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Thermoelastic Problems of Multilayered Cylinders

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Thermoelastic problems of long multilayered cylinders manufactured from isotropic materials are considered. The steady state thermal field corresponding to constant difference between temperatures outside and inside the assembly is used in the analysis. Both the case of a hollow cylinder as well as that of a solid multilayered assembly are discussed. A closed form exact solution is shown for an arbitrary number of layers if the properties of constituent materials remain unaffected by temperature. Various strategies leading to an analytical solution are proposed for the case where material properties depend on temperature. The stresses in a dual-coated fiber subject to a uniform temperature are determined as a particular case.

Introduction

Multilayered cylinders subject to the action of a nonuniform thermal field represent both theoretical as well as practical interest. Examples of such structures include pipes in projective coatings or multilayered pressure vessels where temperature of fluid or gas inside the pipe (vessel) differs from that of the outside environment. An example of a solid multilayered cylinder is a coated wire where temperature of the core can increase due to electric current. Coated fibers used in fiber optics represent another example.

Thermal fields acting on multilayered cylindrical assemblies are often axisymmetric. Axisymmetric thermoelastic problems are analyzed and solved for a single-layered cylinder in classical books on the theory of elasticity (Timoshenko and Goodier, 1970). A large number of studies dealing with both axisymmetric as well as asymmetric thermoelasticity of multilayered or non-isotropic cylindrical tubes has been recently published. Several thermoelastic problems for multilayered and composite shells were formulated by Ambartsumian (1974). Kalan and Tauchert considered thermal stresses in a single-layered orthotropic cylinder subject to an asymmetric thermal field (1978). Axisymmetric thermal stresses in multilayered angle-ply composite cylinders in a uniform thermal field were studied by Hyer and Rousseau (1987). A theory of elasticity solution was presented for an asymmetric thermal problem in composite tubes subject to a circumferential gradient of temperature by Hyer and Cooper (1986). Kardomateas (1989) considered transient thermal stresses in a single-layered orthotropic cylinder by assumption that material properties are independent of temperature. Suhir and Sullivan (1989) studied interface stresses in bi-annular cylindrical assemblies subject to a uniform temperature.

Numerical solutions of thermoelastic problems of non-isotropic cylindrical assemblies have also been published. Mention here the paper by Blandford, et.al. (1988), and Thangaratnam, et.al. (1988).

Another related area of research deals with thermoelastic problems of solid cylinders. Avery and Herakovich (1986) presented an analytical solution based on the theory of elasticity approach for the stresses in an orthotropic fiber surrounded by an isotropic matrix and subject to a uniform temperature. Suhir solved a number of problems related to applications in electronics (1988, 1989). For example, thermal stress and stability of dual coated optical fibers were considered (Suhir, 1988). Another study presented the analysis of solder joints in finite circular cylinders subject to low temperature (Suhir, 1989).

Research of single-layered cylinders with temperature-dependent properties has been outlined in the review of Noda (1986). In particular, incompressible elastic cylindrical shells in a thermal field symmetric with respect to the axis were first considered by Hilton (1952) and Trostel (1958). The closed form solution of the plane strain problem was obtained. The solutions illustrated by Noda (1986) for compressible elastic cylinders include:

-closed form solution in the case where one of Lame's elastic coefficients is a power function of radius;

-approximate solutions utilizing the Gauss hypergeometric equations in cases where one of Lame's elastic coefficients is an exponential function of radius as well as where this coefficient is a power function of a sum of a constant and the radius.

Perturbation technique was applied to obtain solutions in case where shear modulus is dependent on temperature. Transversely isotropic single-layered cylinders were considered by Hata and Atsumi (1968) who employed a perturbation method to obtain an approximate solution.

In this paper strategies for an analytical solution of the axisymmetric thermal stress problem in multilayered cylinders with a constant difference between temperatures outside and inside the assembly are discussed. The solutions are obtained by an assumption that cylinders are in the state of plane strain. It is shown that if the material properties of each constituent layer can be supposed to remain constant, an exact solution is available. Numerical results are presented for a double-coated optical fiber in a uniform thermal field.

