Theory and Applications of Graphs

Volume 3 | Issue 2 Article 2

2016

On Minimum Spanning Subgraphs of Graphs With Proper Connection Number 2

Zhenming Bi
zhenming.bi@wmich.edu

Gary Chartrand
Western Michigan University, gary.chartrand@wmich.edu

Garry L. Johns
glj@svsu.edu

Ping Zhang
ping.zhang@wmich.edu

Recommended Citation

Bi, Zhenming; Chartrand, Gary; Johns, Garry L.; and Zhang, Ping (2016) "On Minimum Spanning Subgraphs of Graphs With Proper Connection Number 2," *Theory and Applications of Graphs*: Vol. 3 : Iss. 2 , Article 2.
DOI: 10.20429/tag.2017.030202
Available at: https://digitalcommons.georgiasouthern.edu/tag/vol3/iss2/2

On Minimum Spanning Subgraphs of Graphs With Proper Connection Number 2

Cover Page Footnote

We greatly appreciate the valuable suggestions made by anonymous referees that resulted in an improved paper.

Abstract

An edge coloring of a connected graph G is a proper-path coloring if every two vertices of G are connected by a properly colored path. The minimum number of colors required of a proper-path coloring of G is called the proper connection number $\mathrm{pc}(G)$ of G. For a connected graph G with proper connection number 2, the minimum size of a connected spanning subgraph H of G with $\mathrm{pc}(H) = 2$ is denoted by $\mu(G)$. It is shown that if s and t are integers such that $t \geq s + 2 \geq 5$, then $\mu(K_{s,t}) = 2t - 2$. We also determine $\mu(G)$ for several classes of complete multipartite graphs G. In particular, it is shown that if $G = K_{n_1,n_2,\ldots,n_k}$ is a complete k-partite graph, where $k \geq 3$, $r = \sum_{i=1}^{k-1} n_i \geq 3$ and $t = n_k \geq r^2 + r$, then $\mu(G) = 2t - 2r + 2$.

1 Introduction

An edge coloring of a connected graph G is a *rainbow coloring* of G if every two vertices are connected by a path where no two edges are colored the same (a *rainbow path*). A connected graph G with a rainbow coloring is a *rainbow-connected graph*. The minimum number of colors required for a rainbow coloring of a connected graph G is its *rainbow connection number* $\mathrm{rc}(G)$. These concepts were introduced and studied by Chartrand, Johns, McKeon and Zhang in 2006 resulting in the 2008 paper [3]. Since then, much research has been done on these topics. In fact, a book [5] has been written on rainbow connection in graphs.

An edge coloring of a graph G is a *proper coloring* of G if every two adjacent edges of G are assigned distinct colors. The minimum number of colors required of a proper edge coloring of G is its *chromatic index*, denoted by $\chi'(G)$. An edge coloring of a connected graph G is called a *proper-path coloring* if every two vertices of G are connected by a properly colored path (a *proper path*). The minimum number of colors required of a proper-path coloring of G is called the *proper connection number* $\mathrm{pc}(G)$ of G. Therefore, $\mathrm{pc}(G) \leq \chi'(G)$ for every connected graph G. This concept was independently introduced and studied in [1, 2]. Recently, much research has been done on these concepts and, in fact, there is a dynamic survey on this topic due to Li and Magnant [4].

In [1] it was shown that every complete multipartite graph that is neither a complete graph nor a star has proper connection number 2. In fact, many connected graphs have been shown to have proper connection number 2. On the other hand, a graph can have a large proper connection number. For example, $\mathrm{pc}(T) = \Delta(T)$ for every tree T. Indeed, we have the following observation (see [1]).

Observation 1.1 *Let G be a nontrivial connected graph containing bridges. If the maximum number of bridges incident with a single vertex in G is b, then* $\mathrm{pc}(G) \geq b$.

While trees can have large proper connection number, this is not true for 2-connected graphs. The following theorem is due to Borozan, Fujita, Gerek, Magnant, Manoussakis, Montero and Tuza in [2].

Theorem 1.2 *If G is a 2-connected graph that is not complete, then* $\mathrm{pc}(G) \leq 3$.

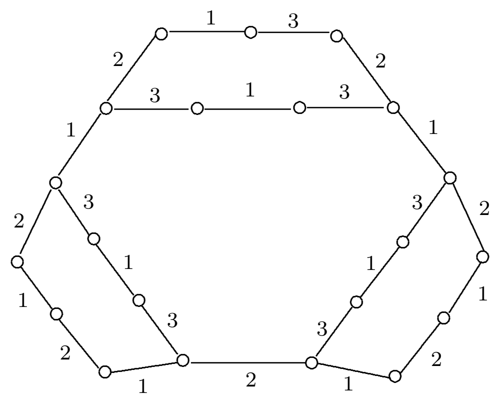

Figure 1: A 2-connected graph G with proper connection number 3

The upper bound 3 in Theorem 1.2 cannot be improved as it was shown in [2] that 2-connected graphs with proper connection number 3 exist. The 2-connected graph G in Figure 1 has proper connection number 3. A proper-path 3-coloring of G is shown in Figure 1.

The graph G in Figure 1 is, of course, also 2-edge connected. That is, there exist 2-edge connected graphs with proper connection number 3. While verifying that the proper connection number of the graph G of Figure 1 is 3 may require some time, there is a class of 2-edge connected graphs with proper connection number 3, the verification of which is more immediate.

A graph G is a *friendship graph* if every two distinct vertices of G have a unique common neighbor. A friendship graph has odd order $2k+1$ for some positive integer k. The friendship graph of order 3 is K_3 and the friendship graph of order $2k+1 \geq 5$ is obtained by identifying one vertex from each of k triangles.

Proposition 1.3 *If G is the friendship graph of order 7, then* $\text{pc}(G) = 3$.

Proof. Label the vertices of G as shown in Figure 2(a). First, the edge coloring c' where $c'(wx_1) = c'(wy_1) = c'(wz_1) = 1$, $c'(wx_2) = c'(wy_2) = c'(wz_2) = 2$ and $c'(x_1x_2) = c'(y_1y_2) = c'(z_1z_2) = 3$ is a proper-path 3-coloring of G and so $\text{pc}(G) \leq 3$.

