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Elementary School Students Quantitative Reasoning: Processing Whole Numbers and Proportions

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Abstract

Elementary school-aged children have great difficulty reasoning proportionally and struggle with fractions and decimals, theoretically because proportions do not abide by the same principles as more familiar whole number quantities. The present study examines individual differences in proportional reasoning and whole number representations and tests a prediction for a non-linearity in the development of relations between the two. Pre-kindergarten through fifth-grade students completed a battery of computerized tasks, including a proportional reasoning task, “which is more?” and “which is #?” whole number comparison tasks, and symbolic and non-symbolic numerical line-estimation tasks. The results indicate that though younger children’s performance on each of the whole number comparison and number line estimation tasks were significantly positively correlated, performance on each was negatively correlated with performance on the proportional reasoning task. By contrast, older children’s performance on the proportional, whole number comparison, and number line estimation tasks were all positively correlated. These findings support the proposal that better counting abilities early in development interfere with early proportional reasoning capacities, though the two are positively related later in development.

Elementary School Students Quantitative Reasoning:

Processing Whole Numbers and Proportions

An understanding of proportionality and fractions is essential for acquiring the mathematic concepts crucial to progress in the science, technology, engineering, and mathematics (STEM) disciplines (Ball, 1993; Carpenter et al., 1993; Lesh et al., 2003; Pitkethly & Hunting, 1996; Sophian, 2007). Current cognitive developmental psychological research and theory suggest that earlier mathematical aptitudes, particularly numerical estimation, predict understanding proportionality and fraction operations in the middle school years (Jordan et al., 2013), and understanding of proportionality and fractions in the middle school years predicts success in advanced mathematics in secondary school (Siegler et al., 2011; 2012; Siegler & Lortie-Forgues, 2014; Siegler & Pyke, 2013). The relations between proportional and whole number quantitative skills earlier in development, however, are less clear.

Whereas a long line of studies demonstrated notable advances in proportional and probabilistic reasoning between 5- and 12-years of age (Brainerd, 1981; Chapman, 1975; Davies, 1965; Haseman, 1981; Noelting, 1980; Piaget & Inhelder, 1951/1975), more recent studies have found that children are able to solve proportional reasoning problems at younger ages if special accommodations are made in how problems are presented to children (Empson, 1999; Falk et al., 1980; Falk & Wilkening, 1998; Goswami, 1989; Spinillo & Bryant, 1991; Sophian, 2000). As one specific example, children are able to demonstrate some intuition for proportionality if visual-perceptual stimuli represented with continuous quantities are used, though they remain challenged by stimuli that are represented with discrete, countable quantities (Boyer et al., 2008; Boyer & Levine, 2011; 2015; Jeong et al., 2007). This suggests that discrete, countable, whole number quantities potentially interfere with proportional reasoning in early childhood, which is consistent with the assertion that proportions and fractions challenge children specifically because they do not adhere to the principles that govern whole number operations (e.g., 3/4 > 4/6, though neither 3 > 4, nor 4 > 6; Siegler et al., 2012; Van Dooren, Lehtinen, & Verschaffel, 2015), and oftentimes engage in a natural number bias that adversely affects rational number task performance (Van Hoof, Verschaffel, & Van Dooren, 2015). Analyses of the development of non-symbolic proportionality and explicit whole number understanding in the elementary school years, however, have remained largely independent, and it is presently unknown how individual differences in each area of quantitative reasoning are related.

The goal of the present study is to address this gap in the knowledge base and advance our understanding of the development of basic mental processes that have direct relevance for STEM education, by characterizing the relations among children's understanding of proportionality, whole number comparison, and numerical estimation. The primary hypothesis of the study, generated from the literature reviewed above, is that proportional reasoning and explicit whole number skills will be non-linearly developmentally related. We predict that early elementary school students who are skilled in solving whole number quantity problems will have particular difficulty with proportional reasoning problems (i.e., a negative correlation between whole number and proportional reasoning performance), but that more skilled children in the later elementary-school years will understand when to apply whole number and proportional operations and be more likely to demonstrate mastery of both (i.e., a positive correlation between whole number and proportional reasoning performance).

