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Genericity and Singularities of Robot Manipulators

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***Abstract*—We study the kinematic singularities of robot manipulators from the point of view of the theory of singularities. We examine the notion of a "generic" kinematic map, whose singularities form smooth manifolds of prescribed dimension in the joint space of the manipulator. For three-joint robots, an equivalent algebraic condition for genericity using Jacobian determinants is derived. This condition lends itself to symbolic computation and is sufficient for the study of decoupled manipulators. Orientation and translation singularities of manipulators are studied in detail. A complete characterization of orientation singularities of robots with any number of joints is given. The translation singularities of the eight possible topologies of three-joint robots are studied and the conditions on the link parameters for nongenericity are determined.**

I. Introduction

THE kinematics of robot manipulators is important in almost all areas of robotics, including dynamics, control, and motion planning. Of particular interest is the differential kinematic map, commonly known as the manipulator Jacobian, which plays a central role in trajectory planning, velocity and force control, and the numerical solution to the inverse kinematics problem (e.g., [5], [29]). Since the Jacobian is the best linear approximation to the kinematic map at a configuration, the manipulator's performance is profoundly affected by the value of the Jacobian. In particular, if the Jacobian is singular, the kinematic map may not be invertible. Further, the singular manipulator cannot impose any velocities of the end-effector reference frame in certain directions. This causes local control methods such as resolved rate control [31] and operational space control [11] to fail at a singularity. The robot is also able to withstand, in principle, infinite forces along the same directions. Determining the sets of singular points of various ranks and the images of these singular points is thus of importance.

The problems of determining the singularities of robot manipulators and of coping with them have received considerable attention such as in [2], [3], [7], [14], [15], [18], [20], [22], [25], [27], and [30]. The problem of computing the image of the set of singular points is generally studied under the rubric of the "manipulator workspace problem" (e.g., [9], [13], [16], [24]). Using the techniques discussed in the literature, one can compute the singular configurations of manipulators. For example, Sugimoto *et al.* [25] described a general procedure for calculating singular configurations using screw theory. They also presented an important result that characterizes the instantaneous joint screws in a singular configuration. Gorla [7] was able to get expressions for the set of singular points by assuming that link twists were multiples of $\pi/2$. Wang and Waldron [30], following earlier work [27], derived an algorithm using screw theory for computing the "singularity field" of the manipulator, i.e., given numerical values of three joint angles θ_2, θ_3, and θ_4 of a six-joint manipulator, they showed how to compute the angle θ_5 that makes the manipulator singular. Their algorithm can be used to numerically trace out singular points of a given manipulator geometry. Burdick [3] presented a detailed analysis of singularities using screw theory. Burdick also showed the significance of manipulator singularities in the design of robot manipulators. Shamir [22] provided an analytic tool to determine if the singularities are avoidable or unavoidable for three-joint manipulators operating in two-dimensional space.

In this paper we are concerned with a somewhat different goal. We would like to study the qualitative properties of manipulator singularities so as to come to grips with the following types of questions:

- Given a class of manipulators (e.g., the class of all RRP regional manipulators or the class of all possible revolute manipulators used for orientation), what statements can we make about the singularities of all manipulators of this class? Can we say anything about almost all manipulators in the class? More specifically, how do we set about classifying these manipulators by the nature of the singularities? Such information would be valuable to a design engineer who has narrowed down the manipulator design to the given class and needs to know what to expect.
- What are the topological and geometric properties of the set of singular points? Can they behave badly or do they form "nice" smooth manifolds? For example, if the set of singular points has kinks and intersections, it is difficult to trace it out numerically. Smooth manifolds do not have such problems.
- If the singular points form manifolds, what are the dimensions of these manifolds? This is important because low-dimension manifolds cannot separate the joint space into disconnected regions, whereas high-dimension manifolds (manifolds of codimension 1) can.
- How low can the rank of the Jacobian drop? The rank of the Jacobian is the number of degrees of freedom the end-

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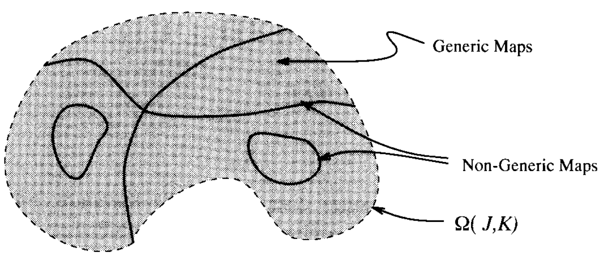

Fig. 1. Generic and nongeneric functions.

effector of the manipulator has locally. Hence, the lower the rank of the Jacobian, the more constrained the motion of the end-effector. It would be useful to have guarantees that certain low-rank singularities cannot occur in a class of manipulator designs.

All these questions can be answered within the framework of the theory of singularities, first developed in the 1950's by Whitney [32]. Recent treatments of these ideas can be found in [1] and [6]. Our goal in this paper is to examine the singularities of manipulators from this viewpoint. We do not address the question of stability in this paper, but we would like to point out that it is closely related to the other questions listed above (see [6] for more details).

Our study is briefly as follows. Suppose we are given a class of manipulators or, more generally, a class $\Omega(\mathcal{J}, \mathcal{K})$ of smooth functions from the joint space $\mathcal{J}$ to the task space $\mathcal{K}$. Almost all functions in $\Omega(\mathcal{J}, \mathcal{K})$ have well behaved singularities. Such functions are called "generic." The remaining functions form thin sets in $\Omega(\mathcal{J}, \mathcal{K})$ (see Fig. 1).

We first describe the properties of generic manipulators. Then we characterize the nongeneric manipulators and subject them to closer study. In particular, we try to derive algebraic conditions on the link parameter values for these nongeneric manipulators. Thus, we achieve a classification of manipulators according to the behavior of their singularities.

The paper is organized as follows. Section II provides the definitions of relevant terms used in the paper. We introduce the notion of genericity in Section III and present some differential-geometric results pertaining to the singularities of generic manipulators. A generic property is the formalization of the idea that almost all members of a class have the given property. Our definition of genericity will apply to the class of smooth functions from the joint space of the robot to its task space. A particularly useful characteristic of the singularities of generic manipulators is that they form a collection of smooth manifolds in the joint space of the manipulator. Further, the dimension of each manifold is related to the rank of the Jacobian at all points in the manifold. For three-joint robots, we derive an algebraic condition for genericity using Jacobian determinants (Section IV). This condition is sufficient for characterizing the singularities of decouplable manipulators, i.e., manipulators that can be separated into a three-joint translating part and an orienting part.

Next, we study the orientation and translation singularity problems separately in detail. The sets of singular points thus obtained are subsets of the singularities of the manipulator used for both translation and rotation simultaneously. In Section V, we consider a manipulator used only for orienting the end-effector. The singular sets can be described completely in this case, for robots with any number of joints. Then we consider three-joint regional manipulators, i.e., manipulators used only for translating the end-effector (Section VI). In the simpler cases, a complete description of the singularities and conditions for genericity are given. For the more complicated cases, the singularities are examined after making certain assumptions about the link parameters.

II. Preliminaries

This section provides background definitions for the sake of completeness. The concepts here may be familiar to many, but we suggest that the reader at least skim this section to become familiar with the terminology. In the following, a robot manipulator is taken to be any open linkage, i.e., a sequence of rigid bodies connected by joints, which are assumed to be either prismatic (sliding) or revolute (turning). Since we are only concerned with singularities of the chain, we shall ignore any joint limits that may be present.

The joint space $\mathcal{J}$ of a manipulator is the space of all joint variables $(q_1, q_2, \ldots, q_n)$ of the manipulator. The variables are defined in the usual sense of Denavit and Hartenberg [4]. If the manipulator has r revolute joints and $n - r$ prismatic joints, the joint space is actually $\mathcal{T}^r \times \Re^{n-r}$, where $\mathcal{T}^r$ is an r-torus, $\mathcal{T}^r = \mathcal{S}^1 \times \cdots \times \mathcal{S}^1$. Since the distinction is not important for our purposes, we shall consider $\mathcal{J}$ to be $\Re^n$, the space of n-tuples of real numbers, with the understanding that, for revolute joints, $q_i + 2k\pi \equiv q_i$. We shall denote by $\boldsymbol{q} = (q_1\, q_2 \, \ldots \, q_n)^T$ a point in the joint space. The joint space is a configuration space, i.e., by specifying $\boldsymbol{q}$, we completely specify the configuration of the robot. Hence, we will speak of $\boldsymbol{q}$ as a *configuration* of the robot.

A robot manipulator's motion is typically required in terms of the motion of a reference frame E attached to the manipulator. The task space $\mathcal{K}$ of a manipulator is the space of all required rigid motions of E. For typical tasks, the task space is the six-dimensional space of rigid translations and rotations, $\Re^3 \times \mathcal{SO}(3)$. However, the task space is actually defined by the application. For example, if one is only interested in the translation of the end-effector, the task space is $\Re^3$. An element of the task space is called a *position.* Note that this may have both a *translational* part belonging to $\Re^3$ and a *rotational* part belonging to $\mathcal{SO}(3)$. This is *not* standard terminology, since none exists at the present time. In the literature, positions have also been called locations, displacements, motions, configurations, transformations, etc., which unfortunately convey different meanings to different readers.

The kinematic map of a manipulator is the map $\boldsymbol{\kappa} : \mathcal{J} \rightarrow \mathcal{K}$, which maps a configuration $\boldsymbol{q}$ of the robot to the position of the end-effector reference frame E. The map can be considered to be the Cartesian product of two maps

$$\boldsymbol{\kappa} = \begin{pmatrix} \boldsymbol{\kappa}_t \\ \boldsymbol{\kappa}_r \end{pmatrix} \tag{1}$$

where $\kappa_t : \mathcal{J} \to \Re^3$ and $\kappa_r : \mathcal{J} \to \mathcal{SO}(3)$. κ_t will be called the translation map and κ_r will be called the rotation, or orientation, map.

The derivative $D_q\kappa$ of the kinematic map at a configuration q is a linear map from the tangent space of $\mathcal{J}$ at q to the tangent space of $\mathcal{K}$ at $\kappa(q)$. When represented as a matrix in coordinates, it is commonly known as the *manipulator Jacobian*. Methods for computing the Jacobian matrix may be found in [19], [27], [31].

A manipulator is said to be *singular* at a configuration q if $D_q\kappa$ is singular, i.e., if it is not of maximal rank. The configuration q is then called a *singular point* and its image $\kappa(q)$ is called a *singular image*. Some authors also call a singular point a *critical* point and a singular image a *critical value*, especially when dealing with real-valued maps. A point in $\mathcal{J}$ is called a *regular point* if it is not a critical point. A point in $\mathcal{K}$ is called a *regular value* if it is not the image of a singular (critical) point.

In this paper, *we shall always assume that the dimension of $\mathcal{J}$ is at least as large as that of $\mathcal{K}$*, i.e., we shall deal with manipulators with at least as many degrees of freedom as required by the task. Hence, a configuration q is singular if and only if rank$(D_q\kappa)$ is less than the dimension of $\mathcal{K}$.

III. Singularities of Generic Maps

In this section, we introduce the important concept of genericity of a smooth map and related results from the field of differential topology, and demonstrate their relevance to the singularities of kinematic maps. The book by Golubitsky and Guillemin [6] provides more information for the reader. Elementary definitions of smooth manifolds, tangent spaces, etc., can be found in textbooks such as [8].

In general, the types of singular sets that can occur depend on the actual map. Of particular interest are maps whose singular points form smooth manifolds in the domain. Smooth manifolds have several important properties, including the fact that they can be traced out by local methods. *Generic maps* constitute a class of maps whose singular points form smooth manifolds. Further, the dimension of each singular manifold is related to the rank of singular points in the manifold by a simple formula.