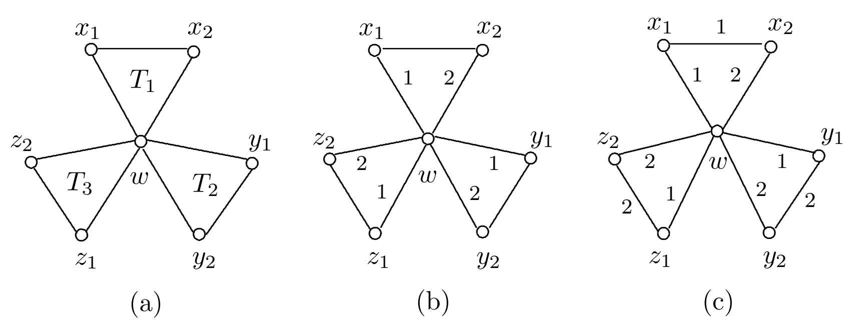

Figure 2: The friendship graph of order 7

Next, we show that $\text{pc}(G) \geq 3$. Assume, to the contrary, that there is a proper-path 2-coloring c of G with the colors 1 and 2. Denote the triangle with vertices x_1, x_2, w by T_1, the triangle with vertices y_1, y_2, w by T_2 and the triangle with vertices z_1, z_2, w by T_3, as shown in Figure 2(a). We claim that in each of the three triangles T_1, T_2, T_3, the two edges incident with w are assigned different colors by c.

Assume, to the contrary, that one of T_1, T_2, T_3 has its two edges incident with w colored the same, say $c(wx_1) = c(wx_2) = 1$. Since there is a proper $x_1 - y_1$ path and there is a proper $x_1 - z_1$ path, at least one edge incident with w in T_2 and T_3 is colored 2, say $c(wy_1) = c(wz_1) = 2$. Since G contains a proper $y_1 - z_1$ path, either wy_2 or wz_2 is colored 1, say $c(wy_2) = 1$. Since G contains a proper $x_1 - y_2$ path, $c(y_1y_2) = 1$. Thus, (y_1, w, z_2, z_1) is the unique proper $y_1 - z_1$ path, which implies that $c(wz_2) = 1$ and $c(z_1z_2) = 2$. However then, there is no proper $x_1 - z_2$ path, which produces a contradiction. Thus, as claimed, in each of three triangles T_1, T_2, T_3, the two edges incident with w are assigned different colors by c. Hence, we may assume that $c(wx_1) = c(wy_1) = c(wz_1) = 1$ and $c(wx_2) = c(wy_2) = c(wz_2) = 2$, as shown in Figure 2(b).

Since G contains a proper $x_1 - y_1$ path, it follows that $c(x_1x_2) = 1$ or $c(y_1y_2) = 1$, say the former. There is now only one possible proper $x_2 - z_2$ path, namely, (x_2, w, z_1, z_2), which implies that $c(z_1z_2) = 2$. However then, there is only one proper $x_2 - y_2$ path, namely (x_2, w, y_1, y_2), which implies that $c(y_1y_2) = 2$. Thus, we arrive at the coloring of G shown in Figure 2(c). However then, there is no proper $y_1 - z_1$ path, producing a contradiction. Hence, $\text{pc}(G) \geq 3$ and so $\text{pc}(G) = 3$. ∎

The proof of Proposition 1.4 and the proper-path 3-coloring of G described in this proof can be extended to give the following result.

Corollary 1.4 *Every friendship graph of order at least* 7 *has proper connection number* 3.

We have noted that many graphs have proper connection number 2. Certainly, many 2-connected graphs have proper connection number 2. If G is a noncomplete connected graph containing a connected spanning subgraph H such that $\text{pc}(H) = 2$, then $\text{pc}(G) = 2$ as well. In fact, every noncomplete connected supergraph F of H with $V(F) = V(H)$ also has proper connection number 2. This suggests the following concept. For a connected graph G with $\text{pc}(G) = 2$, let $\mu(G)$ denote the minimum size of a connected spanning subgraph H of G with $\text{pc}(H) = 2$. In this context, we refer to a spanning subgraph H of G with $\mu(G)$ edges as a *minimum spanning subgraph* of G. In what follows, we determine $\mu(G)$ for some familiar graphs G with $\text{pc}(G) = 2$.

2 Complete Bipartite Graphs

We now investigate this concept for complete multipartite graphs that are neither a star nor a complete graph, beginning with complete bipartite graphs $K_{s,t}$ with $2 \leq s \leq t$. Since $K_{s,t}$ contains a Hamiltonian path if and only if $t - s \leq 1$, it follows that this minimum size is $t + s - 1$ for these graphs. It therefore suffices to consider those graphs $K_{s,t}$ with $t - s \geq 2$. We begin with the graphs $K_{2,t}$ where $t \geq 4$.

Theorem 2.1 *For an integer* $t \geq 4$, $\mu(K_{2,t}) = 2t - 2$.

Proof. Let $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, \ldots, w_t\}$ be the two partite sets of $K_{2,t}$ and let $H = K_{2,t} - \{u_1w_2, u_2w_1\}$. Thus, the size of H is $2t-2$. We show (1) $\text{pc}(H) = 2$ and (2) H has the minimum size of a connected spanning subgraph of $K_{2,t}$ with proper connection number 2.

First, we show that $\text{pc}(H) = 2$. Define the edge coloring $c : E(H) \to \{1, 2\}$ by

$$c(u_iw_j) = \begin{cases} 1 & \text{if either } i = 1 \text{ and } j = 1 \text{ or } 3 \le j \le t-1 \text{ or } (i,j) = (2,t) \\ 2 & \text{if either } i = 2 \text{ and } 2 \le j \le t-1 \text{ or } (i,j) = (1,t). \end{cases}$$

To verify that c is a proper-path coloring of H, we show that every two vertices of H are connected by a proper path. Let x and y be two *nonadjacent* vertices of H.

⋆ If $\{x, y\} = \{u_1, u_2\}$, then (u_1, w_3, u_2) is a proper $u_1 - u_2$ path.

If $\{x, y\} = \{u_1, w_2\}$, then (u_1, w_t, u_2, w_2) is a proper $u_1 - w_2$ path.

If $\{x, y\} = \{u_2, w_1\}$, then (u_2, w_t, u_1, w_1) is a proper $u_2 - w_1$ path.

⋆ Let $x = w_i$ and $y = w_j$ where $1 \le i < j \le t$. First, suppose that $x \in \{w_1, w_2\}$.

If $x = w_1$ and $y = w_j$ for $2 \le j \le t-1$, then $(w_1, u_1, w_t, u_2, w_j)$ is a proper $w_1 - w_j$ path.

If $x = w_1$ and $y = w_t$, then (w_1, u_1, w_t) is a proper $w_1 - w_t$ path.

If $x = w_2$ and $y = w_j$ for $3 \le j \le t-1$, then $(w_2, u_2, w_t, u_1, w_j)$ is a proper $w_2 - w_j$ path.

If $x = w_2$ and $y = w_t$, then (w_2, u_2, w_t) is a proper $w_2 - w_t$ path.

Next, suppose that $x = w_i$ where $3 \le i \le t-1$.

If $y = w_j$ for $i+1 \le j \le t-1$, then $(w_i, u_1, w_t, u_2, w_j)$ is a proper $w_i - w_j$ path.

If $y = w_t$, then (w_i, u_1, w_t) is a proper $w_i - w_t$ path.

Hence, c is a proper-path coloring of H and so $\text{pc}(H) = 2$.

