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Stability of Discrete-Time Matrix Polynomials

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Stability of Discrete-Time Matrix Polynomials

Kanh T. Ngo and Kelvin T. Erickson

***Abstract*—This paper derives conditions for the stability of discrete-time systems that can be modeled by a vector difference equation, where the variables are $m \times 1$ vectors and the coefficients are $m \times m$ matrices. Stability of the system is related to the locations of the roots of the determinant of a real $m \times m$ matrix polynomial of nth order. In this case, sufficient conditions for the system to be stable are derived. The conditions are imposed on the ∞-norm of two matrices constructed from the coefficient matrices and do not require the computation of the determinant polynomial. The conditions are the extensions of one of the Jury sufficient conditions for a scalar polynomial. An example is used to illustrate the application of the sufficient conditions.**

***Index Terms*—Matrix polynomials, model predictive control, stability analysis.**

I. Introduction

A linear discrete-time system can be modeled by a vector difference equation

$$\begin{aligned} A_n y(k) + A_{n-1} y(k-1) + \\ \cdots + A_1 y(k-n+1) + A_0 y(k-n) \\ = B_1 u(k-1) + \cdots + B_q u(k-q) \end{aligned} \tag{1}$$

where $y(k)$'s are $m \times 1$ vectors that represent the outputs of the systems, $u(k)$'s are $r \times 1$ vectors that represent the inputs of the system, A_i's are $m \times m$ matrices, and B_i's are $m \times r$ matrices. The right side of (1) is a linear combination of the values of past inputs, from q samples before the current time up to the last sample before the current time. These values do not affect the stability of the system. Stability of the system is related only to the left side of the equation. Therefore, for stability analysis, we consider a simpler form of (1), namely

$$\begin{aligned} A_n y(k) + A_{n-1} y(k-1) + \\ \cdots + A_1 y(k-n+1) + A_0 y(k-n) \\ = f(k) \end{aligned} \tag{2}$$

where now $f(k)$ represents the right side of (1).

II. Stability Analysis

As discussed in the previous section, the system is modeled by the vector difference equation (2). We will discuss stability for three different forms of the leading coefficient (square) matrix A_n. Section II-A will present conditions for stability when A_n is the identity matrix. Next, stability results are derived when A_n is a general nonsingular matrix in Section II-B. Section II-C will discuss stability when A_n is singular. An illustrative example is presented in Section III.

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A. The Leading Coefficient is the Identity Matrix

When A_n is the identity matrix, stability is related to the number of roots inside the unit circle of the characteristic polynomial

$$\begin{aligned} g(z) = \det[I_m z^n + A_{n-1} z^{n-1} + \cdots \\ + A_2 z^2 + A_1 z + A_0] \end{aligned}$$

which is the determinant of a matrix polynomial.

For real scalar polynomials, one common approach is to use the necessary and sufficient conditions of Jury [1] on the coefficients of the polynomial. Other conditions are derived by Kalman [2], [3] and Parks [4], among others. These conditions are expressed by the positiveness of the principal minors of a matrix or, equivalently, by the positive definiteness of a related matrix. In either case, the matrix elements are rational functions of the polynomial coefficients. Therefore, these conditions are often not suitable for control system synthesis. In practice, some other sufficient conditions—although more conservative—are used to design controllers.

When the characteristic polynomial is the determinant of a matrix polynomial, two difficulties arise in applying the necessary and sufficient conditions for scalar polynomials. First, the closed forms of the coefficients of the characteristic polynomial as functions of the coefficient matrices are still unknown. Second, even if those functions can be derived, they are definitely very complicated. As a result, since the effects of the coefficient matrices on the coefficients of the characteristic polynomial are unknown, the synthesis problem is almost impossible. Ahn [5] and Hmamed [6] formulated a sufficient condition for the stability of discrete matrix polynomials. However, these methods involve checking the positive definiteness of a large matrix and hence are still not suitable for control system synthesis.

This section presents a set of sufficient conditions applied directly to the coefficient matrices. These conditions reduce to one of the Jury sufficient conditions when the matrices are scalars. Thus, the result is an extension of one of the Jury sufficient conditions to matrix polynomials. Furthermore, the conditions are not more conservative than those in the case of scalar polynomials.