Considering that these generic maps have such nice singularities, one may wonder if such maps are rare. In fact, the opposite is true and hence the title of "generic." As a consequence of the Thom Transversality theorem, it turns out that these maps are residual in the space of smooth maps (see [6]).

Let $\mathcal{L}$ be the space of all linear maps from the tangent space to $\mathcal{J}$ at q, denoted $T_q\mathcal{J}$, to the tangent space to $\mathcal{K}$ at $\kappa(q)$, denoted $T_{\kappa(q)}\mathcal{K}$. Let $\dim(\mathcal{J}) = j$ and $\dim(\mathcal{K}) = k$. With local coordinates on $\mathcal{J}$ and $\mathcal{K}$, $\mathcal{L}$ is isomorphic to the space of $k \times j$ matrices. Note that $\mathcal{L}$ is a vector space under scalar multiplication and addition of matrices, isomorphic to $\Re^{jk}$.

We denote by $\mathcal{L}_r$ the set of points in $\mathcal{L}$ of rank r. It is well known that each $\mathcal{L}_r$ is a manifold with $\text{codim}(\mathcal{L}_r) = (j - r)(k - r)$, where codim is the codimension of a submanifold in its containing manifold. Thus, the $\mathcal{L}_r$ partition $\mathcal{L}$, i.e., with $\nu = \min\{j, k\}$

$$\mathcal{L}_0 \cup \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_\nu = \mathcal{L} \tag{2}$$

and

$$\mathcal{L}_r \cap \mathcal{L}_s = \emptyset \quad \text{for } r \neq s. \tag{3}$$

Further, the limit points of $\mathcal{L}_r$ not in it are in some $\mathcal{L}_s, s > r$. Such a set of manifolds is called a "manifold collection."

Definition 1: Let $f : \mathcal{M} \to \mathcal{N}$ be a smooth map between manifolds $\mathcal{M}$ and $\mathcal{N}$. The map f is *transversal* to a submanifold $\mathcal{U}$ of $\mathcal{N}$ if and only if for each point $x \in f^{-1}(\mathcal{U})$

$$\text{Image}(D_x f) + T_{f(x)}\mathcal{U} = T_{f(x)}\mathcal{N}. \tag{4}$$

Also, f is transversal to a manifold collection $\{\mathcal{L}_i\}$ in $\mathcal{N}$ if and only if f is transversal to each $\mathcal{L}_i$.

We write $f \pitchfork \mathcal{U}$ to indicate that f is transversal to $\mathcal{U}$. Transversality is one of the most important concepts in differential topology. We note some of the applications below.

Theorem 1 (Preimage Theorem): Let f, $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{U}$ be as above. Then the preimage $f^{-1}(\mathcal{U})$ is a submanifold of $\mathcal{M}$ and

$$\text{codim}(f^{-1}(\mathcal{U})) = \text{codim}(\mathcal{U}). \tag{5}$$

The Jacobian $D_q\kappa$ is a linear map from $T_q\mathcal{J}$ to $T_{\kappa(q)}\mathcal{K}$. We can view the collection of $D_q\kappa$, for all $q \in \mathcal{J}$, as a map from $\mathcal{J}$ to the space of linear maps from $T_q\mathcal{J}$ to $T_{\kappa(q)}\mathcal{K}$, viz., $\mathcal{L}$. We will denote this by $D\kappa : \mathcal{J} \to \mathcal{L}$, with $D\kappa(q) = D_q\kappa$. The map $D\kappa$ is smooth.

Definition 2 [6]: A kinematic map κ of a manipulator is *generic*[1] if $D\kappa \pitchfork \{\mathcal{L}_i\}$. We shall call a manipulator generic if it has a generic kinematic map.

Proposition 1: Let $\mathcal{S}_r \subset \mathcal{J}$ be the set of all singular points of rank r and let $\kappa : \mathcal{J} \to \mathcal{K}$ be generic with $\dim(\mathcal{J}) = j$ and $\dim(\mathcal{K}) = k$. Then $\mathcal{S}_r$ is a smooth submanifold of $\mathcal{J}$. Further, if $\mathcal{S}_r$ is not empty

$$\text{codim}(\mathcal{S}_r) = (j - r)(k - r). \tag{6}$$

Proof: Since κ is generic, $D\kappa \pitchfork \mathcal{L}_r$, $\text{codim}(\mathcal{L}_r) = (j - r)(k - r)$. From the preimage theorem, $\mathcal{S}_r = D\kappa^{-1}(\mathcal{L}_r)$ is a smooth manifold of $\mathcal{J}$ and $\text{codim}(\mathcal{S}_r) = \text{codim}(\mathcal{L}_r) = (j - r)(k - r)$. ■

Thus, Proposition 1 guarantees that, for generic kinematic maps, the singular points of various ranks form smooth manifolds. Further, it may be used to determine the dimension of each singular manifold. Observe that one way for κ to be generic is to have no singular points at all. In this case, all $\mathcal{S}_r$ for $r < k$ will be empty. The above proposition describes the dimension of $\mathcal{S}_r$ when it is not empty.

An important application of Proposition 1 is that it allows us to preclude the existence of generic singularities of certain low ranks. If $(j - r)(k - r) > j$, then the codimension of $\mathcal{S}_r$ is greater than the dimension of the joint space. Hence, a rank r singular point cannot exist.

We examine three examples of singularities of generic kinematic maps below.

[1] Also called *one*-generic, to distinguish it from genericity with respect to higher derivatives.

Example 1: A three-joint generic manipulator used for translation only or orientation only. Here, the dimension of the joint space j is 3 and the dimension of the task space k is also 3. Hence, if $\mathcal{S}_2$ is not empty, dim $(\mathcal{S}_2) = 2$, and both $\mathcal{S}_1$ and $\mathcal{S}_0$ have to be empty. Therefore, only the rank 2 singularity is possible.

Example 2: A six-joint generic manipulator used for both translation and orientation. Here, the dimension of the joint space is 6 and the dimension of the task space is also 6. Hence, if singularities of ranks 4 and 5 exist, dim $(\mathcal{S}_5) = 5$, dim $(\mathcal{S}_4) = 2$, and the smaller rank singular sets are empty.

Example 3: An eight-joint generic manipulator used for both translation and orientation. This robot is redundant and has two extra degrees of freedom. Here, if the manipulator can become singular, dim $(\mathcal{S}_5) = 5$, dim $(\mathcal{S}_4) = 0$ and smaller rank singularities cannot occur.

IV. A Condition for Genericity

We saw in Section III that generic mappings possess several desirable properties. However, it is difficult to determine if a map is generic using only the definition of genericity. In this section, we derive an algebraic criterion for determining if a three-joint robot ($j = 3$) in a three-dimensional task space ($k = 3$) is generic. This criterion lends itself well to symbolic computation and has been implemented in MACSYMA.

The restriction to three-joint manipulators does not limit us in analyzing the singularities of many general spatial manipulators. Almost all current manipulators can be decoupled into a translational part (a "regional structure") and an orienting wrist [12], [17], [21]. This design is widespread since it makes the inverse kinematic solution in closed form tractable [21]. The translational part corresponds to the large links toward the base of the manipulator, while the orienting part corresponds to the small terminal links that constitute a wrist. Our results will allow us to analyze each part separately. Also, some manipulator tasks are predominantly either translational tasks (as in gross motions of the manipulator) or rotating tasks (as in turning a screw). Our results will allow us to analyze the suitability of manipulators for such tasks.

Lemma 1: Let $A = (a_{lm})$ be an $n \times n$ matrix, and A^* be the matrix of cofactors of A. A and $A^* \in \mathcal{L}$, where $\mathcal{L}$ is the space of $n \times n$ matrices. Let $\det : \mathcal{L} \to \Re$ be the determinant function. Then

$$D_A \det = A^*. \tag{7}$$

Proof: Let a_{lm} be the l, m element of A. Therefore, the determinant of A can be written as

$$\det(A) = a_{lm}\text{cofactor}(a_{lm}) + \text{terms not involving } a_{lm}. \tag{8}$$

Therefore

$$\frac{\partial}{\partial a_{lm}} \det(A) = \text{cofactor}(a_{lm}). \tag{9}$$

Hence

$$D_A \det = (\text{cofactor}(a_{lm})) = A^*. \tag{10}$$

■

When $j = k = n$, $\mathcal{L}$ is a manifold isomorphic to $\Re^{n^2}$. We have seen that $\mathcal{L}_{n-1}$, the set of all matrices of rank $n-1$, is a submanifold of $\mathcal{L}$ with codimension $(n-(n-1))(n-(n-1)) = 1$. Hence, there is a one-dimensional vector space normal to the tangent space of $\mathcal{L}_{n-1}$. With the identification of $\mathcal{L}$ with $\Re^{n^2}$, a vector in $\mathcal{L}$ is just another $n \times n$ matrix.

Lemma 2: The normal subspace to $\mathcal{L}_{n-1}$ at $A \in \mathcal{L}_{n-1}$ is spanned by $D_A \det = A^*$.

Proof: Since $\mathcal{L}_{n-1}$ is a singular set, $\det(A) = 0$. Further, since A is of rank $n-1$, at least one cofactor of $A \neq 0$. Therefore, $D_A \det = A^* \neq 0$. So 0 is a regular value of the function det. Hence, locally $\mathcal{L}_{n-1}$ is an (n^2-1)-dimensional surface and $D_A \det \neq 0$ is normal to it.

In the proof of the theorem below, we use the following identity that relates derivatives of determinants in $\mathcal{L}$ to derivatives in $\mathcal{J}$.

Lemma 3:

$$\left(\frac{\partial}{\partial q_i} D\boldsymbol{\kappa}\right)\Big|_{\boldsymbol{q}} \cdot D_{\boldsymbol{q}}\boldsymbol{\kappa}^* = \left(\frac{\partial}{\partial q_i} \det D\boldsymbol{\kappa}\right)\Big|_{\boldsymbol{q}} \tag{11}$$

where $\cdot$ is the usual inner product in $\Re^{n^2}$.

Proof: Let d_{lm} be an element of $D\boldsymbol{\kappa}$ and d_m be a column.

$$\begin{aligned}\frac{\partial}{\partial q_i} \det [d_1 \dots d_n]\Big|_{\boldsymbol{q}} &= \sum_m \det \left[d_1 \dots \frac{\partial}{\partial q_i} d_m \dots d_n\right]\Big|_{\boldsymbol{q}} \\ &= \sum_l \sum_m \left(\frac{\partial}{\partial q_i} d_{lm}\right) \text{cofactor}(d_{lm})\Big|_{\boldsymbol{q}} \\ &= \left(\frac{\partial}{\partial q_i} D\boldsymbol{\kappa}\right)\Big|_{\boldsymbol{q}} \cdot D_{\boldsymbol{q}}\boldsymbol{\kappa}^*. \end{aligned} \tag{12}$$

■

We now use these facts to show the main result of this section.

Theorem 2—Manipulator Genericity: For a three-joint robot, with a three-dimensional task space, $\boldsymbol{\kappa}$ is generic if and only if $D_{\boldsymbol{q}} \det(D\boldsymbol{\kappa}) \neq 0$ for all $\boldsymbol{q}$ for which $\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}) = 0$.

Proof: By definition, $\boldsymbol{\kappa}$ is generic if and only if $D\boldsymbol{\kappa} \pitchfork \{\mathcal{L}_i\}, i = 0, 1, 2, 3$. $D\boldsymbol{\kappa}$ is obviously transversal to $\mathcal{L}_3$. Now, $\text{codim}(\mathcal{L}_0) = 9$ and $\text{codim}(\mathcal{L}_1) = 4$, while $\dim(\text{T}_q\mathcal{J}) = 3$. Therefore, the only way for $D\boldsymbol{\kappa}$ to be transversal to $\mathcal{L}_0$ and $\mathcal{L}_1$ is to avoid them altogether. Hence

genericity ⇔ (a) Only rank 2 singularities can occur

AND

(b) $D\boldsymbol{\kappa} \pitchfork \mathcal{L}_2$.

