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Induced Subgraph Saturated Graphs

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Abstract

A graph G is said to be *H-saturated* if G contains no subgraph isomorphic to H but the addition of any edge between non-adjacent vertices in G creates one. While induced subgraphs are often studied in the extremal case with regard to the removal of edges, we extend saturation to induced subgraphs. We say that G is *induced H-saturated* if G contains no induced subgraph isomorphic to H and the addition of any edge to G results in an induced copy of H. We demonstrate constructively that there are non-trivial examples of saturated graphs for all cycles and an infinite family of paths and find a lower bound on the size of some induced path-saturated graphs.

1 Introduction

In this paper we address the problem of graph saturation as it pertains to induced graphs, in particular paths and cycles. We begin with some background and definitions, and complete Section 1 with statements of the main theorems. In Section 2 we demonstrate that there are non-trivial induced saturated graphs for an infinite family of paths, and prove lower bounds on the number of edges in possible constructions. We continue on to demonstrate results regarding induced cycles in Section 3 and claws in Section 4.

A number of results were discovered using the SAGE mathematics software [11], an open source mathematics suite.

Throughout we use K_n to denote the complete graph on n vertices, C_n the cycle on n vertices, $K_{n,m}$ the complete bipartite graph with parts of order n and m, and P_n the path on n vertices. All graphs in this paper are simple, and by $\overline{G}$ we mean the complement of the graph G. If u, v are nonadjacent vertices in G then $G + uv$ is the graph G with edge uv added. Given graphs G and H the graph join $G \vee H$ is composed of a copy of G, a copy of H, and all possible edges between the vertices of G and the vertices of H. The *graph union* $G \cup H$ consists of disjoint copies of G and H. The graph kH consists of the union of k copies of H. In particular a *matching* is a collection of pairwise disjoint edges, denoted kK_2. The *order* $n(G)$ and *size* $e(G)$ of G are the numbers of its vertices and edges, respectively. A vertex v in a connected graph G is a *cut vertex* if its removal results in a disconnected graph. If G has no cut vertex then it is *2-connected*, and a maximal 2-connected subgraph of G is a *block*. Note that a cut edge is also a block.

For simple graphs G and H we say that G is *H-saturated* if it contains no subgraph isomorphic to H but the addition of any edge from $\overline{G}$ creates a copy of H. We refer to G as the *parent graph*. The study of graph saturation began when Mantel and students [9] determined the greatest number of edges in a K_3-free graph on n vertices in 1907, which was generalized by Turán in the middle of the last century [12] to graphs that avoid arbitrarily large cliques. Erdős, Hajnal, and Moon then addressed the problem of finding the fewest number of edges in a K_m-saturated graph [3]. In particular, they proved the following theorem.

Theorem 1.1. *For $m \geq 3$ and $n \geq m$, the unique smallest graph on n vertices that is K_m-saturated is $K_{m-2} \vee \overline{K_{n-m+2}}$. This graph contains $\binom{m-2}{2} + (n-m+2)(m-2)$ edges.*

Since then, graph saturation has been studied extensively, having been generalized to many other families of graphs, oriented graphs [7], topological minors [5], and numerous other properties. A comprehensive collection of results in graph saturation is available in [4].

Given a graph G and a subset X of vertices of G, the subgraph *induced by* X is the graph composed of the vertices X and all edges in G among those vertices. We say that a subgraph H of G is an *induced subgraph* if there is a set of vertices in G that induces a graph isomorphic to H. We say that G is H*-free* if G contains no induced subgraph isomorphic to H.

Finding induced subgraphs of one graph isomorphic to another is a traditionally difficult problem. Chung, Jiang, and West addressed the problem of finding the greatest number of edges in degree-constrained P_n-free graphs [1]. Martin and Smith created the parameter of *induced saturation number* [10]. We include their definition below for completeness.

Definition 1.2 (Martin, Smith 2012)**.** *Let T be a graph with edges colored black, white, and gray. The graph T* realizes *H if the black edges and some subset of the gray edges of T together include H as an induced subgraph. The* induced saturation number *of H with respect to an integer n is the fewest number of gray edges in such a graph T on n vertices that does not realize H but if any black or white edge is changed to gray then the resulting graph realizes H.*

In this paper we only consider adding edges to a simple non-colored graph.