Next, we show that H has the minimum size of a connected spanning subgraph of $K_{2,t}$ with proper connection number 2. Suppose that there is a connected spanning subgraph F of $K_{2,t}$ having less than $2t-2$ edges for which $\text{pc}(F) = 2$. Necessarily, at least three vertices of W have degree 1 in F. It cannot occur that three vertices of degree 1 in F are adjacent to the same vertex of U, for otherwise $\text{pc}(F) \ge 3$ by Observation 1.1. Hence, we may assume that the vertices w_i, $1 \le i \le 3$, have degree 1 in F and that u_1w_1, u_1w_2, u_2w_3 are edges of F. Any proper-path coloring $c : E(F) \to \{1, 2\}$ of F must assign distinct colors to u_1w_1 and u_1w_2, say $c(u_1w_1) = 1$ and $c(u_1w_2) = 2$. We may assume, without loss of generality, that $c(u_2w_3) = 1$. Let P be a proper $w_1 - w_3$ path in F. Thus, $P = (w_1, u_1, w_j, u_2, w_3)$ for some integer $j \ge 4$. Since $c(w_1u_1) = c(u_2w_3) = 1$ and P is a proper path, it follows that $c(u_1w_j) \ne 1$ and $c(w_ju_2) \ne 1$ and so $c(u_1w_j) = c(w_ju_2) = 2$, which is a contradiction. ■

Next, we determine $\mu(K_{s,t})$ for all integers s and t with $t \ge s+2 \ge 5$.

Theorem 2.2 *If s and t are integers with $t \ge s+2 \ge 5$, then $\mu(K_{s,t}) = 2t-2$.*

DOI: 10.20429/tag.2017.030202

Proof. Let $G = K_{s,t}$ with partite sets

$$U = \{u_1, u_2, \ldots, u_s\} \text{ and } W = \{w_1, w_2, \ldots, w_t\}.$$

Write $t = sq + r$ for integers q and r where $0 \le r \le s-1$.

First, we construct a connected spanning subgraph H of G of size $2t-2$ such that $\text{pc}(H) = 2$. We consider two cases, according to whether $r = 0$ or $1 \le r \le s-1$.

Case 1. $r = 0$. Partition the set W into $q \ge 2$ subsets $W_1, W_2, \ldots, W_q$ where $|W_i| = s$ for $1 \le i \le q$ such that $W_1 = \{w_1, w_2, \ldots, w_s\}$ and $W_2 = \{w_{s+1}, w_{s+2}, \ldots, w_{2s}\}$.

- ⋆ First, suppose that $s = 3$. If $q = 2$, let $Q_1 = (u_1, w_3, u_2, w_4, u_1) = C_4$ and $Q_2 = (u_2, w_5, u_3, w_6, u_2) = C_4$. If $q \ge 3$, then for each integer i with $3 \le i \le q$, let $Q_i = C_6$ be a Hamiltonian cycle of the subgraph $G[U \cup W_i] = K_{3,3}$ of G induced by $U \cup W_i$. Let H be the spanning subgraph of G with

$$E(H) = E(Q_1) \cup E(Q_2) \cup \cdots \cup E(Q_q) \cup \{u_1w_1, u_1w_2\}. \tag{1}$$

- ⋆ Next, suppose that $s \ge 4$. Let $U' = U - \{u_1, u_2\}$ and $W' = W_1 - \{w_1, w_2\}$. Let $Q_1 = C_{2s-4}$ be a Hamiltonian cycle in the subgraph $G[U' \cup W'] = K_{s-2,s-2}$ of G induced by $U' \cup W'$ and for each integer i with $2 \le i \le q$, let $Q_i = C_{2s}$ be a Hamiltonian cycle in the subgraph $G[U \cup W_i] = K_{s,s}$ of G induced by $U \cup W_i$. Let H be the spanning subgraph of G whose edge set is described in (1). Hence, the size of H is $2t-2$.

It remains to show that $\text{pc}(H) = 2$. We first define an edge coloring $c : E(H) \to \{1, 2\}$. For $1 \le i \le q$, let c be a proper edge coloring of the even cycle Q_i. Also, let $c(u_1w_1) = 1$ and $c(u_1w_2) = 2$. Next, we show that c is a proper-path coloring of H. Let x and y be two *nonadjacent* vertices of H. If x and y lie on some even cycle Q_p for some $p \in \{1, 2, \ldots, q\}$, then there is a proper $x - y$ path in H. Thus, we may assume that x and y do not lie on a common even cycle Q_i where $1 \le i \le q$.

First, suppose that $s = 3$. Since x and y do not lie on a common even cycle Q_i where $1 \le i \le q$, it follows that $x, y \in U \cup \{w_1, w_2, \ldots, w_6\}$. Let H' be the subgraph of H induced by $U \cup \{w_1, w_2, \ldots, w_6\}$. We may furthermore assume that the edges of H' are colored as shown in Figure 3, where a solid edge and a dashed edge are colored differently. If $x, y \in U$, say $x = u_1$ and $y = u_3$, then $(u_1, w_3, u_2, w_5, u_3)$ is a proper $x - y$ path. Thus, we may assume that at least one of x and y does not belong to U, say $y \in W$. Then $y = w_i$ for some i with $i = 1, 2, \ldots, 6$. A case-by-case argument shows that there is a proper $x - y$ path in H' (and so in H). For example, (u_1, w_4, u_2, w_6) is a proper $u_1 - w_6$ path and $(w_1, u_1, w_4, u_2, w_6)$ is a proper $w_1 - w_6$ path.

Next, suppose that $s \ge 4$. Since every two distinct vertices of U lie on the even cycle Q_2, it follows that at least one of x and y does not belong to U, say $y \in W$. We consider two subcases, according to whether $x \in U$ or $x \in W$.

Subcase 1.1. $x \in U$. First, suppose that $y = w_i$ for $3 \le i \le s$. Then $x \in \{u_1, u_2\}$. Since x lies on Q_2 and y is adjacent to a vertex z on Q_2, there are two $x - y$ paths passing through z, one of which is proper. Next, suppose that $y \in \{w_1, w_2\}$. Since x and y are nonadjacent, $x = u_i$ for some i with $2 \le i \le s$. Then x lies on Q_2 and y is adjacent to the vertex u_1 on Q_2. Then there are two $x - y$ paths passing through u_1, one of which is proper.

