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Observability Conditions for Target States with Bearing-Only Measurements in Three-Dimensional Case

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Abstract— **In target tracking with a passive sensor such as infrared seekers, angle-only information is determined. In this case no information about the range of the target is provided and unobservable system is resulted. This paper studies the observability of discrete time three-dimensional bearing–only target tracking. The target is assumed to be moving in a straight line while a single moving observer (own-ship) measures its relative (elevation and azimuth) bearing angles. By transforming the inherently nonlinear bearing measurements into a pseudo-linear form, a linear Least Squares (LS) estimator is formulated. Observability is then analyzed by studying the solvability of the associated LS problem. This approach has the advantage of providing simple unobservability conditions as opposed to other approaches in the continuous time setup. In this paper, most of motivation and problem formulation is taken from [9]. It is shown that for no maneuvering target, at least three independent measured bearings for system observability is required.**

I. Introduction

TWO objectives in angle-only tracking may be considered. The first is simply trying to maintain the target in the field of view [1], and the second is to obtain estimation of the target state vector as defined with three dimensional coordinate system. In this case due to lack of range information we may have a large uncertainty volume. Such a system may be observable only for certain type of target dynamics. For example, when a telescope is used to track a satellite this situation constitutes an observable system and target range can be estimated. However, because the satellite trajectory is influenced by the earth gravity, large estimation error is resulted [2]. Without range information for applying extended Kalman filter (EKF) in bearing-only tracking, a new tracking approach is proposed in [3] which consist of a set of weighted EKF, each with a different initial range estimate. All EKF are performed in parallel with a weighting update and range is estimated based on normalized weighting that approaches to unity. In this manner there is always potential for having mistake in the range estimation, even though with correct initial range and the EKFs may become unstable at any instant and should be removed from tracking process. This approach needs a high processing power while have not good accuracy for range estimation. A basic principle of position estimation using angle-only measurements is that the observer motion dynamics must be one derivative higher than that of the target and that a component of this motion must be perpendicular to the line of sight. Specifically, for example, a constant, nonzero velocity observer can estimate the position of a stationary target and an accelerating observer can estimate the position and velocity of a constant velocity target. In this study, it is assumed that bearings-only tracking involves a single moving observer (own-ship) that passively tracks a target assumed to be traveling with a constant velocity in no maneuvering situation, and subsequently estimates target position and velocity. State observability of the target depends on target and own-ship relative position geometry. Most previous studies on observability condition considered continuous time cases [4-8]. The results obtain are not easily applicable in own-ship maneuver optimization and generating observable conditions for the target is much complicated. In practice, most systems take discrete form and observability properties are much easier to obtain. In [9], it is shown that for two-dimensional case, if target travels on a straight line at constant velocity, the bearing-only target tracking system is unobservable if one of the following conditions is satisfied:

1. The number of bearing measurements is less than four.
2. Bearing tangent is constant during the entire tracking process, i.e. $\tan\beta_i = \text{Const}$.
3. Own-ship travels along a straight line at a constant speed.
4. In case of no own-ship maneuver, three bearing measurements are independent and the fourth angle can be calculated from the three measured angles, so any bearing can be deduced by the three bearings. Thus in the viewpoint of observability, three bearings can sufficiently reflect the observability information. For this reason bearing other than the three independent ones can be defined as null bearings, which leads to the following statement:
   "For a bearing-only target tracking arbitrarily pick three bearings from a known bearings sequence, the system is unobservable if all other bearings are null"

The observability problem in continuous time and three dimensional cases is studied in [4] and [5]. This paper is an extension for [9] in the discrete time form and three dimensional case.