The studies of thermoelasticity of multilayered and non-isotropic cylinders usually disregard the fact that the coefficients of thermal expansion depend on the stresses in the material. This fact was brought to the attention of the author by Prof. Bert (Bert, 1990). The relationships between the coefficients of thermal expansion and the stresses were developed for orthotropic materials by Ungar, et.al. (1964). Such relationships were not used in the present paper. However, the present solution is still applicable (as well as other solutions cited above), although they have to be iterative. This means that the coefficients of thermal expansion must be corrected at each step according to the level of stresses obtained in the previous iteration.

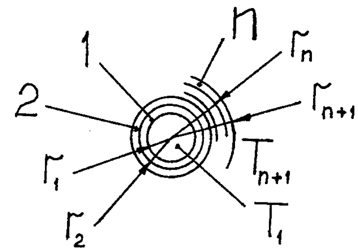

Fig. 1. Cross section of a multilayered cylinder with the numbers of layers. If the cylinder is solid, the number of the core is 0.

Governing Equations

Consider a multilayered hollow or solid cylinder shown in Fig. 1. Temperature inside the hollow cylinder remains constant and equal to T_1. In the case of a solid cylinder it is assumed that the core retains constant temperature T_1. In both cases temperature outside the assembly is equal to T_{n+1}. The steady state thermal field within $r_1 \le r \le r_{n+1}$ is described by the conduction equation (Ingersoll, et. al., 1948) written here by assumption that the thermal conductivity remains constant within each layer.

$$\nabla^2 T = \frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0 \tag{1}$$

The solution of this equation for the j-th layer is

$$T = B_j \ell n r + P_j \tag{2}$$

where $r_j \le r \le r_{j+1}$

$$B_j = \frac{T_j - T_{j+1}}{\ell n r_j - \ell n r_{j+1}}, \qquad T_j = T(r_j)$$

$$P_j = \frac{T_j \, \ell n r_{j+1} - T_{j+1} \, \ell n r_j}{\ell n r_{j+1} - \ell n r_j} \tag{3}$$

The values of T_j $(1<j\le n)$ can be obtained from the continuity requirements. In this problem the length of the assembly is much larger than its external radius. The ends of the assembly are assumed to be restrained so that the axial strain $\epsilon_z = 0$ and the cylinder is in the state of plane strain. Although this assumption simplifies the analysis, other axial boundary conditions could be also incorporated in a manner shown by Timoshenko and Goodier (1970).

The problem being axisymmetric, the strain-displacement relationships are

$$\epsilon_r = \frac{du}{dr} \qquad \epsilon_\theta = \frac{u}{r} \tag{4}$$

where u is the radial displacement.

The Hookean relations written here for the j-th layer

$$E_j\epsilon_z = \sigma_z - \nu_j(\sigma_r + \sigma_\theta) + E_j\alpha_j T = 0$$

$$E_j\epsilon_r = \sigma_r - \nu_j(\sigma_\theta + \sigma_z) + E_j\alpha_j T$$

$$E_j\epsilon_\theta = \sigma_\theta - \nu_j(\sigma_r + \sigma_z) + E_j\alpha_j T \tag{5}$$

yield the axial stresses in the form

$$\sigma_z = \nu_j(\sigma_r + \sigma_\theta) - E_j\alpha_j T \tag{6}$$

and the plane constitutive equations known in the theory of elasticity (Dugdale and Ruiz, 1971):

$$\sigma_r = \frac{E_j}{(1+\nu_j)(1-2\nu_j)}\left[(1-\nu_j)\frac{du}{dr} + \nu_j\frac{u}{r}\right] - \frac{E_j\alpha_j T}{1-2\nu_j}$$

$$\sigma_\theta = \frac{E_j}{(1+\nu_j)(1-2\nu_j)}\left[\nu_j\frac{du}{dr} + (1-\nu_j)\frac{u}{r}\right] - \frac{E_j\alpha_j T}{1-2\nu_j} \tag{7}$$