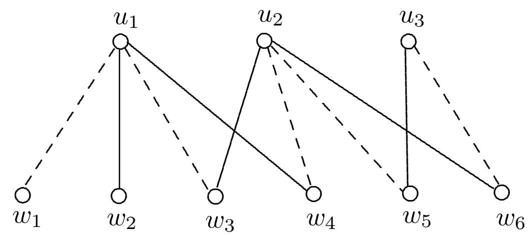

Figure 3: A subgraph H' in H when $s = 3$

Subcase 1.2. $x \in W$. We may assume that $x = w_i$ and $y = w_j$ where $1 \leq i < j$. First, suppose that $i \in \{1, 2\}$. If $(i, j) = (1, 2)$ or $j \geq s + 1$, then (w_i, u_1, w_j) is a proper $w_i - w_j$ path in H. Next, suppose that $3 \leq j \leq s$. Then x is adjacent to u_1 of Q_2 and y is adjacent to a vertex z of Q_2 (where it is possible that $u_1 = z$). There are two $x - y$ paths passing through z, one of which is proper. Next, suppose that $i \geq 3$. Hence, $x \in W_a$ and $y \in W_b$ where $1 \leq a < b \leq q$. Then x is adjacent to a vertex z on Q_b and y lies on Q_b. Thus, there are two $x - y$ paths passing through z, one of which is proper.

Case 2. $1 \leq r \leq s - 1$. Partition the set W into $q + 1$ subsets $W_0, W_1, W_2, \ldots, W_q$ where $|W_0| = r$ and $|W_i| = s$ for $1 \leq i \leq q$ such that $W_0 = \{w_1, w_2, \ldots, w_r\}$ and $W_1 = \{w_{r+1}, w_{r+2}, \ldots, w_{r+s}\}$.

- ⋆ For $r = 1$, let $W' = W_1 - \{w_2\}$ and $U' = U - \{u_1\}$. Now, let $Q_1 = C_{2s-2}$ be a Hamiltonian cycle in the subgraph $G[U' \cup W'] = K_{s-1,s-1}$ of G induced by $U' \cup W'$ and for $2 \leq i \leq q$ if $q \geq 2$, let $Q_i = C_{2s}$ be a Hamiltonian cycle in the subgraph $G[U \cup W_i] = K_{s,s}$ of G induced by $U \cup W_i$. Let H be the spanning subgraph of G whose edge set is described in (1).

- ⋆ For $r = 2$, let $Q_i = C_{2s}$ be a Hamiltonian cycle in the subgraph $G[U \cup W_i] = K_{s,s}$ for $1 \leq i \leq q$. Let H be the spanning subgraph of G whose edge set is described in (1).

- ⋆ For $r = 3$, let $Q_0 = (u_1, w_3, u_2, w_4, u_1) = C_4$. Let $U' = U - \{u_1\}$ and $W' = W_1 - \{w_4\}$. Now let $Q_1 = C_{2s-2}$ be a Hamiltonian cycle in the subgraph $G[U' \cup W'] = K_{s-1,s-1}$ of G induced by $U' \cup W'$. For each integer i with $2 \leq i \leq q$, let $Q_i = C_{2s}$ be a Hamiltonian cycle in the subgraph $G[U \cup W_i] = K_{s,s}$ of G induced by $U \cup W_i$. Let H be the spanning subgraph of G whose edge set is

$$E(H) = E(Q_0) \cup E(Q_1) \cup \cdots \cup E(Q_q) \cup \{u_1w_1, u_1w_2\}. \tag{2}$$

- ⋆ For $r \geq 4$, let $W' = W_0 - \{w_1, w_2\}$ and $U' = \{u_1, u_2, \ldots, v_{r-2}\}$. Now, let $Q_0 = C_{2r-4}$ be a Hamiltonian cycle in the subgraph $G[U' \cup W'] = K_{r-2,r-2}$ of G induced by $U' \cup W'$ and for $1 \leq i \leq q$, let $Q_i = C_{2s}$ be a Hamiltonian cycle in the subgraph $G[U \cup W_i] = K_{s,s}$ of G induced by $U \cup W_i$. Let H be the spanning subgraph of G whose edge set is described in (2).

In each case, the size of H is $2t - 2$. We define an edge coloring $c : E(H) \to \{1, 2\}$ such that each even cycle Q_i ($1 \leq i \leq q$ or $0 \leq i \leq q$) is properly colored and $c(u_1w_1) = 1$ and

$c(u_1w_2) = 2$. An argument similar to that in Case 1 shows that c is a proper-path coloring of H and so $\mathrm{pc}(H) = 2$.

Next, we show that the minimum size of a connected spanning subgraph of $K_{s,t}$ with proper connection number 2 is $2t-2$. Suppose that there is a connected spanning subgraph F of $K_{s,t}$ having less than $2t-2$ edges but $\mathrm{pc}(F) = 2$. Necessarily, at least three vertices of W have degree 1 in F. It cannot occur that three vertices of degree 1 in F are adjacent to the same vertex of U, for otherwise, $\mathrm{pc}(F) \geq 3$ by Observation 1.1. We may assume that the vertices w_i, $1 \leq i \leq 3$, have degree 1 in F. Suppose that e_1, e_2, e_3 are the three pendant edges that are incident with w_1, w_2 or w_3, respectively. Any proper-path coloring $c : E(F) \to \{1, 2\}$ of F must assign the same color to two of e_1, e_2, e_3, say $c(e_1) = c(e_2) = 1$. Thus, e_1 and e_2 cannot be adjacent, say $e_1 = u_1w_1$ and $e_2 = u_2w_2$. Let P be a proper $w_1 - w_2$ path in F, say $P = (w_1, u_1, w_i, \ldots, w_j, u_2, w_2)$ for some integers $i, j \in \{3, 4, \ldots, t\}$. Let $P' = P - \{w_1, u_1, u_2, w_2\}$ be the subpath of P. Then P' has even length and the edges of P' are colored alternately 2 and 1. However then, $c(w_ju_2) = c(u_2w_2) = 1$, which is a contradiction. Therefore, the minimum size of a connected spanning subgraph of $K_{s,t}$ with proper connection number 2 is $2t-2$. ■

Combining Theorems 2.1 and 2.2, we obtain the following.

Corollary 2.3 *If s and t are integers with $t \geq s+2 \geq 4$, then $\mu(K_{s,t}) = 2t-2$.*

The proof of Theorem 2.2 gives rise to the following useful corollary.

Corollary 2.4 *Let F be a connected spanning subgraph of the complete bipartite graph $K_{s,t}$ with partite sets U and W, where $|U| = s$, $|W| = t$ and $1 \leq s \leq t$ and $t \geq 3$. If at least three vertices of W have degree 1 in F, then $\mathrm{pc}(F) \geq 3$.*

3 Complete Multipartite Graphs

We now look at complete k-partite graphs for $k \geq 3$. Let $G = K_{n_1,n_2,\ldots,n_k}$ be the complete k-partite graph of order $n = \sum_{i=1}^{k} n_i$, where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ and $k \geq 3$. Since G contains a Hamiltonian path if and only if $n_k \leq \sum_{i=1}^{k-1} n_i + 1$, it follows that $\mu(G) = n-1$ in this case. Thus, it suffices to consider the case when $n_k \geq \sum_{i=1}^{k-1} n_i + 2$. We begin by establishing an upper bound for $\mu(G)$ for such complete k-partite graphs G.