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II. THREE DIMENSIONAL PROBLEM FORMULATION

Tracking angles ϑ and φ are considered as shown in Fig. 1 and the following relationships hold:

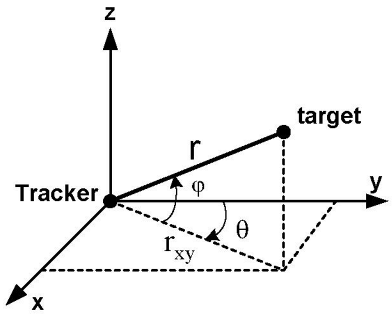

Fig. 1 Tracking angles

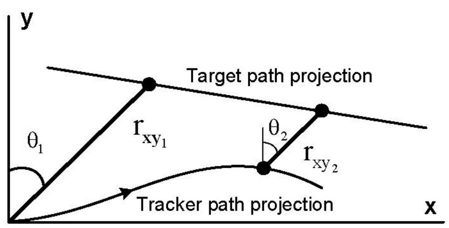

Fig. 2 Target path projection on XY plane

$$r_{xy} = r\cos\varphi = \sqrt{x^2(t) + y^2(t)} \tag{1}$$

$$r = \sqrt{x^2(t) + y^2(t) + z^2(t)} \tag{2}$$

$$\vartheta(t) = \tan^{-1}\frac{x(t)}{y(t)} \tag{3}$$

$$\varphi(t) = \tan^{-1}\frac{z(t)}{\sqrt{x^2(t)+y^2(t)}} \tag{4}$$

Fig. 2 represents in XY plane the own-ship and the target path projection. At the instant of the first angle measurement, own-ship is considered in the origin. Target movement has a constant velocity and a straight path. The target position component at instant i is $[x_T(i) \quad y_T(i) \quad z_T(i)]$ and its velocity component is $[v_{Tx} \quad v_{Ty} \quad v_{Tz}]$. Usually no constraint is imposed on own-ship motion. In addition, it is assumed that own-ship position $[x_o(i) \quad y_o(i) \quad z_o(i)]$ is known with no error. If bearing measurement is also error free, then target own-ship geometry can be described as follows:

$$z_T(0) + t_i v_{Tz} - z_o(i) - r_i \sin\varphi_i = 0 \tag{5}$$

$$x_T(0) + t_i v_{Tx} - x_o(i) - r_i \cos\varphi_i \sin\vartheta_i = 0 \tag{6}$$

$$y_T(0) + t_i v_{Ty} - y_o(i) - r_i \cos\varphi_i \cos\vartheta_i = 0 \tag{7}$$

Equations. (5-7) are the xyz projections of the target own-ship geometry, φ_i and ϑ_i are the tracking angles in ith measurement, $x_T(0)$, $y_T(0)$ and $z_T(0)$ are the initial target positions.

In practice, measurement errors are inevitable and the right hand side of the (5-7) is not zero, hence:

$$z_T(0) + t_i v_{Tz} - z_o(i) - r_i \sin\varphi_i = e_z(i) \tag{8}$$

$$x_T(0) + t_i v_{Tx} - x_o(i) - r_i \cos\varphi_i \sin\vartheta_i = 0 \tag{9}$$

$$y_T(0) + t_i v_{Ty} - y_o(i) - r_i \cos\varphi_i \cos\vartheta_i = e_y(i) \tag{10}$$

Equations. (8-10) are the basic equations representing relative motion and geometrical relationship between own-ship and target. For convenience φ_i and ϑ_i are still used to represent measured azimuth and elevation (bearings) with measurement errors. Eliminate range r_i in (8-10), use (8) and (9) yields:

$$\begin{aligned}&(\sin\varphi_i)x_T - (\cos\varphi_i \sin\vartheta_i)z_T + (t_i \sin\varphi_i)v_{Tx}\\ &- (t_i \cos\varphi_i \sin\vartheta_i)v_{Tx} = z_1(i) + e_1(i)\end{aligned} \tag{11}$$

Where:

$$z_1(i) = z_o(i)\cos\varphi_i \sin\vartheta_i - x_o(i)\sin\varphi_i \tag{12}$$

Which can be viewed as the artificial (mathematical) measurement and

$$e_1(i) = e_x(i)\sin\varphi_i - e_z(i)\cos\varphi_i \sin\vartheta_i \tag{13}$$

is the artificial measurement error. In a similar manner, use (8) and (10) yield:

$$\begin{aligned}&(\sin\varphi_i)y_T - (\cos\varphi_i \cos\vartheta_i)z_T + (t_i \sin\varphi_i)v_{Ty}\\ &- (t_i \cos\varphi_i \cos\vartheta_i)v_{Tz} = z_2(i) + e_2(i)\end{aligned} \tag{14}$$