Method

Participants

Participants were 161 students (84 girls, 77 boys) recruited from pre-K through fifth-grade classrooms in three public elementary schools and one private elementary school in a rural school district in the Southeastern United States. An additional 15 children were tested, but were not included in the analyses due to a failure to complete one or more of the included tasks (N = 4, 2.2%) or due to producing outlier data in one or more of the tasks (i.e., performance ± 3.0 *SD* the age-group mean, N = 11, 6.3%). The remaining participants were divided into two groups via a median split on age: younger group, M = 6.5-years, *SD* = .84-years, *Range* = 4.8- to 7.8-years; older group, M = 9.6-years, *SD* = 1.2-years, *Range* = 7.9- to 12.1-years. The study was approved by the Georgia Southern University Institutional Review Board and the Bulloch County School District, and all students had written parental consent to participate.

Procedure

We tested participants during regular or after school hours, in familiar rooms adjacent to their classrooms, on a Dell Latitude laptop computer with a 15.6” HD LED screen. Participants completed a series of five experimental tasks in a randomized order.

The proportional reasoning task, adapted from Boyer and colleagues (2008), was presented with the cover story of a teddy bear character who mixes juice and water quantities to produce a target concentration juice mix. Participants were asked to choose which of two choice alternatives is proportionally equivalent to the target mix. All trials were presented with a discrete target proportion and discrete choice alternatives (i.e., demarcating lines on the stimuli could be used to count units and mathematically solve each problem). For example, in one trial, the target displayed nine total units, three red, six light blue (i.e., a 3/9 proportion), and participants chose between a correct proportional match that displayed three total units, one red, two light blue (i.e., a 1/3 proportional match), and a foil, which displayed four total units, three red, one blue (i.e., a 3/4 mismatch, but a whole number match for the target's red units; see Figure 1A for an example screenshot). Participants completed 16 trials of the proportional equivalence judgment task.

The explicit whole number "which is more?" and "which is #?" comparison tasks, adapted from Halberda and Feigenson (2008), presented children with two different colored arrays of dots, and they were asked to select the array that contained more dots (see Figure 1B) or the array that displayed a stated number of dots (see Figure 1C). The intent of the tasks was to assess explicit whole number operations, and, therefore, timing was open-ended and children were not prevented from counting the units, though they also were not prompted to do so. Participants completed 20 trials, four trials each with 3:1, 2:1, 1.33:1, 1.2:1, and 1.14:1 inter-array numerosity ratios, of both tasks. Dot array stimuli in each task were created with a customized computer algorithm that randomized the size and placement of each of the dots (i.e., effectively randomizing cumulative contour, total surface area, and stimulus density), and randomly presented each set as red, orange, green, or blue.

The symbolic and non-symbolic numerical estimation tasks, adapted from Booth and Siegler (2006), presented children with either an array of dots (see Figure 1D) or an Arabic numeral (see Figure 1E) and a number line with "0" marked at the far left and "100" at the far right, and they were asked to place a mark on the line to register their estimation of the number of dots or numeral. Participants completed 24 trials of each task. The dot arrays in the non-symbolic task were created with a customized computer algorithm that randomized the size and placement of each dot, and randomly presented each array as red, orange, green, and blue, and the number stimuli in the symbolic task were presented in bold, 72 point, courier new font presented in the center of the computer screen.

Results

Performance scores were derived for each of the tasks, calculated as the proportion of correct responses on the proportional equivalence task, as an inverse efficiency score (RT/proportion correct) on the “which is more?” and “which is #?” tasks, and as each participant’s mean absolute deviation from the target response on the non-symbolic and symbolic number line estimation tasks subtracted from 1 (i.e., so that higher values would indicate higher accuracy), all of which were normalized within the younger and older age groups with z-score transformation.

Younger students showed statistically significant positive correlations between most of the whole number quantity tasks (i.e., “which is more?”, “which is #?”, non-symbolic, and symbolic number line estimation); however, they exhibited negative correlations between the proportional reasoning task and each of the whole number quantity tasks. Of particular note, there was a marginally significant negative correlation between proportional reasoning and “which is #?” and non-symbolic number line estimation, and a significant negative correlation between proportional reasoning and symbolic number line estimation (see Table 1). These correlations imply that there is general coherence in younger students processing of the variety of whole number quantity tasks, but that those who were better at counting and estimating whole number values did less well on the proportional reasoning task.