First look at (b). Let $\boldsymbol{q} \in D\boldsymbol{\kappa}^{-1}(\mathcal{L}_2)$. $D\boldsymbol{\kappa} \pitchfork \mathcal{L}_2 \Leftrightarrow \exists \underset{\sim}{v} \in \text{T}_q\mathcal{J}$ such that $(D_{\boldsymbol{q}}D\boldsymbol{\kappa})(\underset{\sim}{v}) \cdot D_{\boldsymbol{q}}\boldsymbol{\kappa}^* \neq 0$. Note that $D_{\boldsymbol{q}}\boldsymbol{\kappa}^*$ is the normal to $\mathcal{L}_2$ at $D_{\boldsymbol{q}}\boldsymbol{\kappa}$, from Lemma 2. This is equivalent to

$$\left(\sum_{i=1}^{3} \left(\frac{\partial}{\partial q_i} D\boldsymbol{\kappa}\right)\Big|_{\boldsymbol{q}} v_i\right) \cdot D_{\boldsymbol{q}}\boldsymbol{\kappa}^* \neq 0$$

i.e.,

$$\sum_{i=1}^{3} \left(\left(\frac{\partial}{\partial q_i} D\boldsymbol{\kappa}\right)\Big|_{\boldsymbol{q}} \cdot D_{\boldsymbol{q}}\boldsymbol{\kappa}^*\right) v_i \neq 0$$

i.e., for some i

$$\left(\frac{\partial}{\partial q_i} D\boldsymbol{\kappa}\right)\Big|_{\boldsymbol{q}} \cdot D_{\boldsymbol{q}}\boldsymbol{\kappa}^* \neq 0$$

from Lemma 10, for some i

$$\frac{\partial}{\partial q_i}\det(D\boldsymbol{\kappa}) \neq 0$$

i.e.,

$$D_{\boldsymbol{q}}\det(D\boldsymbol{\kappa}) \neq 0.$$

Now (a) is equivalent to the condition that $D_{\boldsymbol{q}}\boldsymbol{\kappa}^* \neq 0$ for all $\boldsymbol{q} \in \mathcal{J}$. This is because each 2×2 submatrix of $D_{\boldsymbol{q}}\boldsymbol{\kappa}$ is an element of $D_{\boldsymbol{q}}\boldsymbol{\kappa}^*$; if they are all zero, $D_{\boldsymbol{q}}\boldsymbol{\kappa}$ has rank less than 2. Clearly, $D_{\boldsymbol{q}}\boldsymbol{\kappa}^* \neq 0$ when the robot is nonsingular. When the robot is singular, i.e., when $\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}) = 0$, if $D_{\boldsymbol{q}}\det(D\boldsymbol{\kappa}) \neq 0$, there exists an i such that $((\partial/\partial_{q_i})D\boldsymbol{\kappa})\big|_{\boldsymbol{q}} \cdot D_{\boldsymbol{q}}\boldsymbol{\kappa}^* \neq 0$. Hence, $D_{\boldsymbol{q}}\boldsymbol{\kappa}^* \neq 0$.

Corollary 1: For three-joint robots, genericity implies that the set of singular points is either empty or a regular level surface of dimension 2.

The genericity condition of Theorem 2 has many applications, including the classification of singularities for separable manipulators. For example, using this result it is easy to show (Section VI) that a general SCARA-type manipulator[2] is nongeneric if and only if the axes of joints 2 and 3 intersect.

V. Orientation Singularity

This section discusses the singularities of orientation of a manipulator. We are concerned with the singularities of the restricted map $\boldsymbol{\kappa}_r$.

The Jacobian of an m-joint orienting manipulator corresponds to (see Appendix A for notation)

$$D_{\boldsymbol{q}}\boldsymbol{\kappa}_r = (\,\sigma_0 \boldsymbol{z}_0 \quad \dots \quad \sigma_{m-1}\boldsymbol{z}_{m-1}\,) \tag{13}$$

where

$$\sigma_i = \begin{cases} 0, & \text{if joint } i+1 \text{ is prismatic} \\ 1, & \text{if joint } i+1 \text{ is revolute.} \end{cases}$$

The prismatic joints do not contribute to $D_{\boldsymbol{q}}\boldsymbol{\kappa}_r$ and can be held fixed. If n of the joints in the manipulator are revolute, it is equivalent to a revolute manipulator with n-joints, and

$$D_{\boldsymbol{q}}\boldsymbol{\kappa}_r = (\,\boldsymbol{z}_0 \quad \boldsymbol{z}_1 \quad \dots \quad \boldsymbol{z}_{n-1}\,). \tag{14}$$

Henceforth, we shall assume that all the joints of the manipulator are revolute.

The following lemma indicates the geometric meaning of orientation singularity.

Lemma 4 [3], [10]: The manipulator is singular if and only if all the joint axes are parallel to a plane.

Now we observe some facts about the influence of the link twists α_i on singularities.

Lemma 5: If the number of $\alpha_i \neq 0 \pmod{\pi}$ is less than 2, the robot is always singular. Hence, $\operatorname{rank}(D_{\boldsymbol{q}}\boldsymbol{\kappa}_r) \geq 2$ unless the robot is always singular.

[2] A manipulator with two revolute joints followed by a terminal prismatic joint. In addition, the first two revolute joints are parallel.

Proof: By definition, $\boldsymbol{z}_{i-1} \cdot \boldsymbol{z}_i = \cos\alpha_i$. Therefore, if $\alpha_i = 0 \pmod{\pi}$, then $\boldsymbol{z}_{i-1} = \pm\boldsymbol{z}_i$. Therefore, the number of $\alpha_i \neq 0 \pmod{\pi}$ is less than $2 \Rightarrow$ the number of independent $\boldsymbol{z}_i < 3 \Rightarrow$ the robot is always singular. If some $\cos\alpha_i \neq 0$ there are at least two independent columns of $D_{\boldsymbol{q}}\boldsymbol{\kappa}_r$. Hence, $\operatorname{rank}(D_{\boldsymbol{q}}\boldsymbol{\kappa}_r) \geq 2$. ■

We shall first consider the case when the robot manipulator has no adjacent joints parallel, i.e., no link twists, $\alpha_i = 0 \pmod{\pi}$. The results are generalized to arbitrary manipulators in Theorem 4. In the following, we shall consider the joint space $\mathcal{J}$ to be $\Re^n$ with the understanding that $\theta_i + 2k\pi \equiv \theta_i$.

Theorem 3: The set of all singular points of the manipulator with *no* link twists $\alpha_i = 0 \pmod{\pi}$ is exactly the set of all 2-planes, parallel to the θ_1—θ_n plane, with $\theta_2, \dots, \theta_{n-1} = 0 \pmod{\pi}$.

Proof: First we show that the 2-planes described above consist of only singular points.

Any 3 successive joint axes $\boldsymbol{z}_{i-1}, \boldsymbol{z}_i$, and $\boldsymbol{z}_{i+1}$ lie in a plane if and only if $\boldsymbol{z}_{i-1} \cdot \boldsymbol{z}_i \times \boldsymbol{z}_{i+1} = 0$.

Now, in the coordinate frame of link i we can write $\boldsymbol{z}_{i-1}, \boldsymbol{z}_i$, and $\boldsymbol{z}_{i+1}$ as

$$\boldsymbol{z}_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{15}$$

$$\boldsymbol{z}_{i+1} = \begin{pmatrix} \sin\theta_{i+1}\sin\alpha_{i+1} \\ -\cos\theta_{i+1}\sin\alpha_{i+1} \\ \cos\alpha_{i+1} \end{pmatrix} \tag{16}$$

$$\boldsymbol{z}_{i-1} = \begin{pmatrix} 0 \\ \sin\alpha_i \\ \cos\alpha_i \end{pmatrix}. \tag{17}$$

Therefore, $\boldsymbol{z}_{i-1} \cdot \boldsymbol{z}_i \times \boldsymbol{z}_{i+1} = \sin\theta_{i+1}\sin\alpha_i\sin\alpha_{i+1}$. Since $\sin\alpha_i \neq 0$ and $\sin\alpha_{i+1} \neq 0$, $\boldsymbol{z}_{i-1} \cdot \boldsymbol{z}_i \times \boldsymbol{z}_{i+1} = 0$ if and only if $\theta_{i+1} = 0 \pmod{\pi}$.

Therefore, for points with $\theta_2, \dots, \theta_{n-1} = 0 \pmod{\pi}$, we conclude that all $\boldsymbol{z}_i$ lie in a plane. From Lemma 4, we conclude that these points are singular.

The reverse implication follows by reversing the proof. ■

The above theorem shows that, for manipulators with no successive parallel joints, the set of singular points form 2-planes in the joint space parallel to the θ_1-θ_n plane. This implies that if $n > 3$, the set of all nonsingular configurations is path-connected. A robot can go from a nonsingular configuration to another without going through a singularity. This is an argument for using four-joint wrists instead of the usual three-joint wrists.

If the manipulator has parallel joints also, the singular points continue to form planes, but of higher dimension, as shown in Theorem 4. We first partition the joints of the robot into p sets of adjacent parallel joints by requiring $\boldsymbol{z}_i$ and $\boldsymbol{z}_{i+1}$ to belong to the same set if and only if they are parallel. Let

$$\bar{\theta}_i = \sum_{j \in \text{a set}} \theta_j.$$

Theorem 4: The set of all singular points is exactly the set of planes with $\bar{\theta}_2, \dots, \bar{\theta}_{p-1} = 0 \pmod{\pi}$.

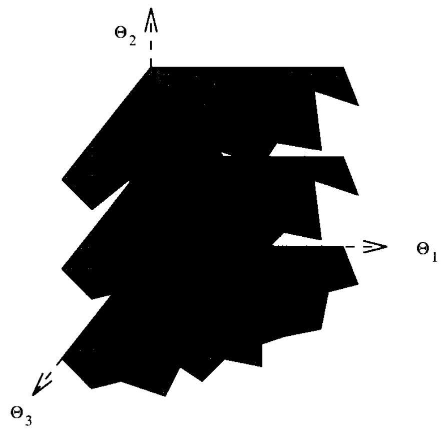

Fig. 2. Singular points of three-joint wrist.

Proof: The manipulator is equivalent to a p-joint manipulator with joint angles $\bar{\theta}_i$, since rotating a joint in a set i does not change the relative orientations of the joint axes unit vectors as long as $\bar{\theta}_i$ is constant.

Therefore, from Theorem 3, the singularities, which depend only on the relative orientations of the joint axes unit vectors, are given by

$$\bar{\theta}_2, \ldots, \bar{\theta}_{p-1} = 0 \pmod{\pi}. \tag{18}$$

■

Corollary: The dimension of the singular surfaces = 2+ the number of pairs of successive parallel joints.

Theorem 4 gives us a complete description of orientation singularities in the joint space of the manipulator. For illustration, we examine these singularities in two specific examples: a three-joint orienting wrist and the PUMA-560 robot manipulator used for orienting.

Example 1: Three-joint wrist with no adjacent parallel joints. The singular sets are 2-planes at $\theta_2 = 0 \pmod{\pi}$ in the joint space (see Fig. 2).

Example 2: The PUMA-560 manipulator used for orientation. This manipulator has one pair of parallel joints (joints 2 and 3). Hence, the singular points form 3-planes with θ_1 and θ_6 arbitrary, $\theta_2+\theta_3 = 0 \pmod{\pi}$ and $\theta_4, \theta_5 = 0 \pmod{\pi}$. Most of these positions lie within the workspace of the PUMA robot.