Proposition 3.1 *Let $G = K_{n_1,n_2,\ldots,n_k}$ be the complete k-partite graph of order $n = \sum_{i=1}^{k} n_i$, where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ and $k \geq 3$. If $n_k \geq \sum_{i=1}^{k-1} n_i + 2$, then $\mu(G) \leq 2n_k - 2$.*

Proof. Let $V_1, V_2, \ldots, V_k$ be the partite sets of G with $|V_i| = n_i$ for $1 \leq i \leq k$ and let F be the complete bipartite graph with partite sets $V_1 \cup V_2 \cup \cdots \cup V_{k-1}$ and V_k. It follows by the proofs of Theorems 2.1 and 2.2 that F contains a connected spanning subgraph H of size $2n_k - 2$ such that $\mathrm{pc}(H) = 2$. Since F is a connected spanning subgraph of G, it follows that G contains a connected spanning subgraph H of size $2n_k - 2$ such that $\mathrm{pc}(H) = 2$ and so $\mu(G) \leq 2n_k - 2$. ■

Next, we describe a class of complete k-partite graphs where $k \geq 3$ such that the upper bound described in Proposition 3.1 is attainable.

Proposition 3.2 *Let $G = K_{n_1,n_2,\ldots,n_k}$ be the complete k-partite graph of order $n = \sum_{i=1}^{k} n_i$, where $1 \le n_1 \le n_2 \le \cdots \le n_k$ and $k \ge 3$. If $n_k = \sum_{i=1}^{k-1} n_i + 2$, then $\mu(G) = 2n_k - 2$.*

Proof. Since G has order n with $n_k = \sum_{i=1}^{k-1} n_i + 2$, it follows by Proposition 3.1 that $\mu(G) \le 2n_k - 2$. Every connected spanning subgraph H of G of size $2n_k - 3$ is a tree. Since G contains no Hamiltonian path, it follows that $\mathrm{pc}(H) = \Delta(H) \ge 3$. Thus, there is no connected spanning subgraph of size $2n_k - 3$ having proper connection number 2. Hence, $\mu(G) \ge 2n_k - 2$ and so $\mu(G) = 2n_k - 2$. ■

We now investigate $\mu(G)$ for the complete 3-partite graphs G. To simplify the notation, let $G = K_{r,s,t}$ be a complete 3-partite graph where $1 \le r \le s \le t$ and $t \ge r + s + 2$. We show that the upper bound described in Proposition 3.1 is attainable when $r = s = 1$ and $t \ge 4$. First, we state a useful observation.

Observation 3.3 *If F is a connected spanning subgraph of a graph G such that $\mathrm{pc}(F) = \mathrm{pc}(G) = 2$, then $\mu(G) \le \mu(F)$.*

Proposition 3.4 *For each integer $t \ge 4$, $\mu(K_{1,1,t}) = 2t - 2$.*

Proof. Since $K_{2,t} \subseteq K_{1,1,t}$, it follows by Observation 3.3 and and Theorem 2.1 that

$$\mu(K_{1,1,t}) \le \mu(K_{2,t}) \le 2t - 2.$$

Thus, it suffices to show that $\mu(K_{1,1,t}) \ge 2t - 2$. Assume, to the contrary, that there is a connected spanning subgraph F of $K_{1,1,t}$ of size $m < 2t - 2$ with $\mathrm{pc}(F) = 2$. Thus, at least three vertices of W have degree 1 in F. Let the partite sets of $K_{1,1,t}$ be $\{u\}$, $\{v\}$ and $W = \{w_1, w_2, \ldots, w_t\}$.

First, suppose that exactly three vertices of W have degree 1 in F. Thus, $m \ge 2t - 3$. Since $m < 2t - 2$, it follows that $m = 2t - 3$ and so $uv \notin E(F)$. However then, F is a bipartite subgraph of $K_{2,t}$ with partite sets $\{u, v\}$ and W. Since F has three vertices of degree 1 in W, it then follows by Corollary 2.4 that $\mathrm{pc}(F) \ge 3$, which is impossible.

Next, suppose that at least five vertices of W have degree 1 in F. Then either u or v is incident with at least three bridges in F and so $\mathrm{pc}(F) \ge 3$ by Observation 1.1, which is a contradiction. Thus, exactly four vertices of W have degree 1 in F and each of u and v is incident with exactly two of these four vertices. Therefore, we may assume that the vertices w_i $(1 \le i \le 4)$ have degree 1 in F and uw_1, uw_2, vw_3, vw_4 are edges of F. Furthermore, if $uv \notin E(F)$, then F is a bipartite subgraph of $K_{2,t}$ with partite sets $\{u, v\}$ and W. Again, since F has four vertices of degree 1 in W, it then follows that $\mathrm{pc}(F) \ge 3$ by Corollary 2.4, which is impossible. Thus, we assume that $uv \in E(F)$.

Any proper-path coloring $c : E(F) \to \{1, 2\}$ of F must assign distinct colors to uw_1 and uw_2, say $c(uw_1) = 1$ and $c(uw_2) = 2$ and distinct colors to vw_3 and vw_4, say $c(vw_3) = 1$ and $c(vw_4) = 2$. Assume, without loss of generality, that $c(uv) = 1$. Any proper $w_1 - w_3$ path P in F must begin with w_1, u and terminate with v, w_3. Necessarily, the only other vertex on P is w_j for some j with $5 \le j \le t$, that is, $P = (w_1, u, w_j, v, w_3)$. Since $c(w_1u) = c(vw_3) = 1$ it follows that $c(uw_j) = c(vw_j) = 2$, which is a contradiction. ■

Next, we show that the upper bound described in Proposition 3.1 can be strict if t is sufficiently large.

DOI: 10.20429/tag.2017.030202

Proposition 3.5 *For an integer $t \geq 11$, if $G \in \{K_{1,2,t}, K_{1,1,1,t}\}$, then*

$$\mu(G) = 2t - 4.$$

Proof. Since $K_{1,2,t}$ is a spanning subgraph of $K_{1,1,1,t}$, it follows by Observation 3.3 that $\mu(K_{1,1,1,t}) \leq \mu(K_{1,2,t})$. Thus, it suffices to show that $\mu(K_{1,2,t}) \leq 2t-4$ and $\mu(K_{1,1,1,t}) \geq 2t-4$ for each integer $t \geq 11$.

We first show that $\mu(K_{1,2,t}) \leq 2t - 4$. Let $G = K_{1,2,t}$, whose partite sets are $\{x, z\}$, $\{y\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. Let F be the connected spanning subgraph of G obtained from the path $P_3 = (x, y, z)$ by joining w_1 and w_2 to x, w_3 and w_4 to y, w_5 and w_6 to z, w_7 to x and z, w_i to x and y for $8 \leq i \leq t-2$, w_i to y and z for $i = t-1, t$. The graph F is shown in Figure 4 for $t = 11$. Since (i) there are six vertices of W of degree 1 in F, namely w_i for $1 \leq i \leq 6$, (ii) the remaining $t-6$ vertices of W have degree 2 in F and (iii) there are exactly two edges in $G[\{x, y, z\}]$, it follows that the size of F is $2t - 4$.