Older students, like younger students, generally showed positive relations between the various whole number quantity tasks, but, in contrast with the younger students, showed consistent positive correlations between proportional reasoning and the whole number quantity tasks. Specifically, there were significant positive correlations between proportional reasoning and “which is #?” and both of the number line estimation tasks (see Table 1). These results imply that older students who did better on whole number tasks have made compensatory gains in their proportional reasoning and their ability to identify when proportional versus whole number operational strategies are appropriate.

Discussion

Early emerging mathematical skills, such as counting and numerically comparing sets of items, are wonderfully important advances in children’s quantitative understanding, which have far ranging implications for the development of processes vital to the STEM disciplines. The data reported here suggest that there is general consistency in the emergence of these sorts of whole number skills, in that there were generally positive correlations between the two whole number comparison tasks and the number line estimation tasks, across both of the tested age groups. The findings, however, also reveal negative correlations between performance on the whole number and proportional reasoning tasks in the early elementary school years, showing some discord in early mathematical skills. Performance across these tasks was positively correlated in the later elementary school years though, as might be expected from findings that an understanding of fractions relates with more general mathematical skills in middle school and beyond (Jordan et al., 2013; Siegler et al., 2011; Siegler & Lortie-Forgues, 2014; Siegler & Pyke, 2013). These findings provide support for the primary hypothesis that proportional and explicit whole number reasoning skills are non-linearly related across development.

Additional data are necessary to draw more robust conclusions, but the preliminary data reported here provide tentative support for the suggestion that understanding of whole numbers interferes with early proportional reasoning (e.g., Boyer et al., 2008; Spinillo & Bryant, 1999).

Our line of reasoning is that better whole number reasoning skills (i.e., counting abilities) early in development are associated with poorer proportional reasoning, because principles that apply to whole number operations do not always align with those that apply to proportional operations (Mix & Paik, 2008; Siegler et al., 2012; Van Dooren et al., 2015), and, indeed, applying whole number principles (e.g., numerical equivalence) can cause errors on proportional reasoning problems (e.g., responding that 3/9 = 3/4 due to their equivalent numerators). We also propose that proportional and whole number reasoning are positively related later in development due to refinement in both whole number and proportional reasoning strategies and better understanding when each is appropriate for either sort of problem.

The present findings raise a challenge for educational efforts: how should we teach children to understand both whole numbers and proportions if the two require conflicting reasoning strategies? Mathematics and science education researchers have advocated using children's intuitive knowledge to scaffold their learning and make use of what they already understand in teaching them challenging concepts (e.g., Fischbein, 1987; 1982; Lesh et al., 2003), which some have advocated to improve children's understanding of proportionality (Ahl et al., 1992; Falk & Wilkening, 1998; Pitkethly & Hunting, 1996). In this sense, teaching proportionality concepts via whole number operations will likely be less effective than situating proportional reasoning in a more intuitive framework. For instance, illustrating proportional relations with non-numerical stimuli, which, as noted above, has been done with continuous quantity illustrations that prohibit whole number counting operations, may be an effective means of teaching proportional concepts (Boyer & Levine, 2015). This, however, will require future research, which must build upon the findings presented here to further examine the most effective means for teaching difficult proportionality concepts.