We now consider the genericity of orientation singularities. From Section III, if an n-joint ($n \geq 3$) orienting robot is generic, then dim $(\mathcal{S}_2) = n - (n-2) = 2$, and lower rank singularities do not exist. By the corollary to Theorem 4, this is not true for a manipulator with some pair of adjacent joints parallel. Hence, such robots are nongeneric. Robots with no adjacent joints parallel have singular sets of the appropriate dimension for genericity. We show below that they are indeed generic.

In the following, let A be a $3 \times n$ matrix, $n \geq 3$ and let a_i denote the ith column of A. Let A_i be the 3×3 submatrix of A given by

$$A_i = [\,a_i \quad a_{i+1} \quad a_{i+2}\,]. \tag{19}$$

Further, no adjacent pair of columns is linearly dependent. Hence, each A_i and A is of at least rank 2.

Lemma 6: Let A be defined as above. A has rank 2 if and only if each A_i has rank 2.

Proof: That each A_i has rank 2 if A has rank 2 is immediate. We need to show the reverse implication.

By hypothesis, the columns of A_1 span a two-dimensional linear space, say $\mathcal{Z}$, and a_2 and a_3 span this space. Since $A_2 = [\,a_2 \quad a_3 \quad a_4\,]$ also has rank 2, $a_4 \in \mathcal{Z}$, and since a_3 and a_4 are linearly independent, they span $\mathcal{Z}$ as well. Proceeding in this manner, all a_i lie in $\mathcal{Z}$ and A has rank 2. ■

Let $f : \mathcal{L} \rightarrow \Re^{n-2}$ be defined as

$$f(A) = \begin{pmatrix} \det A_1 \\ \vdots \\ \det A_{n-2} \end{pmatrix}. \tag{20}$$

Using f instead of the simple determinant, we can use the methods of Section IV to show that the robot with no adjacent joints parallel is generic.

Since A has the property that no adjacent joints are parallel, by continuity there exists an open set $\mathcal{U}$ in $\mathcal{L}$, containing A, with this property. Hence, by Lemma 6

$$\mathcal{L}_2 \cap \mathcal{U} = f\Big|_{\mathcal{U}}^{-1}(0). \tag{21}$$

Now

$$D_A f_i = D_A \det A_i = [\,0 \quad \cdots \quad \boldsymbol{A}_i^* \quad \cdots \quad 0\,]. \tag{22}$$

These are clearly linearly independent and so, locally, $\mathcal{L}_2$ is the zero set of f and the normal space to $\mathcal{L}_2$ at A is spanned by the $D_A f_i$.

$D\boldsymbol{\kappa}_r$ has only rank 2 singularities, so we only need to show that $D\boldsymbol{\kappa}_r \pitchfork \mathcal{L}_2$, i.e., for each $\boldsymbol{q} \in D\boldsymbol{\kappa}_r^{-1}(\mathcal{L}_2)$, and for each $D_{D_{\boldsymbol{q}}\boldsymbol{\kappa}_r} f_i$, there exists a $\underset{\sim}{v} \in T_q\mathcal{J}$ such that $D_{\boldsymbol{q}} D\boldsymbol{\kappa}_r(\underset{\sim}{v}) \cdot D_{D_{\boldsymbol{q}}\boldsymbol{\kappa}_r} f_i \neq 0$. This means that for some j

$$\left(\frac{\partial}{\partial q_j} D\boldsymbol{\kappa}_r\right)\Big|_{\boldsymbol{q}} \cdot [\,0 \quad \cdots \quad [D_{\boldsymbol{q}}\boldsymbol{\kappa}_r]_i^* \quad \cdots \quad 0\,] \neq 0 \tag{23}$$

i.e.,

$$\left(\frac{\partial}{\partial q_j} [D\boldsymbol{\kappa}_r]_i\right)\Big|_{\boldsymbol{q}} \cdot [D_{\boldsymbol{q}}\boldsymbol{\kappa}_r]_i^* \neq 0 \tag{24}$$

i.e.,

$$\frac{\partial}{\partial q_j} \det([D\boldsymbol{\kappa}_r]_i) \neq 0. \tag{25}$$

From (15)–(17), $\det([D\boldsymbol{\kappa}_r]_i)$ is given by

$$\det([D\boldsymbol{\kappa}_r]_i) = \boldsymbol{z}_i \cdot \boldsymbol{z}_{i+1} \times \boldsymbol{z}_{i+2} = \sin\theta_{i+1} \sin\alpha_i \sin\alpha_{i+1}. \tag{26}$$

Hence

$$\frac{\partial}{\partial \theta_{i+1}} \det([D\boldsymbol{\kappa}_r]_i) = \cos\theta_{i+1} \sin\alpha_i \sin\alpha_{i+1} \neq 0 \tag{27}$$

when the robot is singular.

We state the above considerations as the following theorem.

Theorem 5: An n-joint orienting manipulator is generic if and only if no adjacent joints are parallel.

VI. Translation Singularity

This section discusses the singularities of translation of three link manipulators. We are concerned with the singularities of the restricted map $\boldsymbol{\kappa}_t$.

The translation part of the Jacobian of an n-joint manipulator corresponds to (see Appendix A for notation)

$$D_{\boldsymbol{q}}\boldsymbol{\kappa}_t = (\sigma_0 \boldsymbol{z}_0 \times \boldsymbol{p}_0 + \bar{\sigma}_0 \boldsymbol{z}_0 \; \ldots \sigma_{n-1}\boldsymbol{z}_{n-1} \times \boldsymbol{p}_{n-1} + \bar{\sigma}_{n-1}\boldsymbol{z}_{n-1}) \quad (28)$$

where

$$\sigma_i = \begin{cases} 0, & \text{if joint } i+1 \text{ is prismatic} \\ 1, & \text{if joint } i+1 \text{ is revolute} \end{cases}$$

$$\bar{\sigma}_i = \begin{cases} 0, & \text{if } \sigma_i = 1 \\ 1, & \text{if } \sigma_i = 0. \end{cases}$$

If the Jacobian is singular, it has rank less than 3. It is easy to see that if $D_{\boldsymbol{q}}\boldsymbol{\kappa}_t$ has rank $= 0$ at any configuration, then it has rank $= 0$ at all configurations and has no prismatic joints. Therefore, only singularities of rank $= 1$ or 2 occur in nontrivial robots. The following theorem gives us a geometric consideration for a translation singularity to occur. Its corollary describes the situation for rank $= 1$.

Theorem 6 [23], [25], [27]: A manipulator configuration is singular if and only if each revolute joint axis is parallel to or intersects a line through the end-effector point, and each prismatic joint is orthogonal to the line. Further, the direction of the line is a singular direction.

Corollary: A manipulator has a singularity of rank $= 1$ if and only if each revolute joint axis either passes through the end-effector point or lies in a plane of singular directions, and all prismatic joint axes are orthogonal to the plane.

We now proceed to examine the singularities of three-link robots used for translation of the end-effector. The technique used is similar to that of Gorla [7] in that we use the determinant of the Jacobian to obtain an expression for the singular surfaces in joint space. However, unlike [7], we do not assume that all link twists are multiples of $\pi/2$. In many cases we are able to simplify the expressions for the singular manifolds and exhibit the nongeneric cases without any assumptions. For the cases that were not amenable to direct simplification, we examine possible simplifications that can be achieved by making various assumptions on the link parameters. The investigation was carried out using the symbolic algebra system MACSYMA.

By computing the determinant and simplifying, we obtain an expression for the determinant, which we analyze. The determinant may be used to plot the singularities for various values of link parameters. While this is useful for studying specific robots, it is difficult to make more general statements about the effect of link parameters on the singularities. There are as many as seven independent kinematic parameters, so the parameter space is rather large.

For each robot geometry, we attempt to determine conditions on the Denavit–Hartenberg parameters that will make the manipulator nongeneric, and we describe the singularities for the nongeneric manipulators. For the simpler manipulators, we can give a complete characterization of nongeneric manipulators. For the more complicated cases, the set of all possible nongeneric robots does not have a simple characterization. We shall then examine the situation when one or two kinematic parameters are assigned specific values. These represent common design situations as well as those that make the symbolic computations tractable.

Determination of nongenericity is achieved using the solutions of Problems P1, P2, and P3 described in Appendix B. We shall frequently be able to factor the determinant of the Jacobian, in which case we examine each of the factors. We use the fact that a necessary condition for the zero set of such a factor f to fail to be a manifold[3] is that

$$\exists \boldsymbol{q} \quad \text{such that } f = 0 \text{ and } D_{\boldsymbol{q}} f = 0. \quad (29)$$

The techniques of Appendix B can be used for this as well.

The equation for the determinant of the Jacobian is free of the joint variable q_1. This means that if the robot is singular for one value of q_1, it is singular for all values of q_1. Hence, we only need to examine the singular sets in the q_2—q_3 space, with the understanding that the complete set of singular points are cylinders over these sets, in the q_1 direction. In the following, we shall consider the q_2—q_3 space to be a plane, with the understanding that, for revolute joints, q_i is to be taken modulo 2π.

Also, α_3 and d_1 (if the joint variable is θ_1) or θ_1 (if the joint variable is d_1) do not appear in the equation. This is because, for a three-link manipulator, these parameters can be arbitrarily set without affecting the geometry of the manipulator. We simplify our computations by setting θ_1, d_1, and α_3 to zero, leaving seven independent kinematic parameters and two variables.

In the remainder of this section, we examine the singularities of the eight possible three-joint manipulator topologies.

A. PPP Manipulator Singularities

This is a manipulator with all three prismatic joints. The joint variables are d_1, d_2, and d_3. The expression for the determinant simplifies to

$$\det D_{\boldsymbol{q}}\boldsymbol{\kappa}_t = \sin\alpha_1 \sin\alpha_2 \sin\theta_2. \quad (30)$$

The manipulator is either always singular or never singular. If the manipulator is never singular, it is also generic. Likewise, the manipulator that is always singular is nongeneric.

B. PPR Manipulator Singularities

This manipulator has two prismatic joints followed by a terminal revolute joint. The joint variables are d_1, d_2, and θ_3. The expression for the determinant of the Jacobian simplifies to

$$\det D_{\boldsymbol{q}}\boldsymbol{\kappa}_t = a_3 \sin\alpha_1(\cos\theta_2 \sin\theta_3 + \cos\alpha_2 \sin\theta_2 \cos\theta_3). \quad (31)$$

The factor a_3 indicates that if $a_3 = 0$, the end-effector point lies on the axis of revolution of joint 3. Joint 3 cannot contribute any translational velocity to the end-effector

[3] This follows from the preimage theorem.

in this case. Also, $\sin\alpha_1 = 0$ makes the two prismatic joints parallel. The only interesting factor is $\cos\theta_2 \sin\theta_3 + \cos\alpha_2 \sin\theta_2 \cos\theta_3$.

Unless it is identically zero, the zero sets of (31) are singular lines in the d_2—θ_3 plane, at $\theta_3 = \tan^{-1}(-\cos\alpha_2 \tan\theta_2)$, separated in θ_3 by π. These are generic since

$$D_{\boldsymbol{q}} \det D\boldsymbol{\kappa}_t = \begin{pmatrix} 0 \\ 0 \\ \cos\theta_2 \cos\theta_3 - \cos\alpha_2 \sin\theta_2 \sin\theta_3 \end{pmatrix} \neq 0 \tag{32}$$

when (31) is 0. Hence, all nontrivial singularities are generic.