Theorem 1: For the polynomial

$$f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with complex coefficients, all the roots will lie strictly inside the unit circle if the following inequality is satisfied:

$$1 > |a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|.$$

Note that this sufficient condition is one of the sufficient conditions of Jury [1, p. 116], but the following proof will be based on the matrix norm since that approach can be extended to matrix polynomials. (Also see Marden [7, pp. 140, 141].)

Proof: The roots of the polynomial $f(z)$ are the eigenvalues of the companion matrix A below

$$A = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}.$$

The matrix has 1 on the elements just above the diagonal, the coefficients of the polynomial on the last row are as shown, and 0 everywhere else. From the assumption of the theorem we have

$$1 > |a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|.$$

Since we have strict inequality, there exists $\alpha > 1$ so that

$$1 > \alpha(|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|)$$

or

$$1 > \alpha|a_{n-1}| + \alpha|a_{n-2}| + \cdots + \alpha|a_1| + \alpha|a_0|.$$

Now let d be the positive number satisfying $d^{n-1} = \alpha$, then

$$1 < d < d^2 < \cdots < d^{n-1} = \alpha.$$

Thus

$$1 > \alpha|a_{n-1}| + \alpha|a_{n-2}| + \cdots + \alpha|a_1| + \alpha|a_0|$$

or

$$1 > |a_{n-1}| + d|a_{n-2}| + \cdots + d^{n-2}|a_1| + d^{n-1}|a_0|.$$

Now define the matrix D to be

$$D = \mathrm{diag}\{1, d, d^2, \cdots, d^{n-1}\}$$

then

$$D^{-1} = \mathrm{diag}\{1, d^{-1}, d^{-2}, \cdots, d^{1-n}\}.$$

If we multiply D on the left and D^{-1} to the right of A, we have

$$DAD^{-1} = \begin{bmatrix} 0 & d^{-1} & & \\ & & \ddots & \\ & & & d^{-1} \\ -a_0 d^{n-1} & -a_1 d^{n-2} & \cdots & -a_{n-1} \end{bmatrix}.$$

The matrices $K = DAD^{-1}$ and A have the same eigenvalues. Recall that for a given matrix, its spectral radius, i.e., the maximum of the absolute values of its eigenvalues, is always less than or equal to any induced-matrix norm. For the companion matrix A above, it is clear that its induced-∞ norm is equal to

$$\|A\|_\infty = \max(1, |a_{n-1}| + |a_{n-2}| + \cdots + |a_2| + |a_1| + |a_0|)$$

and for the matrix $K = DAD^{-1}$ we have

$$\|K\|_\infty = \max(d^{-1}, |a_{n-1}| + d|a_{n-2}| + \cdots + d^{n-2}|a_1| + d^{n-1}|a_0|).$$

Thus, if the assumption of Theorem 1 is satisfied, there is a value of $d > 1$ so that the ∞-norm of K is strictly less than one, hence its spectral radius is strictly less than one, and all eigenvalues of K (and A) will lie strictly inside the unit circle. □

Now the results for scalar polynomials can be extended to matrix polynomials. First of all, we transform the problem of finding the roots of the determinant of a matrix polynomial into the problem of finding the eigenvalues of a related matrix so that we can apply the same approach as in Theorem 1.

Let $A(z)$ be the matrix polynomial

$$A(z) = I_m z^n + A_{n-1} z^{n-1} + \cdots + A_2 z^2 + A_1 z + A_0 \tag{3}$$

where the A_i's are $m \times m$ matrices, then the roots of the determinant of $A(z)$ are the eigenvalues of the following "block companion" matrix [8]:

$$\mathcal{A} = \begin{bmatrix} 0 & I_m & & \\ & & I_m & \\ & & & \ddots & \\ & & & & I_m \\ -A_0 & & \cdots & & -A_{n-1} \end{bmatrix}. \tag{4}$$

Note that the dimension of $\mathcal{A}$ is $nm \times nm$, and its structure is similar to that of the matrix A in the case of scalar polynomials: the elements 1 are replaced by the $m \times m$ identity matrices, and the scalar coefficients are replaced by the matrices A_i's.