Authors' Notes

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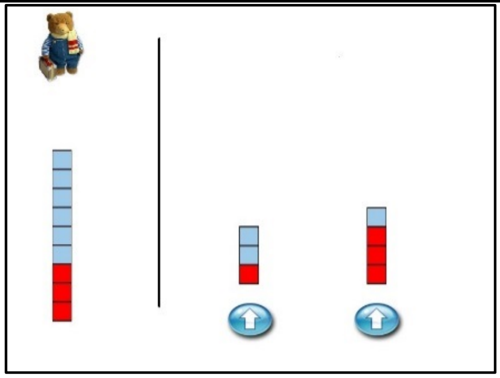
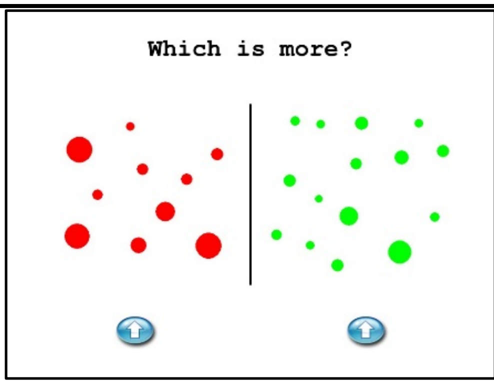
Table 1. Inter-task Pearson product moment correlation coefficients (*p*-values).

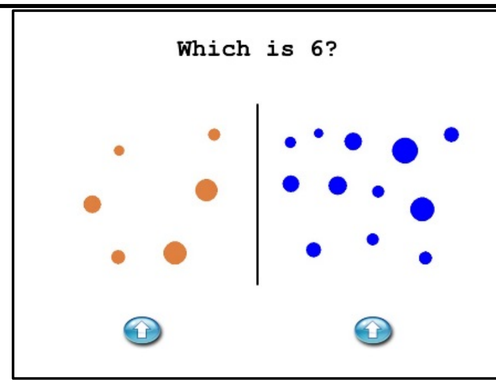
	2	3	4	5
Younger Children (N = 80)				
1. Proportional Reasoning	-.09 (.45)	-.21 (.07)	-.21 (.06)	-.23 (.04)
2. Which is More?	1	.44 (< .001)	.06 (.58)	.23 (.04)
3. Which is #?		1	.27 (.02)	.43 (<.001)
4. Non-symbolic Number Line			1	.63 (<.001)
5. Symbolic Number Line				1
Older Children (N = 81)				
1. Proportional Reasoning	.09 (.44)	.32 (.004)	.26 (.02)	.57 (<.001)
2. Which is More?	1	.52 (<.001)	.12 (.28)	.19 (.10)
3. Which is #?		1	.07 (.54)	.42 (<.001)
4. Non-symbolic Number Line			1	.37 (.001)
5. Symbolic Number Line				1



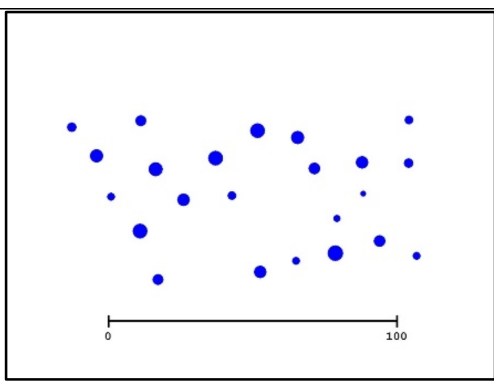
A. Proportional reasoning task. Participants chose which option on the right proportionally matched the target illustrated on the left.



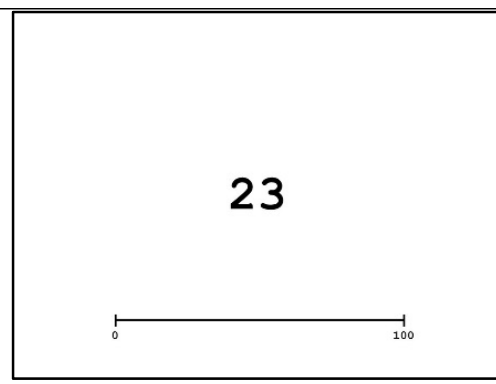
B. "Which is more?" whole number comparison task. Participants chose which of the arrays – red and green here – illustrated more dots.



C. "Which is #?" whole number comparison task. Participants chose which of the arrays –orange and blue here – illustrated the specified number.



D. Non-symbolic number line estimation task. Participants chose where to place a mark on the 0-100 number line that would represent the quantity of dots shown.



E. Symbolic number line estimation task. Participants chose where to place a mark on the 0-100 number line that would represent the numeral shown.

Figure 1. Example screenshots from each of the computerized experimental tasks.