Where

$$z_2(i) = z_o(i)\cos\varphi_i \cos\vartheta_i - y_o(i)\sin\varphi_i \tag{15}$$

and

$$e_2(i) = e_y(i)\sin\varphi_i - e_z(i)\cos\varphi_i \cos\vartheta_i \tag{16}$$

is the artificial measurement and measurement error. z(i) is defined as:

$$z(i) = \begin{bmatrix} z_o(i)\cos\varphi_i \sin\vartheta_i - x_o(i)\sin\varphi_i \\ z_o(i)\cos\varphi_i \cos\vartheta_i - y_o(i)\sin\varphi_i \end{bmatrix} \tag{17}$$

The initial state of the target:

$$X_T(0) = \begin{bmatrix} x_T(0) & y_T(0) & z_T(0) & v_{Tx} & v_{Ty} & v_{Tz} \end{bmatrix}^T \tag{18}$$

is the unknown vector to be solved. Let

$$H(i) = \begin{bmatrix} \sin\varphi_i & 0 & -\cos\varphi_i \sin\vartheta_i & t_i \sin\varphi_i & 0 & -t_i \cos\varphi_i \sin\vartheta_i \\ 0 & \sin\varphi_i & -\cos\varphi_i \cos\vartheta_i & 0 & t_i \sin\varphi_i & -t_i \cos\varphi_i \cos\vartheta_i \end{bmatrix} \tag{19}$$

be the measurement matrix. Then (11) and (14) become

$$z(i) = H(i)X_T(0) - e(i) \quad (20)$$

where e(i) is

$$e(i) = \begin{bmatrix} e_x(i)\sin\varphi_i - e_z(i)\cos\varphi_i \sin\vartheta_i \\ e_y(i)\sin\varphi_i - e_z(i)\cos\varphi_i \cos\vartheta_i \end{bmatrix} \quad (21)$$

Equation (20) is the measurement equation with respect to the i^{th} bearings measurement.

III. A LEAST SQUARE ESTIMATOR

I Because of simple implementation, least square estimator is usually used. When n azimuth and elevation angles ϑ_i and φ_i (i=1,…,n) are sampled, n equations in the form of (20) can be obtained. Combining these n equations yield the following matrix equation

$$Z = HX_T(0) - E \quad (22)$$

Where

$$H = \begin{bmatrix} H^T(1) & H^T(2) & . & . & H^T(n) \end{bmatrix}^T \quad (23)$$

$$Z = \begin{bmatrix} z^T(1) & z^T(2) & . & . & z^T(n) \end{bmatrix}^T \quad (24)$$

$$E = \begin{bmatrix} e^T(1) & e^T(2) & . & . & e^T(n) \end{bmatrix}^T \quad (25)$$

The least square solution satisfies the following equation:

$$[H^T H]X_T(0) = H^T Z \quad (26)$$

Defining $A = H^T H$ and $B = H^T Z$ then we have (27-a), (27-b) and (28). In (27-b) C_i (i=1-6) is the i[th] column of matrix A.

$$A = \sum_{i=1}^{n} \begin{bmatrix} \sin^2\varphi_i & 0 & -\sin\varphi_i\cos\varphi_i\sin\vartheta_i & t_i\sin^2\varphi_i & 0 & -t_i\sin\varphi_i\cos\varphi_i\sin\vartheta_i \\ 0 & \sin^2\varphi_i & -\sin\varphi_i\cos\varphi_i\cos\vartheta_i & 0 & t_i\sin^2\varphi_i & -t_i\sin\varphi_i\cos\varphi_i\cos\vartheta_i \\ -\sin\varphi_i\cos\varphi_i\sin\vartheta_i & -\sin\varphi_i\cos\varphi_i\cos\vartheta_i & 2\cos^2\varphi_i\sin^2\vartheta_i & -t_i\sin\varphi_i\cos\varphi_i\sin\vartheta_i & -t_i\sin\varphi_i\cos\varphi_i\cos\vartheta_i & 2t_i\cos^2\varphi_i\sin^2\vartheta_i \\ t_i\sin^2\varphi_i & 0 & -t_i\sin\varphi_i\cos\varphi_i\sin\vartheta_i & t_i^2\sin^2\varphi_i & 0 & -t_i^2\sin\varphi_i\cos\varphi_i\sin\vartheta_i \\ 0 & t_i\sin^2\varphi_i & -t_i\sin\varphi_i\cos\varphi_i\cos\vartheta_i & 0 & t_i^2\sin^2\varphi_i & -t_i^2\sin\varphi_i\cos\varphi_i\cos\vartheta_i \\ -t_i\sin\varphi_i\cos\varphi_i\sin\vartheta_i & -t_i\sin\varphi_i\cos\varphi_i\cos\vartheta_i & 2t_i\cos^2\varphi_i\sin^2\vartheta_i & -t_i^2\sin\varphi_i\cos\varphi_i\sin\vartheta_i & -t_i^2\sin\varphi_i\cos\varphi_i\cos\vartheta_i & 2t_i^2\cos^2\varphi_i\sin^2\vartheta_i \end{bmatrix} \quad (27\text{-a})$$

