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Robustness Analysis of Hopfield and Modified Hopfield Neural Networks in Time Domain

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Abstract

A variant of Hopfield network, called modified Hopfield network is formulated. This network which consists of two mutually recurrent networks has more free parameters than the well-known Hopfield network. Stability analysis of this network is presented in this study. The analysis is carried out in the time domain with an application of the Lyapunov method and robust control Lyapunov function. The current flow in the network is treated as a 'control'. This 'controller' is shown to guarantee "a practically stabilizing control". Analysis of the Hopfield network is also included for completion.

1 INTRODUCTION

Neural networks, or artificial neural networks (ANN) to be more precise, represent an emerging technology rooted in many disciplines. A neural network is a parallel, distributed information processing structure consisting of several processing elements [?]. Control theory is not only gaining benefits from ANN, but also contributing to the development of ANN. Neural networks can be used to implement controllers and be used as tools to identify system parameters [?]. Good control system performances and good identification results have been reported [?, ?, ?]. However, Narendra [?] has stated that the stability analysis of the system models through neural networks had not been undertaken extensively. He feels that the stability theories built around the linearization of systems would not be adequate to fully describe the behavior of the neural networks, and that new concepts and techniques are required to deal with large uncertainties that the controllers might face in practical applications. In this paper, a class of more general Hopfield-type recurrent neural networks are developed. The stability and robustness properties of these recurrent neural networks, called Modified Hopfield Neural Network are presented.

2 MHNN AND ITS STABILITY

The structure of Modified Hopfield Neural Network (MHNN) is presented in this section. MHNN is a variant of the classical Hopfield networks. Its dynamical model is shown in Figure 1. The advantages of MHNN are readily apparent from its structure: it is mutually recurrent. (The Hopfield networks are self-recurrent.) It can be expanded in layers to suit higher-dimensional applications.

2.1 Stability

The stability of MHNN can be demonstrated by analyzing its dynamics and using an energy function. The network has two clusters of neurons. The right part of the network is characterized by outputs $\phi_1, \phi_2, \cdots, \phi_m$ which are transformed by nonlinear functions f from their states $u_i, u_2, \ldots, u_m$; m is the number of outputs of neurons in the right part. The conductance w^r_{ij} connects the output of the jth neuron in the left part to the input of the ith neuron in the right part, which is indicated in Figure 1 as ■. The superscript r indicates the location of weight w_{ij} in the right part of the network.

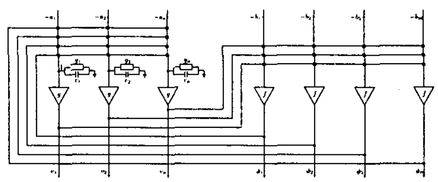

Figure 1: Modified Hopfield Neural Network

The energy function derivation for the dynamical system of MHNN is similar to that of the Hopfield network. Define:

$$\begin{aligned} W^L &= [w^l_{ji}];\ W^R = [w^r_{ij}];\ G = [\frac{1}{R_i}]. \\ A &= [a_1\ a_2\ \cdots\ a_n]^T; \\ B &= [b_1\ b_2\ \cdots\ b_m]; \\ X &= [x_1\ x_2\ \cdots\ x_n]. \end{aligned} \quad (1)$$

It is assumed that all f_i, $i = 1, 2, \dots, n$ are the same, and in effect, can be represented by f. Then, we have the dynamics of the system:

$$C\frac{dX}{dt} = -A - GX - W^L f(W^R V - B), \quad (2)$$

For the network stability proof using the energy function, please see [10].

3 ROBUSTNESS ANALYSIS OF HOPFIELD AND MHNN WITH CLF AND RCLF

3.1 Control Lyapunov Functions

In this subsection, we investigate the stability of the Hopfield networks and MHNN by the use of Control Lyapunov Function (CLF). By [7], a CLF is simply a candidate Lyapunov function whose derivative can be made negative point-wise by the choice of control values. Then we extend the studies to investigate the robustness of the Hopfield and MHNN with a Robust Control Lyapunov Function (RCLF).