With (3) and (4), we can apply the matrix-norm approach in Theorem 1 to find sufficient condition(s) for all eigenvalues of $\mathcal{A}$ to lie strictly inside the unit circle. To facilitate the result, we define the matrix B as follows:

$$B = [A_0 \quad A_1 \quad \cdots \quad A_{n-1}]$$

then it is easy to see that

$$\|\mathcal{A}\|_\infty = \max(1, \|B\|_\infty).$$

Now suppose $\|B\|_\infty < 1$, then as in the arguments of Theorem 1, there is $\alpha > 1$ so that $\alpha\|B\|_\infty$ is still less than one. Let $d > 1$ satisfying $d^{n-1} = \alpha$, and let $\mathcal{D}$ be the matrix

$$\mathcal{D} = \text{block } \{I_m, dI_m, d^2 I_m, \cdots d^{n-1} I_m\}$$

then the matrix $\mathcal{K} = \mathcal{D}\mathcal{A}\mathcal{D}^{-1}$ has the form

$$\mathcal{K} = \mathcal{D}\mathcal{A}\mathcal{D}^{-1} = \begin{bmatrix} 0 & d^{-1}I_m & & \\ & & d^{-1}I_m & \\ & & & \ddots & \\ & & & & d^{-1}I_m \\ -d^{n-1}A_0 & -d^{n-2}A_1 & \cdots & & -A_{n-1} \end{bmatrix}.$$

From the form of $\mathcal{K}$, it can be seen that

$$\|\mathcal{K}\|_\infty \le \max(d^{-1}, \alpha\|B\|_\infty) < 1.$$

Thus, we have just proved the following theorem.

Theorem 2: Let $A(z)$ be the matrix polynomial

$$A(z) = I_m z^n + A_{n-1} z^{n-1} + \cdots + A_2 z^2 + A_1 z + A_0 \tag{5}$$

and define the $m \times nm$ matrix B as

$$B = [A_0 \quad A_1 \quad \cdots \quad A_{n-1}].$$

If $\|B\|_\infty < 1$, then all the roots of the determinant of $A(z)$ lie strictly inside the unit circle. Note that this condition is much simpler than the conditions of Ahn [5] or Hmamed [6], which require checking the positive definiteness of a $mn \times mn$ matrix.

Corollary 1: Under the assumptions of Theorem 2, if the sum of the ∞-norms of the coefficient matrices A_i's is less than one, then all the roots of the determinant of $A(z)$ lie strictly inside the unit circle.

Proof: Let v_j^T be the jth row of the matrix B. Also let w_{ij}^T be the jth row of the matrix A_i, for $0 \le i \le n-1$. Then v_j^T can be expressed as

$$v_j^T = [w_{0j}^T \quad w_{1j}^T \quad \cdots \quad w_{n-1,j}^T].$$

Since the 1-norm of a vector is the sum of the absolute values of its elements, it follows that

$$\|v_j^T\|_1 = \|w_{0j}^T\|_1 + \|w_{1j}^T\|_1 + \cdots + \|w_{n-1,j}^T\|_1 \le \|A_0\|_\infty + \|A_1\|_\infty + \cdots + \|A_{n-1}\|_\infty$$

and

$$\|B\|_\infty = \max_{1 \le j \le m} \|v_j^T\|_1 \le \|A_0\|_\infty + \|A_1\|_\infty + \cdots + \|A_{n-1}\|_\infty.$$

The result then follows from Theorem 2. □

The condition of Corollary 1 is clearly more conservative than that of Theorem 2. This corollary is stated because it is reduced to one of Jury's conditions [1, p. 116] when the A_i's are scalars. Hence, we have, in fact, an extension of a sufficient condition from the scalar polynomials to the matrix polynomials.