Definition 1.3. *Given graphs G and H we say that G is* induced H-saturated *if G does not contain an induced subgraph isomorphic to H but the addition of any edge from $\overline{G}$ to G creates one.*

Note that in Definition 1.3 we allow G to be a complete graph. This case provides for a trivial family of induced H-saturated graphs for any non-complete graph H. Henceforth we will be concerned with determining non-trivial induced H-saturated graphs.

1.1 Main results

We will prove the following results to show the existence of non-complete induced P_m-saturated graphs for infinitely many values of m.

Theorem. *(2.6) For any $k \geq 0$ and $n \geq 14 + 8k$ there is a non-complete induced P_{9+6k}-saturated graph on n vertices. Further, if n is a multiple of $(14 + 8k)$ there is such a graph that is 3-regular.*

As we will see in Section 2.1, these orders are the result of the search for a longest induced path in a class of vertex transitive hamiltonian graphs of small size, visualizable with high rotational symmetry.

Theorem. *(2.11) If G is an induced P_m-saturated graph on n vertices with no pendant edges except a K_2 component, $m > 4$, then G has size at least $\frac{3}{2}(n-2)+1$. This bound is realized when $m = 9 + 6k$ and $n \equiv 2 \mod (14 + 8k)$.*

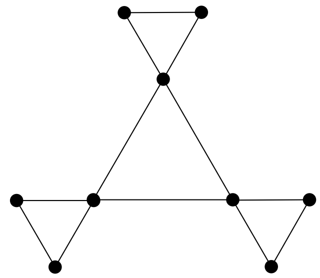

Figure 1: An induced P_5-saturated graph

Theorem. *(2.14) For every integer $k > 0$ there is a non-complete graph that is induced P_{11+6k}-saturated.*

Regarding cycles, we will prove the following theorem.

Theorem. *(3.3) For any $k \geq 3$ and $n \geq 3(k-2)$ there is a non-complete induced C_k-saturated graph of order n.*

Finally, we will demonstrate the following regarding induced claw-saturated graphs.

Theorem. *(4.2) For all $n \geq 12$, there is a graph on n vertices that is induced $K_{1,3}$-saturated and is non-complete.*

2 Paths

2.1 An infinite family of paths

The only induced P_2-saturated graph on $n \geq 2$ vertices is $\overline{K_n}$. The induced P_3-saturated graph of order n with the smallest size is either the matching $\frac{n}{2}K_2$ if n is even, or $\frac{n-1}{2}K_2 \cup K_1$ if n is odd. The case for P_4 is similar, consisting of the matching $\frac{n}{2}K_2$ if n is even and $\frac{n-3}{2}K_2 \cup K_3$ if n is odd. It is also easily seen that the graph of order 9 and size 12 consisting of a triangle with each vertex sharing a vertex with another triangle, as in Figure 1, is induced P_5-saturated, and that the Petersen graph is induced P_6-saturated.

We begin our analysis of induced path-saturated graphs by examining an infinite family of cubic hamiltonian graphs developed by Lederberg [8] and modified by Coxeter and Frucht, and later by Coxeter, Frucht, and Powers [6, 2]. For our purposes we will only consider graphs from this family denoted in LCF (for Lederberg, Coxeter, Frucht) notation by $[x, -x]^a$ with x odd. A graph of this form consists of a 3-regular cycle on $2a$ vertices $\{v_0 v_1 \ldots v_{2a-1}\}$ and a matching that pairs each v_{2i} with v_{2i+x}, with arithmetic taken modulo $2a$. See Figure 2 for an example.

Let G_k denote the graph with LCF notation $[5, -5]^{7+4k}$. Note that the order of G_k is $14 + 8k$. First we find a long induced cycle in G_k.

Fact 2.1. *The graph G_k has an induced cycle of length $8 + 6k$.*

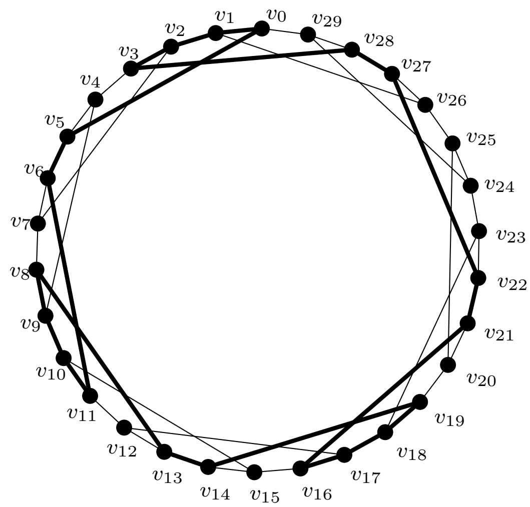

Figure 2: The graph G_2, which has LCF notation $[5, -5]^{15}$, with an induced C_{20}