3.1.1 Stability of Hopfield Network with CLF: The Hopfield network is examined with the help of CLF in this subsection. According to [6], one can have the following system equations for the Hopfield networks

$$\begin{aligned} C\frac{dU}{dt} &= -GU + WV + I \quad (3) \\ V &= f(U) \quad (4) \end{aligned}$$

In order to study them as a system with control, let the state variables U be $x \in R^n$, the control input I be $u \in R^n$, and substitute V in Eqn. (4) into Eqn. (3). The capacity coefficient C is omitted to simplify the equations because it would not affect the stability conclusions.

$$\dot{x} = -Gx + Wf(x) + u \quad (5)$$

The function $V(x) = \frac{1}{2}x^T x$ satisfies

$$\begin{aligned} &\inf_{u \in U} \nabla V(x)^T \cdot f(x, u) \\ = &\inf_{u \in U} [-x^T Gx + x^T W f(x) + x^T u] \\ = &\begin{cases} 0 & \text{when } x = 0 \\ -\infty & \text{when } x \neq 0 \end{cases} \quad (6) \end{aligned}$$

and is therefore a CLF for this system. It is concluded that *this system is globally asymptotically stabilizable.* [7]

3.1.2 Stability Analysis of MHNN with CLF: The mathematical model of MHNN will be used to study its stability. We can rewrite Eqn.(2) as

$$\dot{x} = -Gx - u_1 - W^L f(W^R g(x) - u_2) \quad (7)$$

The function $V(x) = \frac{1}{2}x^T x$ satisfies

$$\begin{aligned} \inf_{u_1, u_2 \in U} \nabla V(x)^T \cdot f(x, u) &= \inf\nolimits_{u_1, u_2 \in U} [-x^T Gx \\ &-x^T u_1 - x^T W^L f(W^R g(x) - u_2)] \quad (8) \\ &= \begin{cases} 0 & \text{when } x = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

and is therefore a CLF for this system. Hence, it is concluded that *MHNN is globally asymptotically stabilizable.* In this case, the stabilizability also can be proved by constructing a particular feedback control; indeed, the controls

$$u_1(x) = -Gx + x - W^L f(W^R g(x)) \quad (9)$$

$$u_2(x) = 0 \quad (10)$$

render $\dot{V}(x)$ negative definite and thus guarantee the global asymptotic stability.

3.2 Robust Control Lyapunov Function

The advantage of defining the CLF is that we can extend the argument to define a Robust Control Lyapunov Function (RCLF) and show the RCLF to be a necessary and sufficient condition for the robust stabilizability of a system with the control inputs in the presence of uncertainties [7].

3.2.1 Robustness Analysis of Hopfield Network:

Either by training or by design and implementation, the neural network weights are the most frequently changed parameters and therefore are considered as disturbances in the studys. Let the Hopfield network be modified with an uncertainty to the weight as: $W = W_n + \delta W$ where W_n denotes nominal weight and δW is an uncertainty. The true system, then, becomes

$$\dot{x} = -Gx + (W_n + \delta W) f(x) + u \quad (11)$$

Suppose the disturbance constraint is such that $W(x, \mathcal{U})$ is bounded for every $x \in \mathcal{X}$. Consider the function $V(x) = \frac{1}{2}x^T x$. If $\alpha_V(x) = \frac{1}{2}x^T x$ is selected, then it has

$$\begin{aligned} & \inf_{u \in \mathcal{U}} \sup_{w \in W(x,u)} [L_f V(x,u,w) + \alpha_V(x)] \\ = & \inf_{u \in \mathcal{U}} \sup_{w \in W(x,u)} \left[-x^T Gx + x^T (W_n + \delta W) f(x) \right. \\ + & \left. x^T u + \frac{1}{2}x^T x\right] = \begin{cases} 0 & \text{when } x = 0 \\ -\infty & \text{when } x \neq 0 \end{cases} \end{aligned} \quad (12)$$

It can be concluded from the definition of RCLFthat V is an RCLF for the uncertain Hopfield network with $c_V = 0$. This shows that the Hopfield network is robust.

3.2.2 Robustness Analysis of MHNN:

Let W^L and W^R of MHNN vary as W in the last subsection, the system uncertain model will become

$$\dot{x} = -Gx - u_1 - (W_n^L + \delta W_n^L) f\left((W_n^R + \delta W_n^R) g(x) - u_2\right)$$

The same suppositions about Y, U and W are used as with the Hopfield network. The same CLF as in the stability study of MHNN is used as an RCLF in this robustness study. Selecting $\alpha_V(x) = \frac{1}{2}x^T x$, one can obtain

$$\begin{aligned} & \inf_{u \in \mathcal{U}} \sup_{w \in W(x,u)} [L_f V(x,u,w) + \alpha_V(x)] \\ = & \inf_{u \in \mathcal{U}} \sup_{w \in W(x,u)} \left[-x^T Gx - x^T u_1 \right. \\ - & x^T (W_n^L + \delta W_n^L) f\left((W_n^R + \delta W_n^R) g(x) - u_2\right) \\ + & \left. \frac{1}{2}x^T x\right] = \begin{cases} 0 & \text{when } x = 0 \\ -\infty & \text{when } x \neq 0 \end{cases} \end{aligned} \quad (13)$$

Therefore, by using the same reasoning, $V(x)$ is an RCLF of MHNN. Thus the MHNN is robust.