In these equations E and ν denote the modulus of elasticity and the Poisson's ratio, α is a function of the coefficient of thermal expansion $\tilde{\alpha}(T)$ chosen from the requirement that

$$\alpha T = \int_{T_0}^{T} \tilde{\alpha}(T)\, dT.$$

The subscript j indicates the number of a layer. Shearing stresses τ_{zr} are excluded from the consideration. This is justified for central portions of long assemblies where one can assume that $\gamma_{zr} = 0$. The equilibrium equation is

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{8}$$

Note that Poisson's ratios of isotropic materials are often insensitive to variations of temperature. Therefore, they are supposed to remain constant. On the other hand, the moduli of elasticity of such materials as steel, alluminum etc. are usually affected by temperature. The effect of temperature on the coefficient of thermal expansion can be also important. Therefore, given the distribution of temperature, i.e. equation (2), one can derive the corresponding analytical relationships $E_j = E_j(r)$ and $\alpha_j = \alpha_j(r)$. Then the substitution of (2) and (7) into (8) yields

$$f_1(r)\frac{d^2u}{dr^2} + f_2(r)\frac{1}{r}\frac{du}{dr} - f_3(r)\frac{u}{r^2} = f_4(r) \tag{9}$$

where

$$f_1(r) = \frac{1-\nu_j}{1+\nu_j} E_j(r)$$

$$f_2(r) = \frac{1-\nu_j}{1+\nu_j} [E_j(r) + r\frac{dE_j(r)}{dr}]$$

$$f_3(r) = \frac{1}{1+\nu_j} [(1-\nu_j)E_j(r) - \nu_j r\frac{dE_j(r)}{dr}]$$

$$f_4(r) = \frac{d}{dr}[E_j(r)\alpha_j(r)T_j(r)] \tag{10}$$

An important particular case is where both the moduli of elasticity and the functions of coefficients of thermal expandion are linear functions of temperature, i.e.

$$E_j = E_{j1} - E_{j2}\,\ell nr$$

$$\alpha_j = \alpha_{j1} - \alpha_{j2}\,\ell nr \tag{11}$$

where E_{ji} and α_{ji} $(i=1,2)$ depend both on material properties of the j-th layer as well as on T_j, T_{j+1}, r_j and r_{j+1}.

Then the coefficients in equation (9) become

$$f_1(r) = \frac{1-\nu_j}{1+\nu_j}(E_{j1}-E_{j2}\,\ell nr)$$

$$f_2(r) = \frac{1-\nu_j}{1+\nu_j}[(E_{j1} - E_{j2}) - E_{j2}\,\ell nr]$$

$$f_3(r) = \frac{1}{1+\nu_j}\{[(1-\nu_j)E_{j1} + \nu_j E_{j2}] - (1-\nu_j)E_{j2}\,\ell nr]$$

$$f_4(r) = \frac{1}{r}[K_2(\ell nr)^2 + K_1\,\ell nr + K_0] \tag{12}$$

where

$$K_2 = 3E_{j2}\alpha_{j2}B_j$$

$$K_1 = 2(P_jE_{j2}\alpha_{j2} - B_jE_{j1}\alpha_{j2} - B_jE_{j2}\alpha_{j1})$$

$$K_0 = B_jE_{j1}\alpha_{j1} - P_jE_{j2}\alpha_{j1} - P_jE_{j1}\alpha_{j2} \tag{13}$$