Proof. Let n be the order of G, and let $C = \{v_0, v_1, v_2, v_3, v_{n-2}, v_{n-3}, v_{n-8}, v_{n-9}\}$. Then, proceed to add 6 vertices at a time to C in the following way until v_5 is included. Let v_p be the last vertex added to C. Add the vertices $\{v_{p-5}, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-7}, v_{p-8}\}$. Once v_5 is added, the graph induced by C is a chordless cycle of order $8 + 6k$ (Figure 2). □

Note that the closed neighborhoods of v_{n-5} and v_{n-6} are disjoint from the cycle C constructed in the proof of Fact 2.1. Therefore, the addition of any edge between any vertex on this cycle and v_{n-5} or v_{n-6} generates an induced path of order $9 + 6k$. A simple reflection that reverses v_{n-5} and v_{n-6} shows that another induced cycle of the same length exists in G_k.

We next must bound the length of induced paths in G_k. Note that a simple counting argument is not sufficient, since in general a 3-regular graph on $14+8k$ vertices may contain an induced path on as many as $10+6k$ vertices as seen in the following construction. Consider a path P on $10+6k$ vertices and an independent set X of $4+2k$ vertices. From each internal vertex in P add a single edge to a vertex in X, and from the endpoints in P add two edges to vertices in X, in such a way as to create a 3-regular graph. The resulting graph has order $14 + 8k$ and an induced path on $10 + 6k$ vertices.

Lemma 2.2. *The graph G_k contains no induced path on more than $8 + 6k$ vertices.*

Proof. First, note that an exhaustive search of G_0 yields a longest induced path of order 7. Consider the case where $k \geq 1$. Let P be a longest induced path in G_k. Let V be the $m = 8 + 6k$ vertices in the cycle C from the proof of Fact 2.1. Let the sets U, X, and Y contain the remaining vertices that have $3, 2$, and 0, neighbors, respectively, in V. Note that $|X| = 4$ and $|Y| = 2$, irrespective of k, and $|U| = 2k$. Consider the induced path P^0 on m vertices in Figure 3. For example, in Figure 2 P^0 would be the path $v_{26}v_1v_2v_3v_{28}v_{29}v_{24}v_{23}v_{22}v_{21}v_{20}v_{19}v_{14}v_{13}v_8v_9v_{10}v_{11}v_6v_5$. We claim P is no longer than P^0.

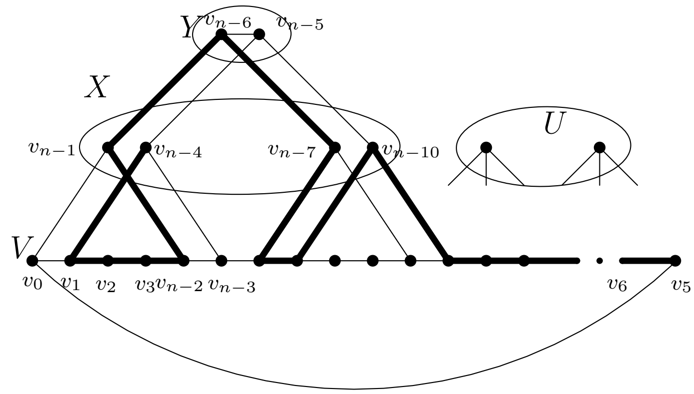

Figure 3: The graph G_k
with longest induced path P^0

For every vertex $u \in U$ in P there is one neighbor (if u is an internal vertex of P) or two neighbors (if u is a terminal vertex) from V not in P. Let us assume that $U_P = \{u_1, \ldots, u_l\} = U \cap P$, and denote by $N(U_P) \subset V$ the neighbors of all vertices in U_P. So P includes l vertices from U and avoids at least l vertices from $N(U_P)$.

Similarly, assume there is a vertex x in $P \cap V$ that is not in P^0. Then either a neighbor of x in $P^0 \cap V$ is not in P, or a route through $X \cup Y$ on P^0 is diverted and hence a vertex from $P^0 \cap (X \cup Y)$ is not in P. In either case, the inclusion of any vertex from U in P leads to at least one fewer vertex from P^0, and hence does not lead to a longer induced path.