4 A ROBUST CONTROLLER

In this section, based on the study of the mathematical models of the Hopfield networks and MHNN, a formulation [8] is proposed to build a family of controllers current to stabilize the Hopfield networks and MHNN. Though the disturbance model is accurate for the Hopfield network, it is approximate for MHNN. Both models reasonably abstract the dynamic characteristics of these two recurrent networks in their operational regions. By doing so, it can be seen in the following analysis that the design take advantage of the network nonlinearity.

4.1 Neural Network System Models with Uncertainty

In order to obtain a robust controller, we first need a generic formulation of the network systems with disturbances. This will help us lead to certain assumptions and inequalities.

Recall that the uncertain Hopfield network model can be written as

$$\dot{x} = -Gx + Wf(x) + \delta W f(x) + u \quad (14)$$

with δW is the uncertainty in weights .

For MHNN, the uncertainties lie in W^L and W^R, therefore the uncertainty model becomes

$$\begin{aligned} \dot{x} &= -Gx - (W^L + \delta W^L) \\ &\quad \cdot f\left((W^R + \delta W^R)g(x) - u_2\right) + u_1 \end{aligned} \tag{15}$$

In general, based on Eqns. (14) and (16) when summarizing the above analyses, one can abstract the uncertainty system equations of both Hopfield networks and MHNN as following:

$$\begin{aligned} \dot{x} &= f(x,t) + \Delta f(x,t,q) + B(x,t)u \\ &\quad + \Delta B(x,t,v)u + C(x,t)w \end{aligned} \tag{16}$$

where, q, v, w are uncertainties. $x, f(\cdot), \Delta f(\cdot) \in R^n$; $B(\cdot), \Delta B(\cdot) \in R^{n\times m}$; $C(\cdot) \in R^{n\times r}$. $f(\cdot), \Delta f(\cdot), B(\cdot), \Delta B(\cdot)$ are continuous in all their arguments.

4.2 Assumptions And Modification of the System

The structure in its current form is complicated for further analysis. Therefore, we make certain (practical) assumptions and observations. Note that the neuro-nonlinearity is bounded.

Assumption 1: For all $(x,t) \in R^n \times R_+$, the following conditions hold:

$$\begin{aligned} &a) \quad \Delta f(x,t,q) = B(x,t)h(x,t,q); \\ &b) \quad \Delta B(x,t,v) = B(x,t)e(x,t,v); \\ &c) \quad C(x,t) = B(x,t)d(x,t); \\ &d) \quad \|e(x,t,v)\| \le 1. \end{aligned} \tag{17}$$

Note that these conditions are properties of the system's structure only. They can guarantee the uncertainty vectors do not influence the dynamics more than the control vector u. As long as these conditions are maintained, the disturbances q and w can always be compensated by control u. The magnitude of e is required to be bounded since the control has no influence on this quantity. It can be verified that these conditions are feasible for the Hopfield networks and MHNN.

With the use of Eqn. (18), Eqn. (16) can be reduced to

$$\dot{x} = f(x,t) + B(x,t)(u + \eta) \tag{18}$$

where $\eta = h(x,t,q) + e(x,t,v)u + d(x,t)w$.

Assumption 2: $f(0,t) = 0$ for all $t \in R$ and moreover, there exist a C^1 function $V(\cdot) : R^n \times I\!R_+ \to R_+$ and class $\mathcal{K}_\infty$ functions γ_1, γ_2 and γ_3 such that for all $(x,t) \in R^n \times I\!R_+$

$$\gamma_1(\|x\|) \le V(x,t) \le \gamma_2(\|x\|). \tag{19}$$

Moreover, defining the Lyapunov derivative $L_f V_0(\cdot) : R^n \times I\!R_+ \to R$ by

$$\frac{\partial V}{\partial t} + \nabla_x V(x,t)^T \cdot f(x,t) \le -\gamma_3(\|x\|) \tag{20}$$

for all pairs $(x,t) \in R^n \times I\!R$. This assumption, in effect, asserts that existence of a Lyapunov function for the uncontrolled nominal function. Consequently, the origin, $x = 0$, is a uniformly asymptotically stable equilibrium point for the uncontrolled nominal system.