A closed form exact solution is available if the properties of each of the constituent materials remain constant, i.e. independent on the radial co-ordinate. In this case, according to Timoshenko and Goodier (1970) displacements and stresses in the j-th layer are

$$u_j = \frac{1+\nu_j}{1-\nu_j}\alpha_j\frac{1}{r}\int_{r_j}^{r}T(r)rdr + \tilde{C}_j r + \frac{\tilde{D}_j}{r}$$

$$\sigma_r^{\,j} = -\frac{\alpha_jE_j}{1-\nu_j}\frac{1}{r^2}\int_{r_j}^{r}T(r)rdr + \frac{E_j}{1+\nu_j}\left(\frac{\tilde{C}_j}{1-2\nu_j} - \frac{\tilde{D}_j}{r^2}\right)$$

$$\sigma_\theta^{\,j} = \frac{\alpha_jE_j}{1-\nu_j}\frac{1}{r^2}\int_{r_j}^{r}T(r)rdr - \frac{\alpha_jE_jT_j(r)}{1-\nu_j} + \frac{E_j}{1+\nu_j}\left(\frac{\tilde{C}_j}{1-2\nu_j} + \frac{\tilde{D}_j}{r^2}\right) \tag{14}$$

where $\tilde{C}_j$ and $\tilde{D}_j$ are constants of integration and $\alpha_j = \tilde{\alpha}_j$. Substituting $T(r)$ from (2) one obtains

$$u_j = \frac{1}{2}\frac{1+\nu_j}{1-\nu_j}\alpha_jB_j\,(r\,\ell nr + C_jr + \frac{D_j}{r})$$

$$\sigma_r^{(j)} = \frac{E_j\alpha_jB_j}{2(1-\nu_j)(1-2\nu_j)}[1-\nu_j + \ell nr + C_j - (1-2\nu_j)\frac{D_j}{r^2}] - \frac{E_j\alpha_jT}{1-2\nu_j}$$

$$\sigma_{\theta(j)} = \frac{E_j\alpha_jB_j}{2(1-\nu_j)(1-2\nu_j)}[\nu_j + \ell nr + C_j + (1-2\nu_j)\frac{D_j}{r^2}] - \frac{E_j\alpha_jT}{1-2\nu_j} \tag{15}$$

where C_j and D_j are new constants of integration. There are 2n constants of integration for a n-layered assembly. These constants can be determined from two boundary conditions at $r = r_1$ and $r = r_{n+1}$ and from 2(n-1) conditions at (n-1) interfaces. The conditions at the j-th interface are the continuity of the displacements and radial stresses:

$$u_j = u_{j-1}, \qquad \sigma_r^{(j)} = \sigma_r^{(j-1)} \tag{16}$$

Boundary conditions at internal and external surfaces of a hollow assembly are

$$\sigma_r^{1}(r_1) = 0 \quad \text{and} \quad \sigma_r^{(n)}(r_{n+1}) = 0 \tag{17}$$

respectively.

If the assembly is solid and the elastic core temperature is constant, $T = T_1$, the equilibrium equation for the core is

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \tag{18}$$

The solution of (18) for a solid cylinder is

$$u = C_0 r \tag{19}$$

where C_0 is a constant.

The number of constants of integration for a solid n-layered cylinder is equal to 2n+1 (obviously, the core is not counted as a layer). These constants can be determined from 2n continuity conditions including the conditions at the interface between the core and the first layer plus the boundary condition at $r = r_{n+1}$.

If one of the surfaces of the assembly is attached to a rigid body, the corresponding boundary condition is $u = 0$.

Strategies for Analytical Solution: Temperature-Dependent Properties

It is interesting to note that numerical approaches based on application of such initial value methods as Runge-Kutta or predictor-corrector are not appropriate for this problem. This is due to difficulties in satisfying the condition of continuity of radial stresses when integrating equation (9) numerically. Therefore, analytical methods of solution can be attractive. Three of such methods are considered in this section.

1. Piece-wise approximation of material properties

This method discussed by Birman (1989) in application to calculations of thermal response of reinforced panels can be used in the present problem as well. Accoridng to the method, the layers are subdivided into a number of narrow rings such that the properties of material within each ring remain constant. Then the analytical solution within a j-th ring is given by (15) while the interface conditions are still (16). Therefore, this approach enables us to obtain a closed form solution of the problem with very high accurasy.