Finally, we consider the possible inclusion of vertices from $X \cup Y$. It is easily seen that no induced path can include all vertices from $X \cup Y$. If P has no vertices from $X \cup Y$ then it is strictly shorter than P^0. If P contains exactly one or two vertices from X then there are at least three vertices in V not in P, making P at least as short as P^0. Any induced path containing 5 vertices from $X \cup Y$ must exclude one from Y as in P^0, and so no other induced path also containing 5 of these vertices will be longer that P^0. Therefore, P^0 is a longest induced path in G_k, and hence G_k does not contain an induced path on more than $8 + 6k$ vertices.

□

Now we demonstrate that the graph G_k is saturated with respect to the property of long induced paths.

Lemma 2.3. *The graph G_k is induced P_{9+6k}-saturated.*

Proof. Once again we let $n = 14 + 8k$, the order of G_k. By Lemma 2.2 there is no induced path of order $9 + 6k$ in the graph G_k. Define the bijections ϕ and ψ on the vertices of G_k by $\phi(v_i) = v_{1-i}$ (reflection) and $\psi(v_i) = v_{i+1}$ (rotation), with arithmetic modulo n. Note that both ϕ^2 and ψ are automorphisms of G_k. Given $v_i, v_j \in V(G_k)$ there is an automorphism of the form $\phi^{2k} \circ \psi^m$ that takes v_i to v_j, where k is some integer and $m \in \{0, 1\}$. Therefore, the graph G_k is vertex transitive under repeated applications of ϕ and ψ. In particular, the induced cycle C_k can be rotated and reflected via these functions to yield a function f such that for any pair x, y of nonadjacent vertices in G_k there is an image $f(C_k)$ so that x is on $f(C_k)$ and neither y nor its neighbors are on $f(C_k)$. Therefore, between any two nonadjacent vertices the addition of any edge creates an induced P_{9+6k}. □

We now generalize to an arbitrary number of vertices.

Lemma 2.4. *The disjoint union H of m copies of G_k, with at most one complete graph on at least 2 vertices, is induced P_{9+6k}-saturated.*

Proof. Since each connected component of H is induced P_{9+6k}-saturated by Lemma 2.3 we need only consider the addition of edges between components. Since G_k is vertex transitive and contains an induced cycle on $(8+6k)$ vertices (Fact 2.1), each vertex in G_k is the terminal vertex of an induced path on $(7+6k)$ vertices. Therefore, any edge between disjoint copies of G_k creates an induced path on far more than the necessary $(8+6k)$ vertices. The addition of an edge between a copy of G_k and a complete component will also result in an induced P_{9+6k} Therefore, H is induced P_{9+6k}-saturated. □

Lemma 2.5. *The join of any complete graph to any induced P_n-saturated graph, for any $n \geq 4$, generates a new induced P_n-saturated graph.*

Proof. Note that joining a clique to a graph does not contribute to the length of the longest induced path except in the most trivial cases of P_1, P_2, and P_3, nor does it add any non-edges to the graph which require testing for saturation. □

Note that joining a complete graph to any induced H-saturated graph, for any non-complete graph H, generates a new induced H-saturated graph. Therefore, we can prove the main result of this section.

Theorem 2.6. *For any $k \geq 0$ and $n \geq 14+8k$ there is a non-complete induced P_{9+6k}-saturated graph on n vertices. Further, if n is a multiple of $(14+8k)$ there is such a graph that is 3-regular.*

Proof. By Lemma 2.3 there is such a graph on $n = 14+8k$ vertices, and Lemmas 2.4 and 2.5 demonstrate that n increases without bound. □

2.2 Lower bounds

As an analogue to Theorem 1.1 by Erdős, Hajnal, and Moon [3], in which smallest K_n-saturated graphs are studied, we now turn our attention to finding the smallest induced P_m-saturated graphs. Assume throughout that $m > 3$.

First we look at some properties of induced P_m-saturated graphs with pendant edges, and then we will turn our attention to graphs with minimum degree two.

Fact 2.7. *If u and v are distinct pendant vertices in an induced P_m-saturated graph G then the distance from u to v is greater than three.*

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Proof. If u and v share a neighbor w then the addition of edge uv cannot create an induced path that includes w, so their distance is at least three. If instead u has neighbor w_u and v has neighbor w_v, with w_u adjacent to w_v, then the added edge uw_v must begin an induced P_m. However, this edge can be replaced in G by vw_v, so G must already contain an induced P_m. Therefore the neighbors of u and v cannot be adjacent. □

Next we examine the neighbor of a pendant vertex in an induced P_m-saturated graph.