4.3 Controller Formulation

A systematic development of a robust controller formulation for the system in Eqn. (16) is presented in this section. Due to the presence of disturbance in the system, it is not always possible to drive it to the equilibrium. However, the controller "practically stabilizes" the system such that the final states remain bounded.

4.3.1 Controller Construction:

The first step is to select functions $\Delta_i(\cdot) : R^n \times I\!R_+ \to R$, $i = 1,2,3$, satisfying:

$$\begin{aligned} \Delta_1(x,t) &\ge \max_{q\in Q} \|h(x,t,q)\| \\ 1 > \Delta_2(x,t) &\ge \max_{v\in Q} \|e(x,t,v)\| \\ \Delta_3(x,t) &\ge \max_{w\in Q} \|d(x,t)w\| \end{aligned} \tag{21}$$

The inequalities in Eqn. (21) is needed so that the system with the uncertainty is controllable. (See system structure in Eqn. (16).)

Now, one may simply select any continuous function $\gamma(\cdot) : R^n \times I\!R_+ \to [0,\infty)$ satisfying:

$$\gamma(x,t) \ge \frac{[\Delta_1(x,t) + \Delta_3(x,t)]^2}{4[1 - \Delta_2(x,t)]\,[C_2 - C_1 L_f V_0(x,t)]} \tag{22}$$

where C_1 and C_2 are any (designer chosen) nonnegative constants such that

a) $C_1 < 1$;

b) either $C_1 \neq 0$ or $C_2 \neq 0$;

c) $C_2 \neq 0$ whenever $\lim_{x\to 0} \frac{[\Delta_1(x,t)+\Delta_3(x,t)]^2}{L_f V_0}$ does not exist;

d) $\frac{C_2}{1-C_1} < (\gamma_3 \circ \gamma_2^{-1} \circ \gamma_1)(\underline{d})$.

where $\underline{d}$ is a disturbance index. Note that these conditions can indeed be satisfied because of the fact that $\lim_{r\to 0} \gamma_i(r) = 0$ for $i = 1, 2, 3$.

This construction then enables one to select a controller, $p_{\underline{d}}$

$$p_{\underline{d}}(x,t) \triangleq -\gamma(x,t) B^T(x,t) \nabla_x V(x,t). \tag{23}$$

4.3.2 Guaranteed Performance:

The controller constructed in Eqn. (23) guarantees practical stabilizability. Note that this construction has steps similar to that in [7], however the formulation and the conclusion are different.

Theorem: The controller in Eqn. (23) practically stabilizes the uncertain dynamical system, Eqn. (16).

$$\begin{aligned} \dot{x} = \; & f(x,t) + \Delta f(x,t,q) + B(x,t)u \\ & + \Delta B(x,t,v)u + C(x,t)w \end{aligned} \tag{24}$$

proof: i) For a given $\underline{d} > 0$ and given uncertainty vectors $(q, v, w) \in \mathcal{Q}$, consider the function $g(\cdot) : R^n \times I\!R_+ \to R^n$ such that

$$g(x,t) = f(x,t) + B(x,t)(p_{\underline{d}}(x,t) + \eta(x,t))$$

As a consequence of the assumptions, $g(\cdot)$ is continuous in x for all $t \in I\!R_+$ there exists a function $m(\cdot) : [a,b] \to R$ such that for all $(x,t) \in E \times [a,b]$

$$\|g(x,t)\| \le m(t)$$

Thus, given (x_0, t_0), E and $[a,b]$ such that $x_0 \in$ *int* E and $t \in [a,b]$, there exists a solution $x(\cdot) : [t_0, t_1] \to R^n$, $x(t_0) = x_0$.

ii) The Lyapunov derivative for the closed loop system obtained with the feedback control, Eqn. (23), is given by:

$$\begin{aligned} L_f V(x) \triangleq \; & L_f V_0(x) + \nabla_x V^T(x)\{\Delta f(x,q) \\ & + [B(x) + \Delta B(x,v)] p_{\underline{d}}(x) + C(x)w\} \end{aligned} \tag{25}$$

By using Assumption 1 in conjunction with Eqn. (23), one can show that

$$\begin{aligned} L_f V(x,t) \; & \le L_f V_0(x,t) \\ & - \Big[1 - \Delta_2(x,t)\Big] \gamma(x,t) \|\phi(x,t)\|^2 \\ & + \Big[\Delta_1(x,t) + \Delta_3(x,t)\Big] \|\phi(x,t)\|. \end{aligned} \tag{26}$$

There are two cases to be considered.