C. RPP Manipulator Singularities

The RPP manipulator has a revolute joint at its base followed by two prismatic joints. The joint variables in this case are θ_1, d_2, and d_3. The Jacobian determinant is

$$\begin{aligned} \det D_{\boldsymbol{q}}\boldsymbol{\kappa}_t = &\sin\alpha_2(a_3 \sin\alpha_1 \sin\alpha_2 \cos\theta_2 \sin\theta_3 \\ &- a_3 \cos\alpha_1 \cos\alpha_2 \sin\theta_3 \\ &+ a_1 \cos\alpha_1 \sin\theta_2 \\ &+ \sin\alpha_1 \cos\alpha_2 \cos\theta_2 d_3 \\ &+ \cos\alpha_1 \sin\alpha_2 d_3 \\ &+ \sin\alpha_1 \cos\theta_2 d_2). \end{aligned} \tag{33}$$

Note that, in general, if joint i is prismatic, the location of the joint axis is not defined, but only the direction is defined. Therefore, we can set a_i to 0 in such a case to simplify the expressions. Henceforth, we shall do this for all cases that involve prismatic joints. With $a_2 = a_3 = 0$, the determinant expression becomes

$$\begin{aligned} \det D_{\boldsymbol{q}}\boldsymbol{\kappa}_t = &\sin\alpha_2(a_1 \cos\alpha_1 \sin\theta_2 \\ &+ \sin\alpha_1 \cos\alpha_2 \cos\theta_2 d_3 \\ &+ \cos\alpha_1 \sin\alpha_2 d_3 \\ &+ \sin\alpha_1 \cos\theta_2 d_2). \end{aligned} \tag{34}$$

This has the form

$$\det D_{\boldsymbol{q}}\boldsymbol{\kappa}_t = u + v d_2 + w d_3 \tag{35}$$

where u, v, and w are constants depending on the link parameters. Therefore

$$D_{\boldsymbol{q}} \det D\boldsymbol{\kappa} = \begin{pmatrix} 0 \\ v \\ w \end{pmatrix} \tag{36}$$

and from the genericity condition, the robot is nongeneric if and only if $u = v = w = 0$. This makes the robot always singular. Therefore, the only nontrivial singularities are generic and form a straight line in the d_2—d_3 plane.

D. PRP Manipulator Singularities

Since the first and the last joints are prismatic, we set $a_1 = a_3 = 0$. The Jacobian determinant of this manipulator is

$$\begin{aligned} \det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) = &a_2 \sin\alpha_1 \cos\alpha_2 \sin\theta_2 \\ &- \sin\alpha_1 \cos\alpha_2 \sin\alpha_2 d_3 \cos\theta_2 \\ &- \cos\alpha_1 \sin^2\alpha_2 d_3. \end{aligned} \tag{37}$$

If $a_2 = 0$, the determinant can be factored as

$$\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) = -\sin\alpha_2 d_3 (\sin\alpha_1 \cos\alpha_2 \cos\theta_2 + \cos\alpha_1 \sin\alpha_2). \tag{38}$$

The factor $\sin\alpha_2$ indicates that if this is zero, the end-effector point always lies on the axis of the revolute joint, making it singular. The next factor always produces a singular line at $d_3 = 0$. If $|\tan\alpha_2/\tan\alpha_1| \leq 1$, the last factor produces a pair of lines at $\theta_2 = \cos^{-1} -(\tan\alpha_2/\tan\alpha_1)$. Since the singular lines intersect, the manipulator is nongeneric, from the manipulator genericity theorem.

We now proceed to show that this is the only possible nontrivial, nongeneric robot. The expression for $\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t)$ has the form $a \sin\theta_2 + b \cos\theta_2 + c$, where

$$a = a_2 \sin\alpha_1 \cos\alpha_2 \tag{39}$$

$$b = -\sin\alpha_1 \cos\alpha_2 \sin\alpha_2 d_3 \tag{40}$$

$$c = -\cos\alpha_1 \sin^2\alpha_2 d_3. \tag{41}$$

Hence, by the solution of Problem P2, there are only two cases to consider for the manipulator to become nongeneric.

Case 1: $a = b = c = 0$ and $a'^2 + b'^2 - c'^2 \geq 0$. If this holds independent of d_3, then the robot is always singular. Hence, b or c should not be identically zero for nontrivial robots. Note that $d_3 = 0$ is always a solution for $b = c = 0$. Consider $a = 0$. If $a_2 = 0$, then

$$a'^2 + b'^2 - c'^2 = (\sin^2\alpha_1 \cos^2\alpha_2 - \cos^2\alpha_1 \sin^2\alpha_2) \sin^2\alpha_2. \tag{42}$$

If $\sin\alpha_2 = 0$, b and c are identically zero and the robot is trivial. If $\sin\alpha_2 \neq 0$, the robot is nongeneric if and only if $\sin^2\alpha_1 \cos^2\alpha_2 - \cos^2\alpha_1 \sin^2\alpha_2 \geq 0$, i.e., $|\tan\alpha_2/\tan\alpha_1| \leq 1$.

If $a_2 \neq 0$, then either $\sin\alpha_1 = 0$ or $\cos\alpha_2 = 0$. If $\sin\alpha_1 = 0$, then $a'^2 + b'^2 - c'^2 = -\sin^4\alpha_2$. Therefore, for a solution to exist, $\sin\alpha_2 = 0$, which makes b and c identically zero. Similarly, if $\cos\alpha_2 = 0$, we require that $\cos\alpha_1 = 0$, which also makes b and c identically zero.

Case 2: $c \neq 0$ and $a^2 + b^2 - c^2 = 0$ and $aa' + bb' - cc' = 0$. By computation

$$\begin{aligned} a^2 + b^2 - c^2 = &{a_2}^2 \sin^2\alpha_1 \cos^2\alpha_2 \\ &+ \sin^2\alpha_1 \cos^2\alpha_2 \sin^2\alpha_2 {d_3}^2 \\ &- \cos^2\alpha_1 \sin^4\alpha_2 {d_3}^2 \end{aligned} \tag{43}$$

$$\begin{aligned} aa' + bb' - cc' = &\sin^2\alpha_1 \cos^2\alpha_2 \sin^2\alpha_2 d_3 \\ &- \cos^2\alpha_1 \sin^4\alpha_2 d_3. \end{aligned} \tag{44}$$

Hence, for these equations to be satisfied

$${a_2}^2 \sin^2\alpha_1 \cos^2\alpha_2 = 0 \tag{45}$$

$$\sin^2\alpha_1 \cos^2\alpha_2 - \cos^2\alpha_1 \sin^2\alpha_2 = 0. \tag{46}$$

But this is almost identical to Case 1 and yields no new nongeneric robots. Therefore, the only possible nontrivial, nongeneric PRP robot has $a_2 = 0$.

E. RRP Manipulator Singularities

This is a manipulator with two revolute joints followed by a terminal prismatic joint. The joint variables are θ_1, θ_2, and d_3. The determinant of the Jacobian simplifies to

$$\begin{aligned}\det(D_{\boldsymbol{q}}\kappa_t) =& \sin\alpha_1 \sin\alpha_2 \sin\theta_2 {d_3}^2 \\ &+ \sin\alpha_1 \cos\alpha_2 \sin\alpha_2 d_2 d_3 \sin\theta_2 \\ &+ a_1 a_2 \cos\alpha_1 \cos\alpha_2 \sin\theta_2 \\ &- a_1 \cos\alpha_1 \cos\alpha_2 \sin\alpha_2 d_3 \cos\theta_2 \\ &+ a_2 \sin\alpha_1 d_3 \cos\theta_2 \\ &+ a_2 \sin\alpha_1 \cos\alpha_2 d_2 \cos\theta_2 \\ &+ a_1 \sin\alpha_1 \sin^2\alpha_2 d_3. \end{aligned} \quad (47)$$

Case 1: $a_1 = 0$. The first two revolute joints intersect in this case. This is an extremely common design, since the joint 2 actuator can be placed along the axis of joint 1, reducing the dynamic load on the joint 1 actuator. Examples of manipulators with this geometry are the Stanford manipulator and the Unimation Unimate. The determinant can be factored as

$$\det(D_{\boldsymbol{q}}\kappa_t) = \sin\alpha_1(d_3 + \cos\alpha_2 d_2)(\sin\alpha_2 d_3 \sin\theta_2 + a_2\cos\theta_2). \quad (48)$$

The first factor indicates that if $\sin\alpha_1 = 0$, the first two joints coincide. The factor $(d_3 + \cos\alpha_2 d_2)$ produces a singular line in the θ_2-d_3 plane at $d_3 = -\cos\alpha_2 d_2$. Physically, this is due to the line from the end-effector to the point of intersection of the first two axes being normal to the axis of joint 3.

The next factor $(\sin\alpha_2 d_3 \sin\theta_2 + a_2\cos\theta_2)$ generally forms a 1-manifold in the θ_2–d_3 plane with two components. We use the solution of Problem P1 to determine when it can fail to be a manifold. Now

$$a = \sin\alpha_2 \sin\theta_2 \quad (49)$$
$$b = a_2 \cos\theta_2 \quad (50)$$
$$a' = \sin\alpha_2 \cos\theta_2 \quad (51)$$
$$b' = -a_2 \sin\theta_2. \quad (52)$$

Hence, $a = b = a' = b' = 0$ only if $a_2 = 0$ and $\sin\alpha_2 = 0$. This makes the robot always singular. Similarly, $a = b = 0$ and $a' \neq 0$ only if $a_2 = 0$ and $\sin\alpha_2 \neq 0$. In this case, the two components of the singular manifold merge to produce lines at $d_3 = 0$ and $\theta_2 = 0 \pmod{\pi}$. This is clearly nongeneric.

Case 2: $\alpha_1 = 0$. The first two revolute joints are parallel. Examples are the Adept robot and many other SCARA-type manipulators. The determinant can be simplified to

$$\det(D_{\boldsymbol{q}}\kappa_t) = a_1\cos\alpha_2(a_2\sin\theta_2 - \sin\alpha_2 d_3 \cos\theta_2). \quad (53)$$

The factor $(a_2\sin\theta_2 - \sin\alpha_2 d_3\cos\theta_2)$ can be analyzed using Problem P2. Here

$$a = a_2 \quad (54)$$
$$b = -\sin\alpha_2 d_3 \quad (55)$$
$$c = 0 \quad (56)$$
$$a' = 0 \quad (57)$$
$$b' = -\sin\alpha_2 \quad (58)$$
$$c' = 0. \quad (59)$$

Since $c = 0$, the manipulator is nongeneric if and only if $a_2 = 0$. The singularities then form lines at $d_3 = 0$ and $\theta_2 = \pi/2 \pmod{\pi}$. These lines clearly intersect.

Case 3: $a_2 = 0$. The last two joints intersect in this case. Note that since joint 3 is prismatic, a_3 has been taken to be zero, i.e., the axis of joint 3 also passes through the end-effector point. An example of a manipulator with this geometry is the Stanford manipulator. The determinant simplifies to

$$\begin{aligned}\det(D_{\boldsymbol{q}}\kappa_t) =& \sin\alpha_2 d_3(\sin\alpha_1 d_3 \sin\theta_2 \\ &+ \sin\alpha_1\cos\alpha_2 d_2 \sin\theta_2 \\ &- a_1\cos\alpha_1\cos\alpha_2\cos\theta_2 \\ &+ a_1\sin\alpha_1\sin\alpha_2). \end{aligned} \quad (60)$$

If $\sin\alpha_2 = 0$, the robot is always singular. The factor d_3 contributes a singular line at $d_3 = 0$, since this value makes the end-effector point lie on the axis of joint 2. The last factor generally forms a 1-manifold in the θ_2–d_3 plane. The exceptions can be derived using the solution of Problem P2. Here

$$a = \sin\alpha_1(d_3 + \cos\alpha_2 d_2) \quad (61)$$
$$b = -a_1\cos\alpha_1\cos\alpha_2 \quad (62)$$
$$c = a_1\sin\alpha_1\sin\alpha_2 \quad (63)$$
$$a' = \sin\alpha_1 \quad (64)$$
$$b' = 0 \quad (65)$$
$$c' = 0. \quad (66)$$

The case $a_1 = 0$ has already been considered, so we exclude it here. Hence, if $a = b = c = 0$, then $\sin\alpha_1 = 0$ and $\cos\alpha_2 = 0$, which makes the robot always singular.