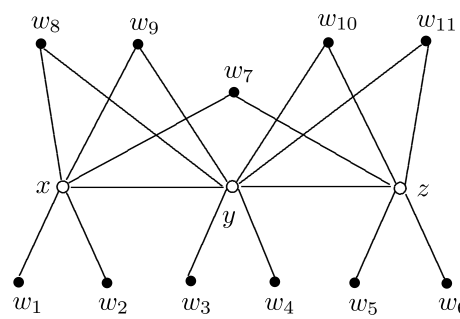

Figure 4: A connected spanning subgraph F of $K_{1,2,11}$ for $t = 11$

Since there is a proper-path 2-edge coloring of F, it follows that $\text{pc}(F) = 2$. Such a proper-path 2-edge coloring of F is shown in Figure 5 for $t = 11$. If $t \geq 12$, then (1) color the two edges $w_i x$ and $w_i y$ ($10 \leq i \leq t-2$) as $w_8 x$ and $w_8 y$, (2) color the two edges $w_{t-1} y$ and $w_{t-1} z$ as $w_{10} y$ and $w_{10} z$ and (3) color the two edges $w_t y$ and $w_t z$ as $w_{11} y$ and $w_{11} z$. It can be verified that for every two nonadjacent vertices u and v of F, there is a proper $u - v$ path in F. For example, if $12 \leq i \leq t-2$, then (w_8, x, w_9, y, w_i) is a proper $w_8 - w_i$ path in F.

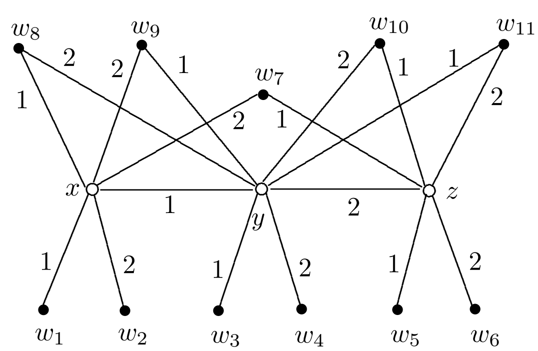

Figure 5: A proper-path 2-edge coloring of F for $t = 11$

Next, we show that $\mu(K_{1,1,1,t}) \geq 2t-4$. Let $G = K_{1,1,1,t}$ whose partite sets are $\{x\}$, $\{y\}$, $\{z\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. Assume, to the contrary, that $\mu(G) \leq 2t-5$. Then there is a connected spanning subgraph F of G of size $m \leq 2t-5$ with $\text{pc}(F) = 2$. We consider three cases, according to the number of edges in the subgraph $G[\{x,y,z\}]$ induced by $\{x,y,z\}$.

Case 1. *The subgraph $G[\{x,y,z\}]$ contains no edge.* Then F is a connected spanning subgraph of $K_{3,t}$ with $\text{pc}(F) = 2$. Since $\mu(K_{3,t}) = 2t-2$, the size of F is at least $2t-2$, which is impossible.

Case 2. *The subgraph $G[\{x,y,z\}]$ contains exactly one edge, say $e = xy$ is an edge of $G[\{x,y,z\}]$.* Since the size of F is at most $2t-5$, at least six vertices of W have degree 1 in F. It cannot occur that three vertices of degree 1 in W are adjacent to a vertex in F, for otherwise, $\text{pc}(F) \geq 3$ by Observation 1.1. Thus, we may assume, without loss of generality, that the vertices w_i $(1 \leq i \leq 6)$ have degree 1 in F and $xw_1, xw_2, yw_3, yw_4, zw_5, zw_6$. Any proper-path coloring $c : E(F) \to \{1,2\}$ of F must assign distinct colors to the two adjacent pendant edges in F, say $c(xw_1) = c(yw_3) = 1$ and $c(xw_2) = c(yw_4) = 2$. Then a proper $w_1 - w_3$ path must be (w_1, x, y, w_3) and so $c(xy) = 2$; while a proper $w_2 - w_4$ path must be (w_2, x, y, w_4) and so $c(xy) = 1$, which is impossible.

Case 3. *The subgraph $G[\{x,y,z\}]$ contains at least two edges.* Then at least seven vertices of W have degree 1 in F. Thus, three vertices of degree 1 in W are adjacent to a vertex in F and so $\text{pc}(F) \geq 3$ by Observation 1.1, which is impossible.

Hence, $\mu(K_{1,1,1,t}) \geq 2t-4$. Since $2t-4 \leq \mu(K_{1,1,1,t}) \leq \mu(K_{1,2,t}) \leq 2t-4$, it follows that $\mu(K_{1,1,1,t}) = \mu(K_{1,2,t}) = 2t-4$. ■

Proposition 3.5 is a special case of a more general result. Let $G = K_{n_1,n_2,\ldots,n_k}$ be the complete k-partite graph, where $k \geq 3$, let $r = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. We now present a formula for $\mu(G)$ when $r \geq 3$ and t is sufficiently large compared to r.

Theorem 3.6 *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph where*

$$k \geq 3,\ r = \sum_{i=1}^{k-1} n_i \geq 3 \text{ and } t = n_k.$$

If $t \geq r^2 + r$, then

$$\mu(G) = 2t - 2r + 2.$$

Proof. Denote the partite sets of G by $V_1, V_2, \ldots, V_k$, where $|V_i| = n_i$ for $1 \leq i \leq k$. First, we show that $\mu(G) \geq 2t-2r+2$. Let H be a minimum connected spanning subgraph of G with $\text{pc}(H) = 2$. Certainly, every vertex of V_k has degree at least 1 in H. Also, at most $2r$ vertices of V_k have degree 1, for otherwise, there are vertices of H incident with three or more pendant edges and so $\text{pc}(H) \geq 3$ by Observation 1.1, contradicting the fact that $\text{pc}(H) = 2$. Thus, at most two vertices of V_k of degree 1 can be adjacent to the same vertex of $\cup_{i=1}^{k-1} V_i$. If any two vertices w', w'' of degree 1 in V_k are incident with edges of the same color, then $H[\cup_{i=1}^{k-1} V_i]$ must contain an edge of the other color. Thus, the size of H must be at least

$$\left(\sum_{w \in V_k} \deg_H w\right) + 2 \geq 2r + 2(t-2r) + 2 = 2t - 2r + 2$$

and so $\mu(G) \ge 2t - 2r + 2$.