It is reasonable to expect some results in terms of the ∞-norm of some related matrix that involves the transposes of the A_i's. To do that, taking the transpose of $A(z)$ in (5), we have

$$[A(z)]^T = I_m z^n + A_{n-1}^T z^{n-1} \cdots + A_2^T z^2 + A_1^T z + A_0^T.$$

Since the determinants of $A(z)$ and its transpose are the same, the roots of det $A(z)$ are the eigenvalues of the following matrix:

$$\mathcal{A}_T = \begin{bmatrix} 0 & I_m & & & \\ & & I_m & & \\ & & & \ddots & \\ & & & & I_m \\ -A_0^T & & \cdots & & -A_{n-1}^T \end{bmatrix}$$

where $\mathcal{A}_T$ is $\mathcal{A}$ with the A_i's replaced by its transposes. Hence, we have the following result.

Theorem 3: Let $A(z)$ be the matrix polynomial

$$A(z) = I_m z^n + A_{n-1} z^{n-1} + \cdots + A_1 z + A_0$$

and define the $m \times nm$ matrix B_T as

$$B_T = [A_0^T \quad A_1^T \quad \cdots \quad A_{n-1}^T].$$

If $\|B_T\|_\infty < 1$, then all the roots of the determinant of $A(z)$ lie strictly inside the unit circle.

Theorem 3 gives another sufficient condition. In Theorem 2, the quantities to be small are related to the elements of the rows of the matrices A_i's. Now Theorem 3 gives a criterion related to the elements of the columns of the matrices A_i's. Furthermore, as an analogous result to Corollary 1, we have the following corollary.

Corollary 2: Under the assumptions of Theorem 3, if the sum of the ∞-norms of the coefficient matrices A_i^T's is less than one, or equivalently if the sum of the 1-norms of the coefficient matrices A_i's is less than one, then all the roots of the determinant of $A(z)$ lie strictly inside the unit circle.

Proof: The proof is identical to the proof of Corollary 1, with B_T in place of B, and A_i^T in place of A_i for $0 \le i \le n-1$. □

Theorems 2 and 3 and their corollaries give extensions from the scalar polynomials to the matrix polynomials in two directions. Roughly speaking, the determinant of a matrix polynomial will have all the roots inside the unit circle if either the rows or the columns of the coefficient matrices are dominated by one.

B. The Leading Coefficient Is A_n—A Nonsingular Matrix

Now we consider the case when A_n in (2) is a general nonsingular matrix. Since A_n is nonsingular, A_n^{-1} exists, and from (2) we have

$$y(k) + A_n^{-1} A_{n-1} y(k-1) + \cdots + A_n^{-1} A_0 y(k-n) = A_n^{-1} f(k).$$

The new system has the leading coefficient matrix equal to the identity matrix so we can apply the results in Section II-A to derive the conditions for stability.

Theorem 4: Define the matrix polynomial $A(z)$ as

$$A(z) = I_m z^n + A_n^{-1} A_{n-1} z^{n-1} + \cdots + A_n^{-1} A_0$$

and the matrix $\mathcal{A}$ as

$$\mathcal{A} = \begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ -A_n^{-1} A_0 & \cdots & \cdots & -A_n^{-1} A_{n-1} \end{bmatrix}$$

then:

1) system (2) is asymptotically stable if and only if all the roots of det $A(z)$ lie strictly inside the unit circle;
2) alternatively, system (2) is asymptotically stable if and only if $\rho(\mathcal{A}) < 1$, where $\rho(\mathcal{A})$ is the spectral radius of $\mathcal{A}$;
3) system (2) is asymptotically stable if $\|B\|_\infty < 1/\|A_n^{-1}\|_\infty$ or if $\|B_T\|_\infty < 1/\|A_n^{-1}\|_1$, where

$$B = [A_0 \quad A_1 \quad \cdots \quad A_{n-1}]$$
$$B_T = [A_0^T \quad A_1^T \quad \cdots \quad A_{n-1}^T].$$

Proof: The result in Part 1) is standard. Stability can be verified without having to find the roots of det $A(z)$ (see Ngo [9]). Part 2) follows from Part 1) and (3) and (4). We prove only Part 3).