Similarly, if $c \neq 0$, then $\sin\alpha_1 \neq 0$. For a solution to exist, it is required that

$$\begin{aligned} a^2 + b^2 - c^2 =& \sin^2\alpha_1(d_3 + \cos\alpha_2 d_2)^2 \\ &+ {a_1}^2(\cos^2\alpha_1\cos^2\alpha_2 \\ &- \sin^2\alpha_1\sin^2\alpha_2) = 0 \end{aligned} \quad (67)$$

$$aa' + bb' - cc' = \sin^2\alpha_1(d_3 + \cos\alpha_2 d_2) = 0. \quad (68)$$

This is possible if and only if $\alpha_1 = \pi/2 \pm \alpha_2 \pmod{\pi}$. In this case the two components of the singular manifold intersect (see Fig. 3).

Case 4: $\alpha_2 = 0$. The last two joints are parallel. The Adept robot and SCARA-type manipulators are examples of this geometry as well. Note that, in general, if $\alpha_i = 0$, the common normal between the the two parallel joints is not uniquely defined. We can set either d_i or d_{i+1} arbitrarily, without loss of generality. In this case, we set d_2 to zero. The determinant reduces to

$$\det(D_{\boldsymbol{q}}\kappa_t) = a_2(a_1\cos\alpha_1\sin\theta_2 + \sin\alpha_1 d_3\cos\theta_2). \quad (69)$$

This equation is similar to that of Case 2 and has similar singularities. The singularities fail to be generic if and only if $a_1\cos\alpha_1 = 0$.

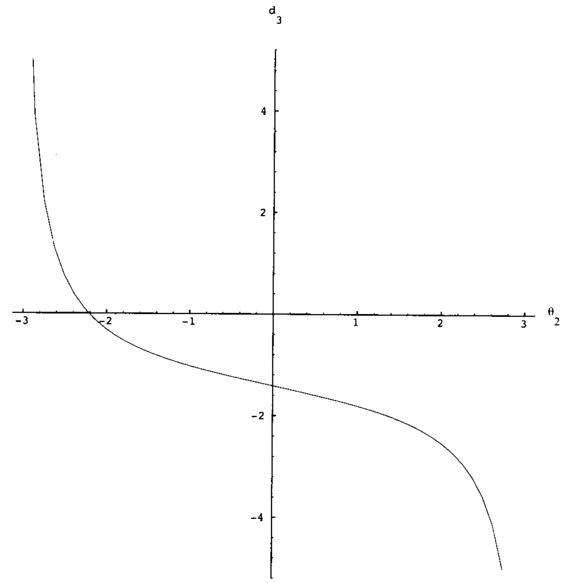

Fig. 3. Zero sets of Case 3 RRP manipulator for $a_1 = 1$, $d_2 = 2$, and $\alpha_1 = \alpha_2 = \pi/4$.

F. RPR Manipulator Singularities

This manipulator has two revolute joints with an intermediate prismatic joint. The joint variables are θ_1, d_2, and θ_3. The determinant of the Jacobian is

$$\begin{aligned}\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) = & a_3(a_3 \sin\alpha_1 \sin\alpha_2 \sin\theta_2 \sin^2\theta_3 \\ & - a_3 \sin\alpha_1 \cos^2\alpha_2 \sin\alpha_2 \cos\theta_2 \cos\theta_3 \sin\theta_3 \\ & + a_3 \cos\alpha_1 \cos\alpha_2 \cos\theta_3 \sin\theta_3 \\ & - a_3 \cos\alpha_1 \cos\theta_3 \sin\theta_3 \\ & + \sin\alpha_1 \cos\alpha_2 d_3 \sin\theta_2 \sin\theta_3 \\ & + \sin\alpha_1 d_2 \sin\theta_2 \sin\theta_3 \\ & - a_1 \cos\alpha_1 \cos\theta_2 \sin\theta_3 \\ & - a_1 \cos\alpha_1 \cos\alpha_2 \sin\theta_2 \cos\theta_3 \\ & - \sin\alpha_1 \cos^2\alpha_2 d_3 \cos\theta_2 \cos\theta_3 \\ & - \sin\alpha_1 \cos\alpha_2 d_2 \cos\theta_2 \cos\theta_3 \\ & - \cos\alpha_1 \cos\alpha_2 \sin\alpha_2 d_3 \cos\theta_3). \qquad (70)\end{aligned}$$

Note that $a_3 \neq 0$ for a nontrivial robot. Henceforth, we shall consider only the remaining factor.

Case 1: $\alpha_1 = \pi/2$. In this case, the determinant can be factored as

$$\begin{aligned}\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) = & (a_3 \sin\alpha_2 \sin\theta_3 + \cos\alpha_2 d_3 + d_2) \\ & (\sin\theta_2 \sin\theta_3 - \cos\alpha_2 \cos\theta_2 \cos\theta_3). \qquad (71)\end{aligned}$$

The first factor, $(a_3 \sin\alpha_2 \sin\theta_3 + \cos\alpha_2 d_3 + d_2)$, always forms a manifold since its derivative with respect to d_2 is not zero. Similarly, the second factor, $(\sin\theta_2 \sin\theta_3 - \cos\alpha_2 \cos\theta_2 \cos\theta_3)$, describes a manifold, unless it is identically zero.

Case 2: $\alpha_2 = 0$. The last two joints are parallel in this case. Hence, we also set $d_3 = 0$ to simplify computations. The determinant simplifies to

$$\begin{aligned}\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) = & \sin\alpha_1 d_2 \sin\theta_2 \sin\theta_3 \\ & - a_1 \cos\alpha_1 \cos\theta_2 \sin\theta_3 \\ & - a_1 \cos\alpha_1 \sin\theta_2 \cos\theta_3 \\ & - \sin\alpha_1 d_2 \cos\theta_2 \cos\theta_3. \qquad (72)\end{aligned}$$

This can be analyzed with Problem P2. Here

$$\begin{aligned} a = & \sin\alpha_1 d_2 \sin\theta_2 \\ & - a_1 \cos\alpha_1 \cos\theta_2 \\ b = & -a_1 \cos\alpha_1 \sin\theta_2 \\ & - \sin\alpha_1 d_2 \cos\theta_2 \\ c = & 0. \qquad (73)\end{aligned}$$

Since c is identically zero, $a'^2 + b'^2 - c'^2 \geq 0$, and the manipulator becomes nongeneric if and only if $a = b = 0$. Computing the resultant of a and b, considered as polynomials in d_2, we get the resultant [26] to be $a_1 \cos\alpha_1 \sin\alpha_1$. The term $\sin\alpha_1$ merely makes the leading coefficients of the polynomials vanish. The manipulator becomes nongeneric only if $a_1 \cos\alpha_1 = 0$. This turns out to be sufficient for nongenericity as well. The determinant can then be factored as $-\sin\alpha_1 d_2 \cos(\theta_2 + \theta_3)$, which results in intersecting lines.

Case 3: $\alpha_2 = \pi/2$. The determinant can be factored as

$$\begin{aligned}\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) = & \sin\theta_3(a_3 \sin\alpha_1 \sin\theta_2 \sin\theta_3 \\ & - a_3 \cos\alpha_1 \cos\theta_3 \\ & + \sin\alpha_1 d_2 \sin\theta_2 \\ & - a_1 \cos\alpha_1 \cos\theta_2). \qquad (74)\end{aligned}$$

The determinant has the form

$$\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) = \sin\theta_3 f. \qquad (75)$$

Therefore, the gradient of $\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t)$ has the form

$$\frac{\partial}{\partial d_2}\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) = \sin\alpha_1 \sin\theta_2 \sin\theta_3 \qquad (76)$$

$$\frac{\partial}{\partial \theta_3}\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) = \sin\theta_3 \frac{\partial f}{\partial \theta_3} + \cos\theta_3 f. \qquad (77)$$

If $f = 0$ and $\sin\theta_3 = 0$, the above equations clearly have a solution and the robot is nongeneric. Now

$$\begin{aligned} f\Big|_{\sin\theta_3 = 0} = & (\pm a_3 \cos\alpha_1 - a_1 \cos\alpha_1 \cos\theta_2) \\ & + \sin\alpha_1 \sin\theta_2 d_2 \qquad (78)\end{aligned}$$

which can always be made zero unless $\sin\alpha_1 \sin\theta_2 = 0$. Hence, the only possible generic manipulators have $\sin\alpha_1 = 0$ or $\sin\theta_2 = 0$. From (78), if $\sin\alpha_1 = 0$, $a_3 = \pm a_1 \cos\theta_2$ is sufficient for nongenericity. Similarly, if $\sin\theta_2 = 0$, $a_3 = a_1$ is sufficient.

By making the substitutions $\sin\alpha_1 = 0$ or $\sin\theta_2 = 0$ and using Problem P3 on the resulting trigonometric polynomials, we see that these conditions are necessary as well.

It is interesting to note that, for the nongeneric cases when $\sin\alpha_1 \sin\theta_2 = 0$, the singular points continue to form manifolds.

G. PRR Manipulator Singularities

This is a manipulator with an initial prismatic joint followed by two revolute joints. The joint variables are d_1, θ_2, and θ_3. The determinant of the Jacobian simplifies to

$$\begin{aligned}\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) =& a_3(a_3 \sin\alpha_1 \cos\alpha_2 \sin\alpha_2 \cos\theta_2 \cos\theta_3 \sin\theta_3 \\ &+ a_3 \cos\alpha_1 \sin^2\alpha_2 \cos\theta_3 \sin\theta_3 \\ &+ a_2 \cos\alpha_1 \sin\theta_3 \\ &+ a_3 \sin\alpha_1 \sin\alpha_2 \sin\theta_2 \cos^2\theta_3 \\ &+ a_2 \sin\alpha_1 \sin\alpha_2 \sin\theta_2 \cos\theta_3 \\ &- \sin\alpha_1 \sin^2\alpha_2 d_3 \cos\theta_2 \cos\theta_3 \\ &+ \cos\alpha_1 \cos\alpha_2 \sin\alpha_2 d_3 \cos\theta_3). \qquad (79)\end{aligned}$$

Here, $a_3 \neq 0$ for a nontrivial robot. As before, we shall consider only the remaining factor.

Case 1: $\alpha_1 = 0$. The first two joints are parallel in this case. The determinant simplifies to

$$\begin{aligned}\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) =& a_3 \sin^2\alpha_2 \cos\theta_3 \sin\theta_3 \\ &+ a_2 \sin\theta_3 \\ &+ \cos\alpha_2 \sin\alpha_2 d_3 \cos\theta_3. \qquad (80)\end{aligned}$$

The above expression is free of θ_2. Hence, the singular points are parallel lines at the roots of the above equation.

Case 2: $\alpha_2 = 0$. The last two joint axes are parallel. The determinant simplifies a great deal to

$$\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) = a_2 \cos\alpha_1 \sin\theta_3. \qquad (81)$$

The singularities of such manipulators are lines in the θ_2–θ_3 plane at $\theta_3 = 0 \pmod{\pi}$, and the nontrivial manipulators are clearly generic.