Next, we show that $\mu(G) \le 2t - 2r + 2$. To verify this, we show that there exists a connected spanning subgraph F of size $2t - 2r + 2$ in G such that $\text{pc}(F) = 2$. Let $\cup_{i=1}^{k-1} V_i = \{v_1, v_2, \ldots, v_r\}$, where the only edges of $F[\cup_{i=1}^{k-1} V_i]$ are $v_{r-2}v_{r-1}$ and $v_{r-1}v_r$. Let $V_k = \{w_1, w_2, \ldots, w_t\}$ where $\deg_F w_i = 1$ for $1 \le i \le 2r$. In particular, for $1 \le i \le r$, w_{2i-1} and w_{2i} are adjacent to v_i. Next, we define an edge coloring $c : E(F) \to \{1, 2\}$ as follows. Let $c(v_{r-2}v_{r-1}) = 1$, $c(v_{r-1}v_r) = 2$, $c(v_i w_{2i-1}) = 1$ and $c(v_i w_{2i}) = 2$ for $1 \le i \le r$. There are $\binom{r}{2}$ distinct pairs of vertices in $\{v_1, v_2, \ldots, v_r\}$. For each such pair $\{v_a, v_b\}$, two vertices w' and w'' in $\{w_{2r+1}, w_{2r+2}, \ldots, w_{r^2+r}\}$ are selected, which are both joined to v_a and v_b, where $c(v_a w') = c(v_b w'') = 1$ and $c(v_a w'') = c(v_b w') = 2$. If $t > r^2 + r$, then each vertex in $\{w_{r^2+r+1}, w_{r^2+r+2}, \ldots, w_t\}$ is joined to two vertices in $\{v_1, v_2, \ldots, v_r\}$, where one edge is colored 1 and the other edge is colored 2. This completes the construction of F and the edge coloring c of F. The subgraph F and the coloring c of F are illustrated in Figure 6 for $K_{2,2,20}$, where each solid edge is colored 1 and each dashed edge is colored 2.

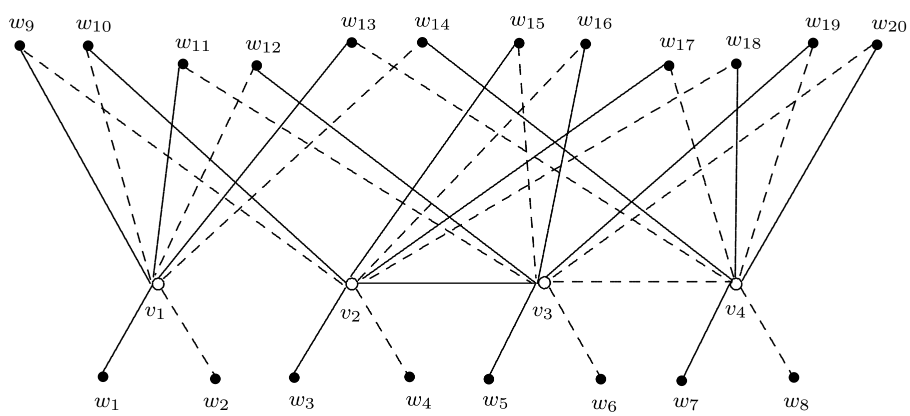

Figure 6: A subgraph F of $K_{2,2,20}$ and an edge coloring of F

We now show that c is a proper-path 2-coloring of F. It remains to show that every two nonadjacent vertices x and y of F are connected by a proper path. This is obvious if there is a proper path of length 2 connecting x and y. Thus, we consider the other possibilities, namely either $x, y \in V_k$ or exactly one of x and y belongs to V_k. First, we consider the situation when $x, y \in V_k$. There are three cases.

Case 1. $\{x, y\} = \{w_i, w_j\}$ *where* $1 \le i \ne j \le 2r$. Suppose that $w_i v_p$ and $w_j v_q$ are edges in F where $1 \le p < q \le r$. We consider three subcases, according to whether $p < q \le r - 3$ or $p, q \ge r - 2$ or $p \le r - 3$ and $q \ge r - 2$.

Subcase 1.1. $p < q \le r - 3$. If $c(w_i v_p) \ne c(w_j v_q)$, say $c(w_i v_p) = 1$ and $c(w_j v_q) = 2$, then there exists a vertex w_s where $s > 2r$ such that $v_p w_s, w_s v_q \in E(F)$ and $c(v_p w_s) = 2$ and $c(w_s v_q) = 1$. Then $(w_i, v_p, w_s, v_q, w_j)$ is a proper $w_i - w_j$ path in F. If $c(w_i v_p) = c(w_j v_q)$, say $c(w_i v_p) = c(w_j v_q) = 1$, then there are vertices w_a and w_b where $a, b > 2r$ such that $v_p w_a, w_a v_{r-1}, v_q w_b, w_b v_r \in E(F)$ with $c(w_a v_{r-1}) = c(w_b v_r) = 1$ and $c(w_a v_p) = c(w_b v_q) = 2$. Thus, $(w_i, v_p, w_a, v_{r-1}, v_r, w_b, v_q, w_j)$ is a proper $w_i - w_j$ path in F.

Subcase 1.2. $p, q \in \{r - 2, r - 1, r\}$. First, assume that $w_i v_{r-2}, w_j v_{r-1} \in E(F)$.

⋆ If w_iv_{r-2} and $v_{r-1}w_j$ are both colored 2, then $(w_i, v_{r-2}, v_{r-1}, w_j)$ is a proper $w_i - w_j$ path in F.

⋆ If w_iv_{r-2} and $v_{r-1}w_j$ are both colored 1, then there is a vertex w_a with $a > 2r$ such that $v_{r-2}w_a, w_av_r \in E(F)$ such that $c(v_{r-2}w_a) = 2$ and $c(w_av_r) = 1$. Hence, $(w_i, v_{r-2}, w_a, v_r, v_{r-1}, w_j)$ is a proper $w_i - w_j$ path in F.

⋆ If w_iv_{r-2} and $v_{r-1}w_j$ are colored differently, say $c(w_iv_{r-2}) = 1$ and $c(v_{r-1}w_j) = 2$, then there is a vertex w_a with $a > 2r$ such that $v_{r-2}w_a, w_av_{r-1} \in E(F)$ such that $c(v_{r-2}w_a) = 2$ and $c(w_av_{r-1}) = 1$. Hence, $(w_i, v_{r-2}, w_a, v_{r-1}, w_j)$ is a proper $w_i - w_j$ path in F.

An argument for the situation where $w_iv_{r-1}, w_jv_r \in E(F)$ is similar. Next, assume that $w_iv_{r-2}, w_jv_r \in E(F)$.

⋆ If $c(w_iv_{r-2}) = 2$ and $c(v_rw_j) = 1$, then $(w_i, v_{r-2}, v_{r-1}, v_r, w_j)$ is a proper $w_i - w_j$ path in F.

⋆ If $c(w_iv_{r-2}) = 1$ and $c(v_rw_j) = 2$, then there is a vertex w_a with $a > 2r$ such that $v_{r-2}w_a, w_av_r \in E(F)$ such that $c(v_{r-2}w_a) = 2$ and $c(w_av_r) = 1$. Hence, $(w_i,\ v_{r-2},\ w_a,\ v_r,\ w_j)$ is a proper $w_i - w_j$ path in F.