We have $\|\mathcal{A}\|_\infty = \max\ (1, \|A_n^{-1} B\|_\infty)$. Using a scaling technique as in the proof of Theorem 2, it follows that $\|A_n^{-1}B\|_\infty < 1$ will imply $\rho(\mathcal{A}) < 1$. Now the result follows since:

$$\|B\|_\infty < \frac{1}{\|A_n^{-1}\|_\infty} \text{ implies } \|A_n^{-1}\|_\infty \|B\|_\infty < 1$$

and

$$\|A_n^{-1}B\|_\infty \le \|A_n^{-1}\|_\infty \|B\|_\infty < 1.$$

Furthermore, since det $A(z) = \det[A(z)^T]$, the A_i's can be replaced by their transposes, and an alternative condition is

$$\|B_T\|_\infty < \frac{1}{\|(A_n^T)^{-1}\|_\infty} = \frac{1}{\|A_n^{-1}\|_1}$$

where $B_T = [A_0^T \ A_1^T \ \cdots \ A_{n-1}^T]$, and the proof is complete. □

Note: Since $X^{-1} = \mathrm{Adj}(X)/\det\ X$, we can replace the conditions in Part 3) by

$$\|B\|_\infty < \frac{\det\ A_n}{\|\mathrm{Adj}(A_n)\|_\infty}$$

or

$$\|B_T\|_\infty < \frac{\det\ A_n}{\|\mathrm{Adj}(A_n)\|_1}$$

which may be numerically better.

C. A_n Is Singular

For simplicity, we will assume that A_0 is nonsingular. Multiplying both sides of (2) by A_0^{-1} we have

$$A_0^{-1} A_n y(k) + \cdots + A_0^{-1} A_1 y(k-n+1) + y(k-n) = A_0^{-1} f(k).$$

Defining new variables

$$z_1(k) = y(k)$$
$$z_2(k) = y(k-1)$$
$$\vdots$$
$$z_n(k) = y(k-n+1)$$

we have

$$\mathcal{A} z(k) = z(k-1) - F(k) \tag{6}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ -A_0^{-1}A_n & \cdots & \cdots & -A_0^{-1}A_1 \end{bmatrix}$$

$$z(k-1) = \begin{bmatrix} z_1(k-1) \\ z_2(k-1) \\ \cdots \\ z_n(k-1) \end{bmatrix}, \quad z(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \\ \cdots \\ z_n(k) \end{bmatrix}$$

$$F(k) = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ A_0^{-1} f(k) \end{bmatrix}.$$

Note that since A_n is singular, so are $A_0^{-1}A_n$ and $\mathcal{A}$. There are conditions on $F(k)$ and the initial condition $z(0)$ so that (6) has a solution. The details can be found in Campbell [10, pp. 181–183]. For stability analysis, we only need to consider the homogeneous equation of (6)

$$\mathcal{A}z(k) = z(k-1) \tag{7}$$

and assume that the initial value $z(0)$ belongs to some consistent set so that (7) has a solution. Let l (a positive integer) be the index of $\mathcal{A}$ (see the Appendix). Since $\mathcal{A}$ is singular, $l > 0$, and we have

$$\begin{aligned} z(k-1) &= \mathcal{A}z(k) = \mathcal{A}^2 z(k+1) = \cdots \\ &= \mathcal{A}^l z(k+l-1) \end{aligned} \tag{8}$$

and

$$\begin{aligned} z(k) &= \mathcal{A}z(k+1) = \mathcal{A}^2 z(k+2) = \cdots \\ &= \mathcal{A}^l z(k+l). \end{aligned} \tag{9}$$

Now multiplying $\mathcal{A}^D$ on the left of (8) yields

$$\begin{aligned} \mathcal{A}^D z(k-1) &= \mathcal{A}^D \mathcal{A} z(k) = \cdots \\ &= \mathcal{A}^D \mathcal{A}^l z(k+l-1) \\ &= \mathcal{A}^D \mathcal{A}^{l+1} z(k+l). \end{aligned}$$

Then using (14) and (15) we have

$$\begin{aligned} \mathcal{A}^D z(k-1) &= \mathcal{A}^D \mathcal{A}^{l+1} z(k+l) \\ &= \mathcal{A}^l z(k+l). \end{aligned} \tag{10}$$

From (9) and (10) it follows that

$$z(k) = \mathcal{A}^D z(k-1). \tag{11}$$

Furthermore, it can be shown that (11) is equivalent to (7) (Campbell [10, p. 182]). But from (11), it is easy to see that the system is asymptotically stable if and only if $\rho(\mathcal{A}^D) < 1$, where ρ denotes the spectral radius.