H. RRR Manipulator Singularities

This is the most difficult and yet common three-joint manipulator. The joint variables are θ_1, θ_2, and θ_3. The determinant of the Jacobian simplifies to the following expression:

$$\begin{aligned}\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) =& a_3(a_2 a_3 \sin\alpha_1 \cos\alpha_2 \sin\theta_2 \sin^2\theta_3 \\ &- a_3 d_2 \sin\alpha_1 \cos\alpha_2 \sin\alpha_2 \sin\theta_2 \cos\theta_3 \sin\theta_3 \\ &+ a_1 a_3 \cos\alpha_1 \cos\alpha_2 \sin\alpha_2 \cos\theta_2 \cos\theta_3 \sin\theta_3 \\ &- a_2 a_3 \sin\alpha_1 \cos\theta_2 \cos\theta_3 \sin\theta_3 \\ &- a_1 a_3 \sin\alpha_1 \sin^2\alpha_2 \cos\theta_3 \sin\theta_3 \\ &- a_2 d_3 \sin\alpha_1 \sin\alpha_2 \sin\theta_2 \sin\theta_3 \\ &- {a_2}^2 \sin\alpha_1 \cos\theta_2 \sin\theta_3 \\ &- a_1 a_2 \sin\alpha_1 \sin\theta_3 \\ &+ a_1 a_3 \cos\alpha_1 \sin\alpha_2 \sin\theta_2 \cos^2\theta_3 \\ &+ a_3 d_2 \sin\alpha_1 \sin\alpha_2 \cos\theta_2 \cos^2\theta_3 \\ &+ d_2 d_3 \sin\alpha_1 \sin^2\alpha_2 \sin\theta_2 \cos\theta_3 \\ &+ a_1 a_2 \cos\alpha_1 \sin\alpha_2 \sin\theta_2 \cos\theta_3 \\ &- a_1 d_3 \cos\alpha_1 \sin^2\alpha_2 \cos\theta_2 \cos\theta_3 \\ &+ a_2 d_2 \sin\alpha_1 \sin\alpha_2 \cos\theta_2 \cos\theta_3 \\ &- a_1 d_3 \sin\alpha_1 \cos\alpha_2 \sin\alpha_2 \cos\theta_3). \qquad (82)\end{aligned}$$

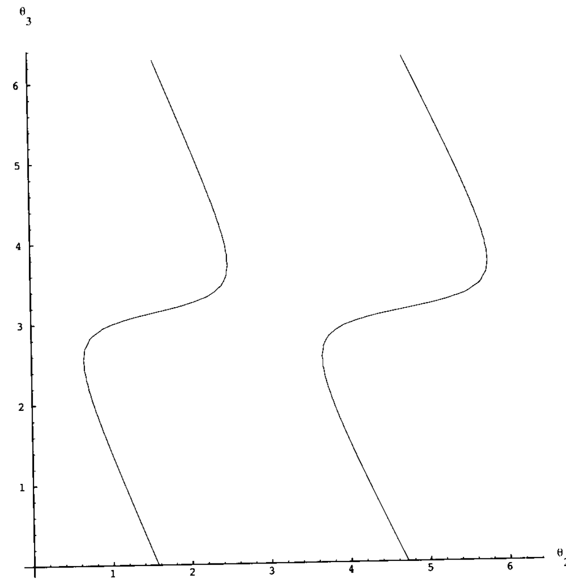

Fig. 4. Zero sets of $(\cos\alpha_2 \sin\theta_2 \sin\theta_3 - \cos\theta_2 \cos\theta_3 - d_3 \sin\alpha_2 \sin\theta_2 - a_2 \cos\theta_2)$ for $a_2 = 1.2$, $d_3 = 0.8$, and $\alpha_2 = 0.1$.

Note that a_3 is a factor. Therefore, any manipulator with $a_3 = 0$ will be singular. Since $a_3 \neq 0$ for a nontrivial manipulator, we may scale all length parameters by a_3 to obtain an expression for the determinant in nondimensional terms. For the rest of this subsection, we shall assume that all length parameters are scaled by a_3.

Case 1: $a_1 = 0$. This is a manipulator in which the first two joint axes intersect. Examples of robots possessing this characteristic are Unimation PUMA, Cincinnati-Milacron T3, Microbot TeachMover, and GE P50. The expression for the determinant can be simplified and factored as

$$\begin{aligned}\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) =& \sin\alpha_1(a_2 \sin\theta_3 - d_2 \sin\alpha_2 \cos\theta_3) \\ &(\cos\alpha_2 \sin\theta_2 \sin\theta_3 - \cos\theta_2 \cos\theta_3 \\ &- d_3 \sin\alpha_2 \sin\theta_2 - a_2 \cos\theta_2). \qquad (83)\end{aligned}$$

The expression for $\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t)$ has three factors. The first factor, $\sin\alpha_1$, indicates that, for a nontrivial manipulator, both a_1 and α_1 cannot be simultaneously zero, since this would make the first two joints coincide. The zero sets of the second factor $(a_2 \sin\theta_3 - d_2 \sin\alpha_2 \cos\theta_3)$ will always form straight lines in the θ_2–θ_3 plane, with two singular lines per 2π rad of θ_3. The zero set of the third factor, $(\cos\alpha_2 \sin\theta_2 \sin\theta_3 - \cos\theta_2 \cos\theta_3 - d_3 \sin\alpha_2 \sin\theta_2 - a_2 \cos\theta_2)$, defines a smooth manifold for almost all parameter values. These are shown as a pair of curved lines in Fig. 4.

An exception occurs if ${a_2}^2 + {d_3}^2 \tan^2\alpha_2 = 1$, when these lines intersect to yield an additional straight line as shown in Fig. 5.

The other exception is when $\cos\alpha_2 = 0, d_3 = 0$ and $|a_2| < 1$. Then the factor can be decomposed as $\cos\theta_2(\cos\theta_3 + a_2)$, producing an additional pair of lines at $\theta_3 = \cos^{-1} a_2$. Note that in this case there can be as many as six distinct singular lines in the configuration space (see Fig. 6).

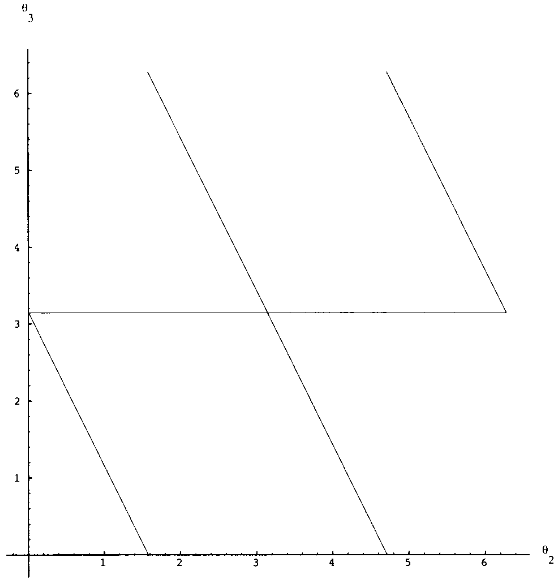

Fig. 5. Zero sets of $(\cos\alpha_2 \sin\theta_2 \sin\theta_3 - \cos\theta_2 \cos\theta_3 - d_3 \sin\alpha_2 \sin\theta_2 - a_2 \cos\theta_2)$ for $a_2 = 1$, $d_3 = 1$, and $\alpha_2 = 0$.

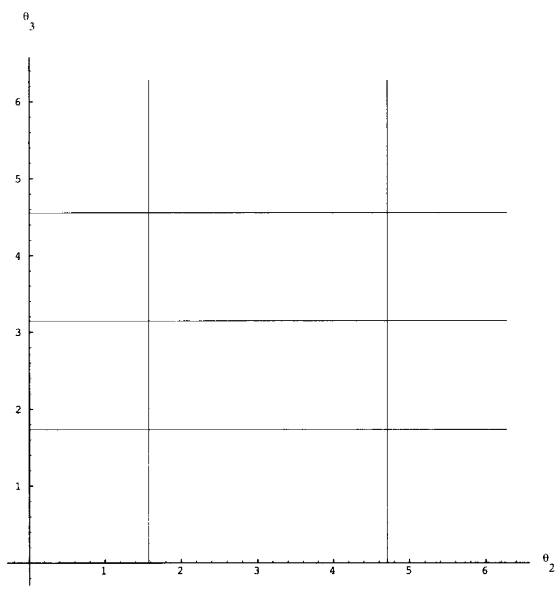

Fig. 6. Zero sets of $\det(D\kappa_t)$ for $a_2 = 0.1, d_2 = 0, d_3 = 0$, and $\alpha_2 = \pi/2$.

Case 2: $\alpha_1 = 0$. This is a manipulator with the first two joints parallel. The determinant of the Jacobian is

$$\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) = a_1 \sin\alpha_2 \cos\theta_3 (\cos\alpha_2 \cos\theta_2 \sin\theta_3 + \sin\theta_2 \cos\theta_3 + a_2 \sin\theta_2 - d_3 \sin\alpha_2 \cos\theta_2). \tag{84}$$

The singular sets are very similar to those of Case 1, since they are both caused by the same fact, i.e., the first two axes are coplanar. First, a_1 is a factor, as in Case 1, since $a_1 = 0$ indicates that the first two joints coincide. Second, $\sin\alpha_2$ is a factor since if it is also 0, all three joints would be parallel. Third, the factor $\cos\theta_3$ results in singular lines which occur at $\theta_3 = \pi/2 \pmod{\pi}$. Finally, the factor $(\cos\alpha_2 \cos\theta_2 \sin\theta_3 + \sin\theta_2 \cos\theta_3 + a_2 \sin\theta_2 - d_3 \sin\alpha_2 \cos\theta_2)$ results in singular sets similar to those of Figs. 4 and 5, except that they are shifted in θ_2 by $\pi/2$ rad.

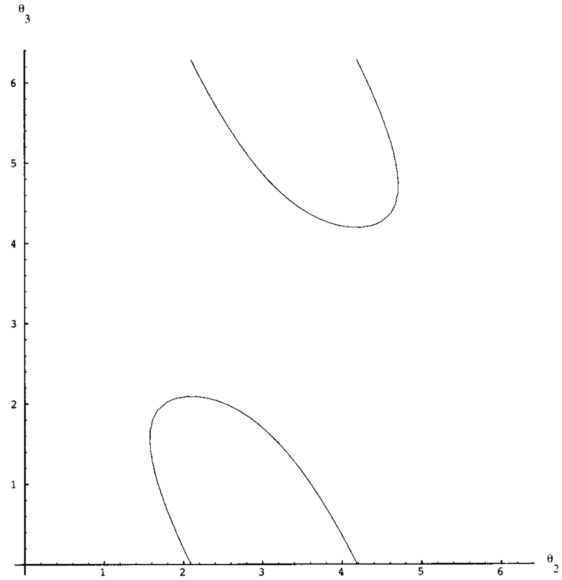

Fig. 7. Zero sets of $(\sin\theta_2 \sin\theta_3 - \cos\theta_2 \cos\theta_3 - a_2 \cos\theta_2 - a_1)$ for $a_1 = 1$ and $a_2 = 1$.

Case 3: $\alpha_2 = 0$. This is a manipulator with the last two joints parallel. This characteristic also exists in the Unimation PUMA, the Cincinnati-Milacron T3, the Microbot Teach-Mover, and the GE P50. The determinant of the Jacobian is

$$\det(D_{\boldsymbol{q}}\boldsymbol{\kappa}_t) = a_2 \sin\alpha_1 \sin\theta_3 (\sin\theta_2 \sin\theta_3 - \cos\theta_2 \cos\theta_3 - a_2 \cos\theta_2 - a_1). \tag{85}$$

In this case, a_2 is a factor, since if both a_2 and $\sin\alpha_2$ are zero, the last two joint axes are identical. The term $\sin\alpha_1$ is a factor since $\sin\alpha_1$ being 0 would make all three joint axes parallel. The factor $\sin\theta_3$ produces singular lines at $\theta_3 = 0 \pmod{\pi}$. This is caused by the last two links being completely extended or completely folded. Finally, the factor $(\sin\theta_2 \sin\theta_3 - \cos\theta_2 \cos\theta_3 - a_2 \cos\theta_2 - a_1)$ results in singular curves as shown in Fig. 7.