Fact 2.8. *Let v be a pendant vertex in a non-complete component of an induced P_m-saturated graph G, with neighbor u. Then u has degree at least four.*

Proof. If $deg(u) = 2$ then the addition of the edge joining its neighbors cannot create a longer induced path than one that includes u. Assume u only has neighbors v, a, and b. If a and b are adjacent then the added edge va must begin an induced P_m that avoids b, but we can then replace va with ua and get an induced path of the same length. If instead a and b are not adjacent then adding edge ab to G does not result in an induced path longer than one containing the path aub. Hence u has at least one other neighbor c. □

For the remainder of the section we will consider non-complete graphs without pendant edges.

Fact 2.9. *If G is induced P_m-saturated and contains a vertex v of degree 2 then the neighbors of v are adjacent.*

Proof. Assume that $deg(v) = 2$ and v has neighborhood $\{u, w\}$. If u is not adjacent to w then the addition of edge uw cannot generate a longer induced path than one originally present in G that includes edges uv, vw. Therefore, u and w must already be adjacent. □

As noted in the beginning of Section 2.1, a matching with possibly an isolated vertex or a connected component isomorphic to K_3 constitute an induced P_3- and P_4-saturated graph, respectively. Note that when $m > 4$ an induced P_m-saturated graph cannot have more than one complete component, as any edge between two such components generates an induced path of order at most 4. We now demonstrate that any induced P_m-saturated graph on n vertices, for $m > 4$, has average degree at least 3 among its non-complete components.

Lemma 2.10. *For $m > 5$ all non-complete connected components of an induced P_m-saturated graph with no pendant edges have average degree at least 3.*

Proof. Let G be an induced P_m-saturated graph. If all vertices of G have degree 3 or more then the result is clear, so let us assume that v is a vertex of G with degree 2 and with neighbors u and w. By Lemma 2.9 u and w are adjacent. Without loss of generality we may assume that $\deg(w) \leq \deg(u)$. We will consider the cases in which $deg(w) = 2$ and $deg(w) > 2$.

First assume that both $\deg(u)$ and $\deg(w)$ are at least 3. We demonstrate that there are sufficiently many vertices of high degree to yield an average degree of at least 3. Let a be a neighbor of u and b a neighbor of w, with $a, b \notin \{u, v, w\}$. If $\deg(w) = 3$ and a and b are distinct then no induced path containing the new edge wa can be longer than an induced path containing the sub-path wua. If instead $a = b$ then the addition of va does not create any induced path not already in G by means of edge wa. Thus, if $\deg(u) \geq \deg(w) \geq 3$ then $\deg(u) \geq \deg(w) \geq 4$.

Now we consider the case in which $\deg(w) = 2$. Vertex u is therefore a cut vertex of G. Note that if there is another block containing u that is isomorphic to K_3 then the addition of an edge between two such blocks does not result in a longer induced path than one already present in the graph. If $\deg(u) = 3$, with u adjacent to a vertex a distinct from w and v, then adding edge va to G does not create any induced path longer than one already present in G that uses the edge ua. So $\deg(u) \geq 4$. Say that $\{v, w, u', w'\}$ are in the neighborhood of u and note that, as above, if $\deg(u') = \deg(w') = 3$ then the the addition of edge $w'a$ shows that G is not induced P_m-saturated. So the graph G with vertices v, w removed must also have average degree at least 3, and therefore G does as well.

Next consider the set T of vertices of degree 2 whose neighbors all have degree at least 3, and the set S composed of neighbors of vertices in T. Since vertices in S all have degree at least 4, if $t = |T| \leq |S| = s$ then the graph has at least as many vertices with degree greater than three than those with degree 2 and we are done. Assume instead that $t > s$. Since the two neighbors of each vertex in T are adjacent, we know that for each vertex in T there are at least 3 edges in the induced graph $< T \cup S >$. Hence the average degree among vertices in $< T \cup S >$ is at least $\frac{6t}{s+t} > \frac{6t}{2t} = 3$. Since all other vertices in G either have degree at least 3 or are part of a distinct triple with total degree at least 9 as shown above, the average degree of any non-complete component of an induced P_m-saturated graph is at least 3.