Note that a standard finite element or a finite difference code would usually require a similar assumption regarding the properties of materials within circular regions of the assembly. The advantage of the proposed method is that it provides a closed form solution of the problem once an approximation of the properties is introduced.

2. Reduction to Gauss equation

If the coefficients of equation (9) are given by (12), the problem can be reduced to a hypergeometric Gauss equation (Whittaker and Watson, 1952; Kamke, 1959).

Consider a j-th layer and introduce nondimensional parameters

$$\bar{r} = \frac{r}{r_{j+1}} \qquad \bar{u} = \frac{u}{r_{j+1}} \tag{20}$$

Obviously, the solution of the conduction equation (1) is

$$T = B_j \ln\bar{r} + P_j \tag{21}$$

where

$$B_j = \frac{T_j - T_{j+1}}{\ln\bar{r}_j} \qquad P_j = T_{j+1} \tag{22}$$

The equilibrium equation (9) becomes

$$f_1(\bar{r})\frac{d^2\bar{u}}{d\bar{r}^2} + f_2(\bar{r})\frac{1}{\bar{r}}\frac{d\bar{u}}{d\bar{r}} - f_3(\bar{r})\frac{\bar{u}}{\bar{r}^2} = f_4(\bar{r}) \tag{23}$$

where $f_i(\bar{r})$ can be obtained from the corresponding functions in (12) by replacement of r with $\bar{r}$. Obviously, the constants E_{ji} and α_{ji} in $f_i(\bar{r})$ are different from those in (12).

If the thickness of the layer is small compared to r_{j+1}, $\ln\bar{r} \approx \bar{r}-1$. Then (23) becomes

$$(k_1\bar{r}-k_2)\bar{r}^2\frac{d^2\bar{u}}{d\bar{r}^2} + (k_3\bar{r}+k_4)\,\bar{r}\frac{d\bar{u}}{d\bar{r}} - (k_5\bar{r}+k_6)\,\bar{u} = k_7\bar{r}^3 + k_8\bar{r}^2 + k_9\bar{r} \tag{24}$$

The coefficients k_i which can be obtained from $f_i(\bar{r})$ by trivial transformations which are omitted for brevity.

The solution of the homogeneous equation obtained from (24) is

$$\bar{u} = \bar{r}^c y\left(\frac{k_1\bar{r}}{k_2}\right) \tag{25}$$

where y(x) is a solution of the hypergeometric equation

$$H(\alpha, \beta, \gamma, y, x) = x(x-1)y'' + [(\alpha+\beta+1)x-\gamma]\, y' + \alpha\beta y = 0 \tag{26}$$

and the constants c, α, β and γ can be calculated as shown by Kamke (1959).

The general solution of (24) is

$$\bar{u} = \phi_2\int\frac{\phi_1 R}{W}\,d\bar{r} - \phi_1\int\frac{\phi_2 R}{W}\,d\bar{r} + C_j\phi_1 + D_j\phi_2 \tag{27}$$

were ϕ_i are solutions of the hypergeometric equation (26) and C_j, D_j are constants of integration.

$$W = \phi_1\frac{d\phi_2}{d\bar{r}} - \phi_2\frac{d\phi_1}{d\bar{r}}$$

$$R = k_7\bar{r}^3 + k_8\bar{r}^2 + k_9 \tag{28}$$

The expressions under the integrals in (27) can be represented by proper algebraic fractions. Integration of such fractions was considered by Fichtengoltz (1966) who proved that any proper fraction can be represented by a finite sum of the following simple fractions:

$$\frac{A}{\bar{r}-a},\ \frac{A}{(\bar{r}-a)^k}\ (k = 2,3,\ldots),\ \frac{M\bar{r}+N}{\bar{r}^2+p\bar{r}+q} \text{ and}$$

$$\frac{M\bar{r}+N}{(\bar{r}^2+p\bar{r}+q)^m} \quad (m = 2,3,\ldots)$$

where A,M,N,a,p and q are real numbers and $p^2<4q$. The methods of calculations of these numbers and the integrals of simple fractions were also shown by Fichtengoltz (1966).