The different limbs of the singular curves touch each other if $\pm a_2 \pm a_1 = 1$ at the points $\theta_2 = 0 \pmod{\pi}$ and $\theta_3 = 0 \pmod{\pi}$ (see Fig. 8).

VII. Conclusions

Some results from differential topology were applied to the manipulator singularity problem. We showed that, for the class of generic robots, important information could be obtained about the ranks of the possible singularities and the differential topology of the set of singular points. We saw that, for

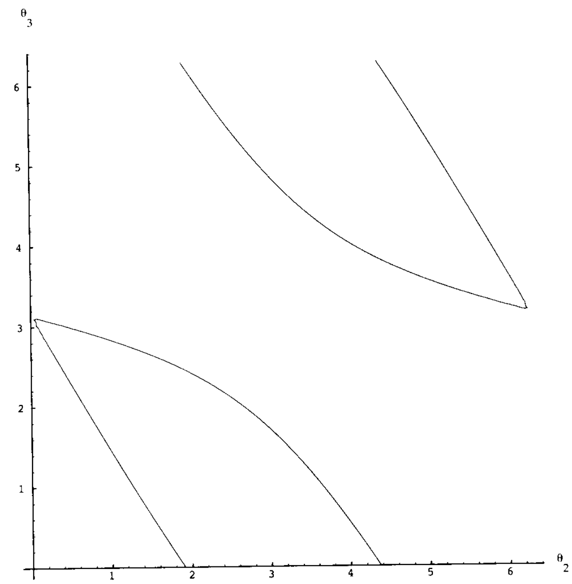

Fig. 8. Zero sets of $(\sin\theta_2 \sin\theta_3 - \cos\theta_2 \cos\theta_3 - a_2 \cos\theta_2 - a_1)$ for $a_1 = 0.5$ and $a_2 = 0.5$.

generic robots, the set of singular points of rank r are smooth manifolds in joint space of codimension $(j-r)(k-r)$, where j is the dimension of joint space and k is the dimension of task space. This result also allows us to automatically exclude certain low-rank singularities from occurring in generic robots. Hence, by designing a robot to be generic, we can eliminate singularities of low rank.

Since generic singularities are so well behaved, the question naturally arises: What types of robots are generic? Genericity was originally defined in terms of transversality of the map $D\kappa$ to a manifold collection in the space $\mathcal{L}$ of all $k \times j$ matrices. This definition is not well suited for determining the values of the kinematic parameters that cause the manipulator to be generic. An equivalent condition for genericity of three-joint manipulators in a three-dimensional task space was derived (Theorem 2). The condition uses the determinant of the Jacobian and its derivatives, and is amenable to symbolic computation. It is directly applicable to the common class of robots that can be decoupled into a translating part and an orienting part. All six-joint manipulators with a so-called "spherical wrist" are of this class. The condition for genericity can be used to analyze the singularities of such robot manipulators.

Two problems concerning manipulator singularities were considered in detail: orientation singularity, when the robot is used only for orienting the end-effector; and translation singularity, when the robot is used only for translating the end-effector. The results can be used for analyzing decoupled manipulators.

Orientation singularities were completely characterized for manipulators with an arbitrary number of joints. It was shown that the number of adjacent parallel joints was critical; the robot is generic if and only if no adjacent joints are parallel. Further, the singular points of generic manipulators form two-dimensional planes in joint space, parallel to the θ_1–θ_n plane, with the intermediate joint angles taking on values of $0 \pmod{\pi}$. Nongeneric manipulators were analyzed by grouping adjacent parallel joints into sets and reducing the manipulator to a generic one. It was shown that the singular points now form planes of higher dimension.

Translation singularities proved to be much more complex. The singularities of the eight possible topologies of three-joint manipulators were studied. Since the singular sets of generic manipulators form manifolds, they can be reliably computed for a given robot. We concentrate our effort on determining the values of the kinematic parameters that make a robot nongeneric and describe the singular sets for nongeneric robots. The manipulator genericity condition produces three nonlinear (trigonometric) equations in two joint variables and as many as seven kinematic parameters, which have to be simultaneously satisfied for the robot to become nongeneric. The joint variables were eliminated from the three equations to determine the values of the kinematic parameters that make the robot nongeneric. Using this procedure, the PPP, PPR, RPP, and PRP manipulators were analyzed, necessary and sufficient conditions on the kinematic parameter values for nongenericity were determined, and the nongeneric singularities described. Such a complete characterization proved to be difficult for RRP, RPR, PRR, and RRR manipulators. For these manipulators, the singularities were examined after assigning values to certain kinematic parameters.

Besides providing insight into the nature of the set of singular points, the results will aid in the kinematic design of robot manipulators. Designers can determine the types of singularities that will occur early in the design cycle. For example, the designer of a PRP manipulator for translation will know that if he or she makes the last two joint axes intersect ($a_2 = 0$), the robot will be nongeneric and will have singular points as described in Section VI. Further, by choosing the link twists judiciously (e.g. $\alpha_1 = 0$, $\alpha_2 = \frac{\pi}{2}$), certain singularities can be eliminated. The designer can also choose joint limits that exclude the singularities from the range of joint motions (e.g., $d_3 > 0$).

Appendix A
Kinematic Parameter Convention

The convention used for the kinematic parameters of robot manipulators is that of Paul [19] and is based on the work of Denavit and Hartenberg [4].

Each joint is assigned an axis. For a revolute joint, the axis is uniquely defined as the axis of revolution of the joint. In the case of prismatic joints, only the direction of the axis is uniquely defined, and the axis is taken to pass through any convenient point. The links are numbered starting from 0 for the base (fixed) link. The joints are numbered starting from 1 for the first joint.

- a_i The *length* of link i, defined as the shortest distance between the axis of joint i and the axis of joint $i+1$.
- α_i The *twist* of link i, defined as the angle between the axis of joint i and the axis of joint $i+1$.

d_i The *offset* of link i, defined as the distance along the axis of joint i, between the foot of the common normal to joint $i-1$ and the foot of the common normal to joint $i+1$.

θ_i The *angle* of joint i, defined as the angle between the common normal to joint $i-1$ and the common normal to joint $i+1$.

It is also customary to associate a coordinate frame with each link. Reference [19] describes the specification of the link coordinate frames. For our purposes, the most important feature of the coordinate frame of link i is that the unit vector $\boldsymbol{z}_i$ is aligned with the axis of joint $i+1$. If joint $i+1$ is revolute, a right-hand rotation about $\boldsymbol{z}_i$ corresponds to a positive rotation of θ_{i+1}; if the joint is prismatic, a displacement along $\boldsymbol{z}_i$ corresponds to increasing d_{i+1}. Also, $\boldsymbol{p}_i$ is the vector from a point on the axis of joint $i+1$ (usually taken to be the origin of link i's coordinate frame) to the origin of the end-effector coordinate frame.

Appendix B
Determining Nongenericity

In this appendix, we derive techniques for determining the conditions on the kinematic parameters of the manipulator, which result in nongenericity. The basic idea follows from manipulator genericity theorem for three-joint robots:

$$\text{nongeneric} \Leftrightarrow \exists \boldsymbol{q} \text{ such that } \det D_{\boldsymbol{q}}\boldsymbol{\kappa} = 0 \text{ and } D_{\boldsymbol{q}} \det D\boldsymbol{\kappa} = 0. \quad \text{(B1)}$$

One way a manipulator can be nongeneric is if it is always singular and (B1) is satisfied for all $\boldsymbol{q}$. Similarly, a manipulator that is never singular is generic. These are the trivial cases. In general, (B1) represents three equations in two joint variables and as many as seven kinematic parameters. This is because the first joint variable q_1 does not appear in the expression for $\det(D\boldsymbol{\kappa})$, since changing q_1 does not change the relative positions of the joint axes. Hence, $(\partial/\partial q_1)(\det D\boldsymbol{\kappa})$ is identically 0.

We determine the conditions on the parameters such that (B1) has a solution. This can be done by eliminating q_2 and q_3 from the equations. For revolute joints, however, trigonometric functions of q_i appear in the equations, and it is difficult to eliminate these in a systematic fashion. We shall next examine three subproblems in determining the link parameters such that (B1) can be satisfied.

The first two problems are of a specific nature and are discussed because they arise in our application and because they can be given necessary and sufficient conditions for having a solution. Problem P3 is applicable to the determination of satisfiability of any system of trigonometric polynomials. However, we can only get necessary but not sufficient conditions. The details of these problems can be found in [18].

Problem P1

Determine necessary and sufficient conditions on a, b, a', b' such that the following system has a solution:

$$ax + b = 0 \quad \text{(B2)}$$

$$a = 0 \quad \text{(B3)}$$

$$a'x + b' = 0. \quad \text{(B4)}$$

Such equations arise when $\det D_{\boldsymbol{q}}\boldsymbol{\kappa}$ has the form $a(q_3)q_2 + b(q_3)$.

Equations (B2), (B3), and (B4) have a common root if and only if

$$a = b = a' = b' = 0 \quad \text{(B5)}$$

and any x is a solution, or

$$a = b = 0 \quad \text{and} \quad a' \neq 0 \quad \text{(B6)}$$

and $x = -b'/a'$ is the solution.

Problem P2

Determine necessary and sufficient conditions on a, b, c, a', b', c' such that the following system has a solution:

$$a \sin t + b \cos t + c = 0 \quad \text{(B7)}$$

$$a \cos t - b \sin t = 0 \quad \text{(B8)}$$

$$a' \sin t + b' \cos t + c' = 0. \quad \text{(B9)}$$

Such equations arise when $\det D_{\boldsymbol{q}}\boldsymbol{\kappa}$ has the form $a(q_3) \sin q_2 + b(q_3) \cos q_2 + c(q_3)$.

Equations (B7), (B8), and (B9) have a common root if and only if

$$a = b = c = 0 \quad \text{and} \quad a'^2 + b'^2 - c'^2 \geq 0 \quad \text{(B10)}$$

or

$$c \neq 0 \quad \text{and} \quad a^2 + b^2 - c^2 = 0 \quad \text{and} \quad aa' + bb' - cc' = 0. \quad \text{(B11)}$$

Problem P3

Determine necessary conditions on the coefficients of a system of trigonometric polynomials, i.e., polynomials with indeterminates $\sin t$ and $\cos t$ such that they have a common solution.

It is clear that any such system can be written in the following canonical form:

$$\begin{aligned} p_0 &= (a_{0,0} + \cdots + a_{0,n_0} \cos^{n_0} t) \\ &\quad + \sin t (b_{0,0} + \cdots + b_{0,m_0} \cos^{m_0} t) = 0 \\ &\ddots \\ p_r &= (a_{r,0} + \cdots + a_{r,n_r} \cos^{n_r} t) \\ &\quad + \sin t (b_{r,0} + \cdots + b_{r,m_r} \cos^{m_r} t) = 0. \quad \text{(B12)} \end{aligned}$$

In this case, we can still use elimination theory [26], [28] with suitable modifications to deal with trigonometric polynomials. The details are provided in [18], where the following theorem is given.

Theorem 7: We can construct a system of polynomials $D_1, \ldots, D_h$ in the coefficients of a system of trigonometric polynomials (see (B12)) such that

$$D_1 = 0, \ldots, D_h = 0 \quad \text{(B13)}$$

is necessary for the system to have a solution.

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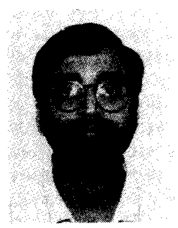

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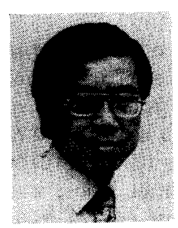

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