Case 1: The pair (x,t) is such that $\Delta_1(x,t) = \Delta_3(x,t) = 0$. It then follows from the preceding inequality that

$$L_f V(x,t) \le L_f V_0(x,t).$$

since $\Delta_2(x,t) < 1$, and $\gamma(x,t) > 0$.

Case 2: The pair (x,t) is such that $\Delta_1(x,t) \neq 0$, or $\Delta_3(x,t) \neq 0$, or both $\Delta_1(x,t)$ and $\Delta_3(x,t) \neq 0$. Then it follows from Eqn. (22) that $\gamma(x,t) > 0$ because $L_f V_0(x,t) < 0$. Moreover, in view of Eqn. (22) and the conditions on C_1 and C_2, it can be shown that after some algebra.

$$L_f V(x,t) \le (1 - C_1) L_f V_0(x,t) + C_2$$

Combining Case 1 and Case 2, and noting that $C_1 < 1$, one can conclude as a consequence of Assumption 2 in Eqn. 20) that

$$\begin{aligned} L_f V(x,t) \; & \le (1 - C_1) L_f V_0(x,t) + C_2 \\ & \le -(1 - C_1)\gamma_3(\|x\|) + C_2 \end{aligned} \tag{27}$$

for all $(x,t) \in R^n \times I\!R_+$.

The interpretation of Eqn. (27) is important. The Lyapunov approach is to show $L_f V(x,t) < 0$. We can see $\gamma_3(\|x\|)$ is positive, and $-(1 - C_1)\gamma_3(\|x\|)$ is always negative. But when C_2 is present, $L_f V(x,t)$ may not always be negative. What we want to show is that, even for these x and C_2 that make $L_f V(x,t)$

positive and lead to the system states diverge, the increasing x will make $-(1-C_1)\gamma_3(\|x\|)$ more negative to balance the positive C_2 so that finally the overall summation go back to a negative quantity again, therefore the system is practically stabilized. It is easy to see from Eqn. (27) that the critical point is:

$$R \triangleq \gamma_3^{-1}(\frac{C_2}{1-C_1}). \tag{28}$$

Consider a solution $x(\cdot):[t_0,t_1]\to R^n, x(t_0)=x_0$ with $\|x_0\|\le r$. Let $\hat{r}\triangleq \max\{r,R\}$ so that $\|x_0\|\le \hat{r}$ and $R\le\hat{r}$. Define

$$\begin{aligned} d(r) &\triangleq (\gamma_1^{-1}\circ\gamma_2)(\hat{r}) \\ &= \begin{cases} (\gamma_1^{-1}\circ\gamma_2)(R) & \text{if } r\le R, \\ (\gamma_1^{-1}\circ\gamma_2)(r) & \text{if } r\ge R. \end{cases} \end{aligned} \tag{29}$$

This $d(r)$ captures the effect of the disturbance on the nonlinear system. After some algebra, we can show there must exist a $t_1 > t_0$ such that

$$\|x(t)\|\le d(r), \qquad \forall t\in[t_0,t_1]. \tag{30}$$

The implication of Eqn. (30) is that in time interval t_0 and t_1, the state is bounded.

The proof for no escape for $x(t)$ can be found in [10].

The above construction has shown that the Hopfield and MHNN are robust with respect to their equilibrium solutions. Note that we have treated the exogenous variable current as the control in the dynamics of the networks. One always can apply extra control quantity to balance the deviation of the states due to the disturbances. The disturbances can make the derivative of Lyapunov function positive and the states diverge. But this change in states make the effect of the disturbances less influential, and make the derivative of Lyapunov function negative again. This behavior of the states staying bounded is called practically stabilizability.

5 CONCLUSIONS

A variant of the Hopfield network called modified Hopfield network has been formulated. A control theoretic approach has been used to study the robustness of the equilibrium of the Hopfield and modified Hopfield networks. Furthermore, theorems have been presented and a method has been formulated to show the bounds of deviations from equilibrium.

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