The constants of integration C_j and D_j can be determined from the conditions (16) and (17).

3. The method of Frobenius (Power series solution)

This method can provide a simpler and less laborious solution than the procedure based on reduction to a Gauss equation. The method requires expansion of the coefficients $f_i(\bar{r})$ in (23) in Maclaurin's series. Suppose, for example, that (24) can provide a satisfactory accurasy. A solution can be sought in the form

$$\bar{u} = \bar{r}^c(a_0 + a\bar{r} + a_2\bar{r}^2 + \ldots) \tag{29}$$

where a_i and c are unknown coefficients. Substituting (29) into the homogeneous equation obtained from (24), combining the like powers of $\bar{r}$ and equating to zero the coefficients of the like powers yield a set of algegraic equations:

$$[-c(c-1)k_2 + ck_4 - k_6]\ a_0 = 0$$

$$[-(c+1)ck_2 + (c+1)k_4 - k_6]\ a_1 + [c(c-1)k_1 + ck_3 - k_5]\ a_0 = 0$$

$$[-(c+2)(c+1)k_2 + (c+2)k_4 - k_6]\ a_2 + [(c+1)ck_1 + (c+1)k_3 - k_5]\ a_1 = 0$$

$$\ldots\ldots\ldots\ldots\ldots \qquad (30)$$

Assuming that a_0 is a constant of integration one can obtain two values c_1 and c_2 from the first equation (30). Then the consequent coefficents can be expressed in terms of a_0 using

$$a_i = \frac{(c_1+i-1)(c_1+i-2)k_1+(c_1+i-1)k_3-k_5}{(c_1+i)(c_1+i-1)k_2-(c_1+i)k_4+k_6}\, a_{i-1} \qquad (31)$$

The similar relationships $\bar{a}_i(a_0)$ can be obtained if c_1 is replaced by c_2 in (31). The values of a_0 can be chosen arbitrary. Then the general solution for the j-th layer is represented by (27) where ϕ_1 and ϕ_2 are given by (29): $\phi_1 = \bar{u}(c_1, a_i)$, $\phi_2 = \bar{u}(c_2, \bar{a}_i)$.

Applications to Fiber Optics

An interesting particular case is the calculation of thermal stresses in a stretched optical fiber. In this problem temperature is usually constant throughout the assembly which represents a solid cylinder with the fiber as a core and one or two layers of coating. The cross section of a dual-coated optical fiber is shown in Fib. 2. The study of dual-coated fibers based on the strength of materials solutions has been recently published by Suhir (1988). The closed form solution obtained in our paper is based on the theory of elasticity and on the assumption that the fiber is stretched and there is no slipping between the core and the layers. Displacements and stresses in the assembly subject to temperature T are:

Fiber: $u_0 = C_0 r$

$$\sigma_r^{(0)} = \sigma_\theta^{(0)} = \frac{E_0}{(1+\nu_0)(1-2\nu_0)} C_0 - \frac{E_0\alpha_0 T}{1-2\nu_0}$$

Layers (j=1,2):

$$u_j = \frac{1+\nu_j}{1-\nu_j}\frac{\alpha_j T}{2r}(r^2-r_j^2) + C_j r + \frac{D_j}{r}$$

$$\sigma_r^{(j)} = -\frac{\alpha_j E_j}{1-\nu_j}\frac{T}{2r^2}(r^2-r_j^2) + \frac{E_j}{1+\nu_j}\left(\frac{C_j}{1-2\nu_j} - \frac{D_j}{r^2}\right)$$

$$\sigma_\theta^{(j)} = \frac{\alpha_j E_j}{1-\nu_j}\frac{T}{2r^2}(r^2-r_j^2) - \frac{\alpha_j E_j T}{1-\nu_j} + \frac{E_j}{1+\nu_j}\left(\frac{C_j}{1-2\nu_j} + \frac{D_j}{r^2}\right) \qquad (32)$$