$$A = \sum_{i=1}^{n} \begin{bmatrix} C_1(i) & C_2(i) & C_3(i) & C_4(i) & C_5(i) & C_6(i) \end{bmatrix} \quad (27\text{-b})$$

$$B = -x_o(0)\sum_{i=1}^{n} C_1(i) - y_o(0)\sum_{i=1}^{n} C_2(i) - z_o(0)\sum_{i=1}^{n} C_3(i) \tag{28}$$

Note that A is a 6×6 real symmetric matrix, (26) becomes:

$$AX_T(0) = B \tag{29}$$

If A is nonsingular, i.e.

$$|A| \neq 0 \tag{30}$$

or equivalently rank(A) = 6 , then a unique solution can be obtained as:

$$X_T(0) = A^{-1}B \tag{31}$$

IV. OBSERVABILITY ANALYSIS

Similar to discrete time dynamic systems, that is for extracting initial system states from a finite sequence of the observed output, observability matrix should be full rank; A is the system observability matrix, and to extract initial target states, matrix A should be full rank. Because the elements of matrix A depend on the measured bearings, there is a direct relationship between the target bearings and the observability matrix. The distance of matrix A from singular condition (the difference of its determinant from zero) is a measure for better system observability.

Conventionally, the relationship between state of own-ship motion and the observability matrix is explored which is complicated. Considering target bearings, provides simple observability conditions that are easier to interpret. . This approach will be adopted in the following observability analysis of the bearings-only tracking system.

Theorem 1:
The bearings-only target tracking system is unobservable if the number of bearing measurements is less than three (In each measurements, two angles, azimuth and elevation are measured).

Proof
If less than three measurements (azimuth and elevation) are available, matrix H has less than six rows so

$$\text{Rank}(H) = \text{Rank}(H^T) < 6 \tag{32}$$

and as a result

$$\text{Rank}(A) = \text{Rank}(H^T H) < 6 \tag{33}$$

therefore A is singular and the system is unobservable. This is a deterministic parameter approach and bearings measurements are supposed to be without error. This assumption is reasonable when measurement errors are small.

Theorem 2
The bearings-only target tracking system is unobservable if for the measured tracking angles; one of the following ratios, during the entire tracking process, remains constant:

$$\frac{\tan\varphi_i}{\sin\vartheta_i} = \text{Const.} \tag{34}$$

or

$$\frac{\tan\varphi_i}{\cos\vartheta_i} = \text{Const.} \tag{35}$$

The special case is when φ and ϑ remains constant. In (34) and (35) φ_i and ϑ_i are the measured tracking angles at the i^{th} sample $(t = t_i)$. Initial bearing can always be considered zero by rotations of the coordinate system.