⋆ If w_iv_{r-2} and v_rw_j are colored the same, say $c(w_iv_{r-2}) = c(v_rw_j) = 1$, then there is a vertex w_a with $a > 2r$ such that $v_{r-2}w_a, w_av_{r-1} \in E(F)$ such that $c(v_{r-2}w_a) = 2$ and $c(w_av_{r-1}) = 1$. Hence, $(w_i, v_{r-2}, w_a, v_{r-1}, v_r, w_j)$ is a proper $w_i - w_j$ path in F.

Subcase 1.3. *$p \le r - 3$ and $q \in \{r - 2, r - 1, r\}$.* Suppose that $q = r - 2$ and so $w_iv_p, w_jv_{r-2} \in E(F)$.

⋆ If w_iv_p and w_jv_{r-2} are colored differently, say $c(w_iv_p) = 1$ and $c(w_jv_{r-2}) = 2$, then there is a vertex w_a with $a > 2r$ such that $c(w_av_p) = 2$ and $c(w_av_{r-2}) = 1$. Hence, $(w_i, v_p, w_a, v_{r-2}, w_j)$ is a proper $w_i - w_j$ path in F.

⋆ If $c(w_iv_p) = c(v_{r-2}w_j) = 1$, then there are vertices w_a and w_b with $a, b > 2r$ such that $v_pw_a, w_av_r, v_{r-1}w_b, w_bv_{r-2} \in E(F)$ where $c(w_av_r) = c(v_{r-1}w_b) = 1$ and $c(w_av_p) = c(v_{r-2}w_b) = 2$. Then $(w_i,\ v_p,\ w_a,\ v_r,\ v_{r-1},\ w_b,\ v_{r-2},\ w_j)$ is a proper $w_i - w_j$ path in F.

⋆ If $c(w_iv_p) = c(v_{r-2}w_j) = 2$, then there is a vertex w_a with $a > 2r$ such that $v_pw_a, w_av_{r-1} \in E(F)$ such that $c(v_pw_a) = 1$ and $c(w_av_{r-1}) = 2$. Hence, $(w_i,\ v_p,\ w_a,\ v_{r-1},\ v_{r-2},\ w_j)$ is a proper $w_i - w_j$ path in F.

The situations for $q = r - 1$ or $q = r$ are similar.

Case 2. *$\{x, y\} = \{w_i, w_j\}$ where $2r < i \ne j \le t$.* Since each of w_i and w_j is incident with two edges of different colors in F, it follows that there exist $p, q \in \{1, 2, \ldots, r\}$ such that $w_iv_p, w_jv_q \in E(F)$ and $c(w_iv_p) \ne c(w_jv_q)$. If $v_p = v_q$, then (w_i, v_p, w_j) is a proper $w_i - w_j$ path in F; while if $v_p \ne v_q$, then there is a vertex w_a with $a > 2r$ such that $v_pw_a, w_av_q \in E(F)$ such that $c(w_iv_p) \ne c(v_pw_a)$ and $c(w_jv_q) \ne c(v_qw_a)$. Thus, $(w_i, v_p, w_a, v_q, w_j)$ is a proper $w_i - w_j$ path in F.

DOI: 10.20429/tag.2017.030202

Case 3. $\{x, y\} = \{w_i, w_j\}$ *where* $1 \leq i \leq 2r$ *and* $2r + 1 \leq j \leq t$. Suppose that $w_i v_p \in E(F)$. Choose v_q such that $c(w_i v_p) \neq c(w_j v_q)$. We may assume that $v_p \neq v_q$. Then there is a vertex w_a with $a > 2r$ such that $v_p w_a, w_a v_q \in E(F)$ where $c(w_i v_p) \neq c(v_p w_a)$. Thus, $(w_i, v_p, w_a, v_q, w_j)$ is a proper $w_i - w_j$ path in F.

Next, we consider the situation when exactly one of x and y belongs to V_k, say $x \in V_k$ and $y = v_j$ for some integer j with $1 \leq j \leq r$. First, suppose that $x = w_i$ where $2r + 1 \leq i \leq t$. Let $w_i v_p, w_i v_q \in E(F)$, where say $c(w_i v_p) = 1$ and $c(w_i v_q) = 2$. Then there is a vertex w_a with $a > 2r$ such that $v_j w_a, w_a v_p \in E(F)$ and $c(w_a v_j) = 1$ and $c(w_a v_p) = 2$. Thus, (w_i, v_p, w_a, w_j) is a proper $w_i - w_j$ path in F. Next, suppose that $x = w_i$ where $1 \leq i \leq 2r$. Let $w_i v_p \in E(F)$, where say $c(w_i v_p) = 1$. Then there is a vertex w_a with $a > 2r$ such that $v_j w_a, w_a v_p \in E(F)$ and $c(w_a v_j) = 1$ and $c(w_a v_p) = 2$. Thus, (w_i, v_p, w_a, w_j) is a proper $w_i - w_j$ path in F.

Hence, c is a proper-path 2-coloring of F and so $\text{pc}(F) = 2$. Therefore, $\mu(G) \leq 2t - 2r + 2$ and so $\mu(G) = 2t - 2r + 2$. ■

What remains then is determining $\mu(G)$ for $G = K_{n_1, n_2, \ldots, n_k}$, where $k \geq 3$, $r = \sum_{i=1}^{k-1} n_i$ and $t = n_k$, when $r + 2 \leq t < r^2 + r$. Of course, the more general problem is that of determining or at least finding bounds for $\mu(G)$ for other connected graphs G not possessing a Hamiltonian path.

Closing Remarks: In a relatively short period after the concept of proper-path colorings in graphs was introduced, it has been studied by many, resulting in numerous beautiful theorems and intriguing conjectures and open questions (such as in the previous paragraph). Li and Magnant's a dynamic survey [4] provides useful information on this topic.

References

[1] E. Andrews, E. Laforge, C. Lumduanhom and P. Zhang, On proper-path colorings in graphs. *J. Combin. Math. Combin. Comput.* To appear.

[2] V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. Manoussakis, L. Monteroa and Z. Tuza, Proper connection of graphs. *Discrete Math.* **312** 2550-2560, 2012.

[3] G. Chartrand, G. L. Johns, K. A. McKeon and P. Zhang, Rainbow connection in graphs. *Math. Bohem.* **133** 85–98, 2008.

[4] X. Li and C. Magnant, Properly colored notions of connectivity - a dynamic survey, *Theory and Applications of Graphs*: Vol. 0: Iss. 1, Article 2. 2015.

[5] X. L. Li and Y. F. Sun, *Rainbow Connections of Graphs.* Springer, Boston, MA, 2012.

[6] V. G. Vizing, On an estimate of the chromatic class of a p-graph. (Russian) *Diskret. Analiz.* **3**, 25–30, 1964