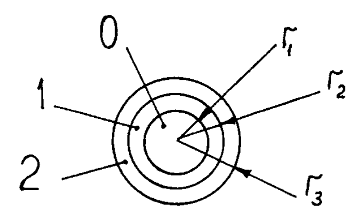

Fig. 2. Cross section of a dual-coated optical fiber
0 = fiber, 1,2 = coatings.

where $\alpha_j = \tilde{\alpha}_j$.
Five constants of integration are found from the continuity conditions (16) for j=1 and 2 and from $\sigma_r^{(2)}(r_3)=0$.

The calculations were performed for an assembly consisting of a fiber coated by silicon (primary coating, layer 1) and nylon (secondary coating, layer 2). The properties of constituent materials and the dimensions of the cross section are given in Table 1. The distribution of radial and circumferential normal stresses is shown in Fig. 3. This Figure also presents the axial stress σ_z calculated from (6). This stress can be used to check stability of the assembly.

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Table 1: Properties of constituent materials in dual-coated fiber

Layer	Number of layer, j	E_j (MPa)	$10^6 r_{j+1}$ (m)	ν_j	$10^6 \alpha_j$ (1/°C)
Fiber (core)	0	65×10^3	62.5	0.16	80
Primary coating	1	18.2	250	0.48	2.6
Secondary coating	2	2139	471	0.33	100

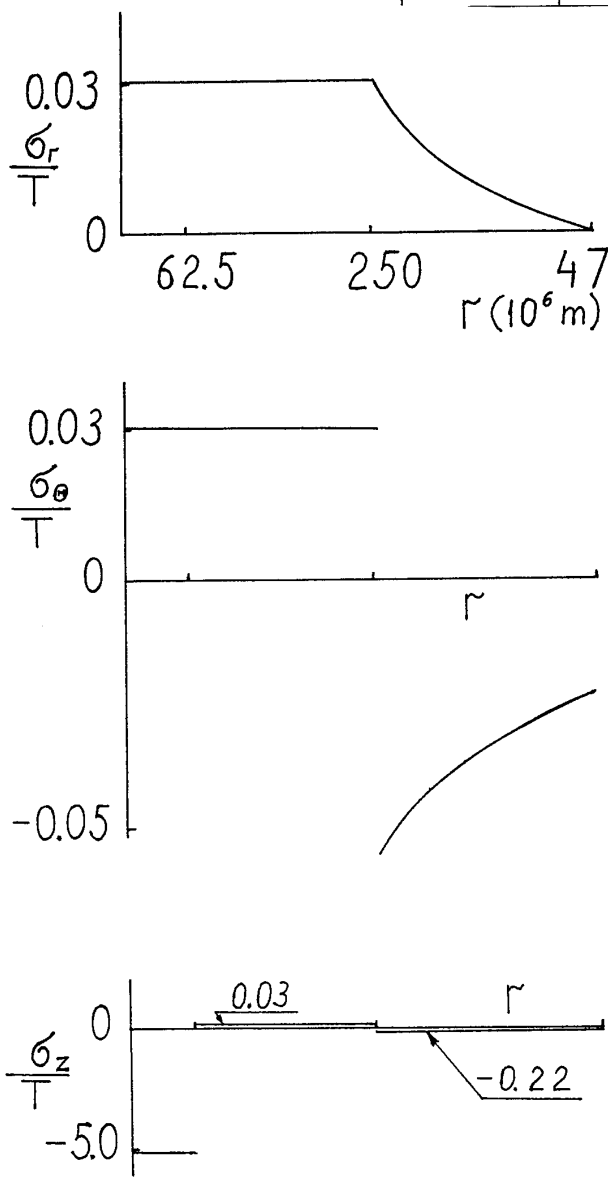

Fig. 3. Stress-to-temperature ratios in a dual-coated optical fiber (stresses in MPa, temperature in °C).

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