Since A_n, hence $\mathcal{A}$, is singular, we cannot expect sufficient conditions in terms of the matrices A_i's, since they would be so conservative that they would be useless [see Theorem 4, Part 3)]. Instead, we try to use another technique to check the condition $\rho(\mathcal{A}^D) < 1$ *without having to calculate* $\mathcal{A}^D$. Thus we have the following theorem.

Theorem 5: When A_n is an $m \times m$ singular matrix but A_0 is nonsingular, then the system is asymptotically stable if and only if the nonzero eigenvalues of the matrix $\mathcal{A}$ in (6) are strictly outside the unit circle.

Proof: There is a nonsingular matrix P so that

$$\mathcal{A} = P \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1} \tag{12}$$

where C is nonsingular and N is a nilpotent of index equal to the index of $\mathcal{A}$. It follows that all eigenvalues of N are zero. Thus in the canonical form of $\mathcal{A}$ in (12), C represents the nonzero eigenvalues of $\mathcal{A}$, and N represents the zero eigenvalues of $\mathcal{A}$. From (12) we have

$$\mathcal{A}^D = P \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

Therefore, $\rho(\mathcal{A}^D) < 1$ if and only if all eigenvalues of C^{-1} are strictly inside the unit circle, if and only if all eigenvalues of C are strictly outside the unit circle, and the proof is complete. □

It is also interesting to note that the eigenvalues of the matrix $\mathcal{A}$ are the roots of the determinant of the following matrix polynomial:

$$\mathcal{A}(z) = I_m z^n + A_0^{-1} A_1 z^{n-1} + \cdots + A_0^{-1} A_n$$

which are also the roots of the determinant of

$$\mathcal{A}_1(z) = A_0 z^n + A_1 z^{n-1} + \cdots + A_n.$$

Let $g(z) = \det\ \mathcal{A}_1(z)$. Then $g(z)$ has nm roots, q of which are zero. Using contour integration (see Ngo [9]), we can check stability as follows.

1) Find the rank of A_n, and let $q = m - \operatorname{rank}(A_n)$. Then q is the number of zero eigenvalues of A_n. Since $A_0^{-1}A_n$ has the same rank as A_n, and $\mathcal{A}$ has the same rank deficiency as $A_0^{-1}A_n, q$ is also the number of zero eigenvalues of $\mathcal{A}$.
2) Find $g(z) = \det\ \mathcal{A}_1(z)$.
3) Find $\int_{|z|=1} g'(z)/g(z)\, dz$, if the value of the integral is equal to q the system is asymptotically stable; if the value of the integral is greater than q, the system is unstable (see Ngo [9]).

The results above are based on the assumption that A_0 is nonsingular. Stability results could also be derived when both A_n and A_0 are singular (see the solution of (2) in Campbell [10] when both A_0 and A_n are singular), but they would be very complicated. In applications, it is usually not too restrictive to assume that at least A_n or A_0 is nonsingular.

III. Illustrative Example

This example is taken from Erickson and Otto [11] and is the problem that motivated the development of these theorems. The process, a distillation column from [12], is a first-order, two-input, two-output system. The transfer function of the process is

$$G(s) = \begin{bmatrix} \dfrac{2.56e^{-s}}{16.7s+1} & \dfrac{-5.67e^{-3s}}{21s+1} \\ \dfrac{1.32e^{-7s}}{10.9s+1} & \dfrac{-5.82e^{-3s}}{14.4s+1} \end{bmatrix}$$

where the time is in minutes. Using model predictive control, the stability of the controller is related to the stability of the matrix polynomial

$$A(z) = A_p z^{N-p} + H_{p+1} z^{N-p-1} + \cdots + H_N$$

where H_k is the 2×2 matrix of the kth impulse response coefficient, $A_p = \Sigma_{i=1}^{p} H_i, p$ is the control horizon, and N is the truncation order of the impulse response. The sampling time is 1 min, and N is chosen equal to 60. When the necessary and sufficient conditions of Jury [1] are applied to $f(z) = \det A(z)$ to determine stability, the minimum control horizon for the first output is $p_1 = 2$, the minimum control horizon for the second output is $p_2 = 4$. When the techniques in Theorems 2 and 3 are used, the minimum values of p_1 and p_2 are 15 and 16, respectively.