□

This leads us to the proof of the lower bound for the size of a class of induced P_m-saturated graphs.

Theorem 2.11. *If G is an induced P_m-saturated graph on n vertices with no pendant edges except for a K_2 component, $m > 5$, then G has size at least $\frac{3}{2}(n-2)+1$. This bound is realized when $m = 9 + 6k$ and $n \equiv 2 \mod (14 + 8k)$.*

Proof. In the graph G all but at most one connected component consists of vertices of average degree at least 3, with potentially one component isomorphic to K_2 or K_3 by Lemma 2.10. Therefore, $e(G) \geq \frac{3}{2}(n-2)+1$. By Lemma 2.4 the graph consisting of disjoint copies of G_k and a K_2 has size $\frac{3}{2}(n-2)+1$ and is induced P_m-saturated. □

2.3 Other path results

For certain induced P_m-saturated graphs we can create induced P_{m+2}-saturated graphs by using the following constructions.

Construction 2.12. *Generate the graph $T_v(G)$ by identifying each vertex in G with one vertex of a distinct triangle. The new graph $T_v(G)$ has order $3n(G)$ and size $e(G) + 3n(G)$ (Figure 4).*

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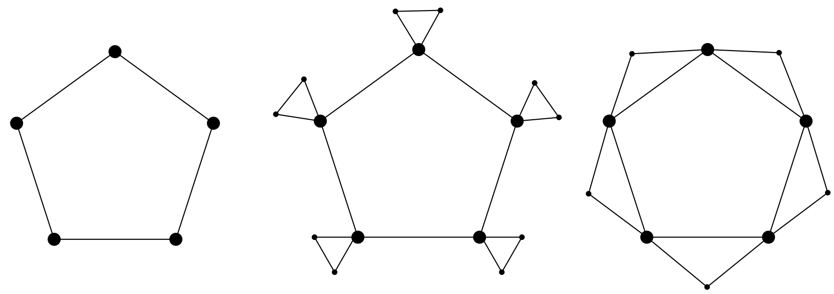

Figure 4: The graphs $C_5, T_v(C_5)$, and $T_e(C_5)$

Construction 2.13. *The graph $T_e(G)$ is composed of the graph G along with a new vertex for each edge of G, adjoined to both endpoints of that edge. The graph $T_e(G)$ has order $n(G) + e(G)$ and size $3e(G)$ (Figure 4).*

Now we will show that both constructions yield the expected results. First, we restate and prove Theorem 2.14 in a different form than that given in Section 1.1.

Theorem 2.14. *The graph $T_v(G_k)$ is induced P_{11+6k}-saturated.*

Proof. First we establish that every vertex v in the graph G_k is the terminal vertex for two induced paths of order $8+6k$, each with a different terminal edge. Let P be the path that begins $\{v_0v_5v_6v_7v_2v_3v_{n-2}v_{n-3}v_{n-4}v_{n-5}v_{n-6}v_{n-7}v_{n-12}\}$. If $i = (n-12)$ then proceed similar to C in Fact 2.1 by adding $\{v_iv_{i-1}v_{i-6}v_{i-5}v_{i-4}v_{i-3}v_{i-8}\}$ to P, repeating until the addition of edge $v_{10}v_9$. By applying the automorphism $\phi^8 \circ \psi$ we get another induced path of order $8+6k$ starting at v_0, P'. Notice that P contains the edge v_0v_5 and P' the edge v_0v_{n-1}. Again, since G_k is vertex transitive, we see that each vertex in G_k is the terminal vertex for two induced paths with distinct terminal edges.

Next, we consider a pair x, y of distinct non-adjacent vertices in G_k. We need only consider the cases in which a pair of nonadjacent vertices are both in the original graph G_k, neither in the original graph G_k, or exactly one is in G_k.

Say that $x, y \in G_k$. Since the addition of edge xy to G_k creates an induced P_{9+6k}, one vertex from each added triangle to the endpoints of this path in $T_v(G_k)$ yields an induced P_{11+6k}. If neither x nor y are in G_k and their neighbors in G_k, say x' and y' respectively, are not adjacent then, since a new edge between x' and y' in G_k generates an induced P_{9+6k}, this path is extended similarly by two edges to create an induced P_{11+6k} in $T_v(G_k)$. If instead $x'y'$ is an edge of G_k, consider the induced P_{8+6k} in G_k that begins at x' and avoids the edge $x'y'$. This extends to an induced P_{9+6k} in $T_v(G_k)$. Since $x'y'$ is not in this path, then the vertex y' is also avoided entirely. The addition of edge xy to $T_v(G_k)$ creates an induced P_{11+6k} beginning with y.

