reading, MA, 1988).
[26] M. E. Rose, *Relativistic Electron Theory* (J. Wiley & Sons, New York, 1961).
[27] R. A. Swainson and G. W. F. Drake, J. Phys. A **24**, 79 (1991).
[28] R. A. Swainson and G. W. F. Drake, J. Phys. A **24**, 95 (1991).
[29] R. A. Swainson and G. W. F. Drake, J. Phys. A **24**, 1801 (1991).
[30] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988).
[31] E. Stückelberg, Helv. Phys. Acta **14**, 588 (1941).
[32] E. Stückelberg, Helv. Phys. Acta **15**, 23 (1942).
[33] R. P. Feynman, Phys. Rev. **76**, 769 (1949).
[34] P. J. Mohr, G. Plunien, and G. Soff, Phys. Rep. **293**, 227 (1998).