This example shows that the tests using Theorems 2 and 3 are simple but conservative, as expected. The theorems give simple tests to apply when both the order of the system and the dimension of the coefficient matrices are large, so that finding the eigenvalues of the matrix $\mathcal{A}$ is impractical.

IV. Conclusion

Results of Jury's sufficient conditions [1] for a scalar polynomial to have its roots inside the unit circle are extended to the determinant of a square matrix polynomial. The conditions on the scalar coefficients in the case of scalar polynomials can be replaced by either the ∞-norm or the 1-norm of the matrix coefficients in case of matrix polynomials. In fact, the conditions are imposed upon the ∞-norm of two matrices constructed from the coefficient matrices, which are definitely less conservative than the conditions on the sum of the norms of the coefficient matrices. Also, because these conditions are imposed directly on the matrix coefficients, they can be used for control system synthesis.

When the leading coefficient matrix is nonsingular, simple sufficient conditions involve the coefficient matrices A_i's and the inverse of the leading coefficient matrix. When the leading coefficient matrix A_n is singular, we no longer have simple sufficient conditions. By assuming the matrix A_0 is nonsingular we can check stability of a new system which is like a "generalized inverse" of the original system. The transformation to the new system involves the Drazin inverse, but we have stability results applied directly to the original system without having to calculate the Drazin inverse.

Appendix
The Drazin Inverse

The Drazin inverse [10] is defined only for square matrices. Given A an $n \times n$ matrix, the Drazin inverse of A, denoted by A^D, is defined to be the unique $n \times n$ matrix satisfying the following conditions:

$$A^D A A^D = A^D \tag{13}$$

$$A A^D = A^D A \tag{14}$$

$$A^{k+1} A^D = A^k \tag{15}$$

where k is the index of A, the smallest positive integer so that $\operatorname{rank}(A^{k+1}) = \operatorname{rank}(A^k)$, and hence $\operatorname{range}(A^{k+1}) = \operatorname{range}(A^k)$. Furthermore, (15) also holds for any integer $l > k$.

There are many ways to calculate A^D from A. For our purpose, it is enough to present the canonical form of A^D. Given A with $\operatorname{index}(A) = k > 0$, there is a nonsingular matrix P so that

$$A = P \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1}$$

where C is nonsingular and $N^k = 0$ (nilpotent of index k). Then

$$A^D = P \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

When A is square, the Drazin inverse and the Moore–Penrose inverse [8], denoted by A^+, do not necessarily coincide. A^+ is equal to A^D if and only if $AA^+ = A^+A$. In that case, it can be shown that the index of A is less than or equal to one.

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Noninteracting Control of Descriptor Systems Involving Disturbances

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***Abstract*— For m-input p-output descriptor systems involving disturbances, it is proven that if the problem of disturbance rejection is solvable via static-state feedback and the input–output transfer function matrix is right invertible, there always exists a static-state feedback control law yielding, simultaneous to disturbance rejection, a triangular input–output relation. The structural properties of the closed-loop system (stability, pole assignment, etc.) are extensively studied.**

***Index Terms*— Decoupling of systems, descriptor systems, singular systems, state feedback.**

I. Introduction

Consider the linear, time-invariant, descriptor system

$$\tilde{E}\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t) + \tilde{D}z(t), \quad y(t) = \tilde{C}x(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $z(t) \in \mathbb{R}^\zeta$ is the vector of unmeasurable disturbances, and $y(t) \in \mathbb{R}^p$ is the performance output vector. The system is assumed to be regular, i.e., $\det\{s\tilde{E} - \tilde{A}\} \not\equiv 0$ (or more widely regularizable). To system (1), apply the static-state feedback law

$$u(t) = \tilde{F}x(t) + \tilde{G}\omega(t), \quad \omega(t) \in \mathbb{R}^p. \tag{2}$$

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