Proof
It can easily be shown if the relation (34) is satisfied then choosing for example i=1 and 2, then:

$$\frac{\tan\varphi_1}{\sin\vartheta_1} = \frac{\tan\varphi_2}{\sin\vartheta_2}$$

then it can be resulted in:

$$\frac{\tan\varphi_1}{\tan\varphi_2} = \frac{\sin\vartheta_1}{\sin\vartheta_2}$$

this results in:

$$\frac{\sin\varphi_1}{\sin\varphi_2} = \frac{\cos\varphi_1 \sin\vartheta_1}{\cos\varphi_2 \sin\vartheta_2}$$

this shows that the ratio of the elements in rows 1 and 3 of the matrix H is constant and this result in rank reduction of H and $A = H^T H$.
For the second case in relation (35), choosing for example i=1 and 2, then:

$$\frac{\tan\varphi_1}{\cos\varphi_1} = \frac{\tan\varphi_2}{\cos\varphi_2} \quad \text{or} \quad \frac{\tan\varphi_1}{\tan\varphi_2} = \frac{\cos\vartheta_1}{\cos\vartheta_2}$$

this results in:

$$\frac{\sin\varphi_1}{\sin\varphi_2} = \frac{\cos\varphi_1 \cos\vartheta_1}{\cos\varphi_2 \cos\vartheta_2}$$

This relation shows that the ratio of the elements in row 2 and 4 of the matrix H is constant and this results in rank reduction of H and $A = H^T H$.

Theorem 3
The bearings-only target tracking system is unobservable if own-ship travels along a straight line at a constant speed.

Proof:
Suppose that the system is observable. In this case we can show that the solution will always be the same as the initial own-ship state i.e.

$$X_T(0) = X_o(0) \tag{36}$$

Where $X_o(0)$ is the initial state of the own-ship, which is defined as:

$$\begin{aligned} X_o(0) &= [x_o(0) \quad y_o(0) \quad z_o(0) \quad v_{ox} \quad v_{oy} \quad v_{oz}] \\ &= [0 \quad 0 \quad 0 \quad v_{ox} \quad v_{oy} \quad v_{oz}] \end{aligned} \tag{37}$$

This obviously contradicts the observability supposition of the system. In the case of no own-ship maneuver, denoting the velocity components v_{ox}, v_{oy} and v_{oz} the matrix B becomes as (38)

$$\begin{aligned} B = &-x_o(0)\sum_{i=1}^{n} C_1(i) - y_o(0)\sum_{i=1}^{n} C_2(i) - z_o(0)\sum_{i=1}^{n} C_3(i) \\ &- v_{ox}\sum_{i=1}^{n} C_4(i) - v_{oy}\sum_{i=1}^{n} C_5(i) - v_{oz}\sum_{i=1}^{n} C_6(i) \end{aligned} \tag{38}$$

Note that matrix B is the linear combination of the six columns of matrix A. Suppose the system is observable, hence $|A| \neq 0$, so the first component of the unknown state vector $X_T(0)$ is

$$x_T(0) = \frac{|\overline{A}|}{|A|} \tag{39}$$

where $\overline{A}$ is the result of replacing the first column of matrix A by the components of matrix B. We can easily obtain $x_T(0) = x_o(0)$ (40)

Similarly,

$$y_T(0) = y_o(0) \tag{41}$$

$$z_T(0) = z_o(0) \tag{42}$$

$$v_{Tx} = v_{ox} \tag{43}$$

$$v_{Ty} = v_{oy} \tag{44}$$

$$v_{Tz} = v_{oz} \tag{45}$$

That is

$$X_T(0) = X_o(0) \tag{46}$$

This completes the proof .

Lemma 1:
In the case of no own-ship maneuver, only three bearings (azimuth and elevation) measurements have sufficient observability information.

Any other bearing than three measured bearings can be calculated by three measured bearings. Lemma 1 can be extended to a more general case: All observability information will be reflected by arbitrarily picking three bearings from a measured bearings sequence.

V. CONCLUSION

For three dimensional cases with constant speed target traveling on a straight line, unobservability conditions are:

1. The number of bearing measurements is less than three.
2. During the entire tracking process, one of the following ratios remains constant i.e.:

$$\frac{\tan\varphi_i}{\sin\vartheta_i} = \text{Const.} \quad \text{or} \quad \frac{\tan\varphi_i}{\cos\vartheta_i} = \text{Const.}$$

3. Tracker has no movement or moving on a straight line with constant speed.

In the case of no own-ship maneuver three measured bearings can sufficiently reflect the observability information.

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