Lastly, consider the case in which x is a vertex of G_k and y is not. Let y' be y's neighbor in G_k and y'' the vertex of degree 2 adjacent to y in $T_v(G_k)$. Again, since there is an induced

P_{9+6k} in $T_v(G_k)$ that begins at x and avoids y', the addition of the edge xy creates an induced P_{11+6k} beginning at y''. □

Theorem 2.15. *If G is K_3-free, induced P_m-saturated, and every vertex is in a component of order at least three, then $T_e(G)$ is induced P_{m+2}-saturated.*

Proof. Just as in the proof of Theorem 2.12 we need to consider the same three cases.

If x, y are nonadjacent vertices, both in G, then the addition of edge xy to G generates an induced P_m. Since none of the neighbors of the end vertices are in the path, it can be extended on both ends to the added vertices, yielding an induced P_{m+2}. If both x and y are new vertices added in the construction of $T_e(G)$, then let the neighbors of x be x', x'' and the neighbors of y y', y''. If adding edge $x'y'$ or $x''y''$ creates an induced P_m that avoids the edges $x'x''$ and $y'y''$ then the addition of edge xy generates an induced P_{m+2}. If instead every induced P_m created by adding either edge $x'y'$ or $x''y''$ includes at least one of the edges $\{x'x'', y'y''\}$ then there is an induced P_m that includes the added edge $x'y''$ that does not since G is K_3-free . Therefore the addition of edge xy is equivalent to adding an edge between a neighbor of each in G, with the induced P_m extended by one edge toward x and one toward y. Lastly, if x is in the original graph G and y is not, then we proceed as above and extend the induced P_m that results from joining x to a neighbor of y not already adjacent to x (which exists since G is K_3-free) by one edge toward y and by another edge at a terminal vertex of the path. Therefore, $T_e(G)$ is induced P_{m+2}-saturated. □

We end this section by noting that a computer search using SAGE [11], in conjunction with Constructions 2.12 and 2.13, has found induced P_m-saturated graphs for all $7 \leq m \leq 30$. The results are listed in Table 5, most in LCF notation. In the interest of space we have omitted the proof that they are saturated, as each is simply a case analysis. Note that there are induced P_m-saturated graphs that cannot be written in the form $[x, -x]^n$, and further some are the result of the operations T_e and T_v. Therefore, not all induced P_m-saturated graph are regular nor the result of a regular graph joined to a complete graph.

3 Cycles

The star $K_{1,(n-1)}$ is induced C_3-saturated, and is in fact the graph on n vertices of smallest size for $n \geq 3$. This is a direct consequence of Theorem 1.1. The largest such graph is $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ due to Mantel [9].

Note that C_5 is trivially both induced C_3-saturated and induced C_4-saturated.

We now show that for all integers $k \geq 3, n \geq 3(k-2)$ there is an induced C_k-saturated graph on n vertices that is non-complete. We begin with another construction.

Construction 3.1. *Define the graph $G[k]$ on $3k$ vertices, $k \geq 3$, in the following way. Let $v_0 v_1 \ldots v_{k-1} v_0$ be a k-cycle, the* internal cycle *of $G[k]$. Add the matching $u_i w_i$ and the edges $u_i v_i, w_i v_i$, and $w_i u_{i+1}$, $0 \leq i \leq (k-1)$ with addition modulo k, the* external cycle *(Figure 6).*

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m	induced P_m-saturated graph
7	$[4, -4, 5, -5]^3$
8, 9	$[5, -5]^7$
10	$[4, 6]^{10}$
11	$T_e([5, -5]^7])$
12, 13	$[5, -5, 9, -9]^5$
14, 15	$[5, 9]^{13}$
16	$[9, 15]^{12}$
17	$T_e([5, 9]^{13})$
18	$[5, 17]^{15}$
19, 20	$[7, 23]^{15}$
21, 22	$[5, -5, 13, -13]^8$
23	$[-17, 9]^{20}$
24	$T_e([5, -5, 13, -13]^8)$
25, 26	$[-15, 15]^{19}$
27	$[-13, 13]^{19}$
28	$[-15, 15]^{21}$
29, 30	$[-13, 13]^{22}$

Figure 5: Induced path-saturated graphs

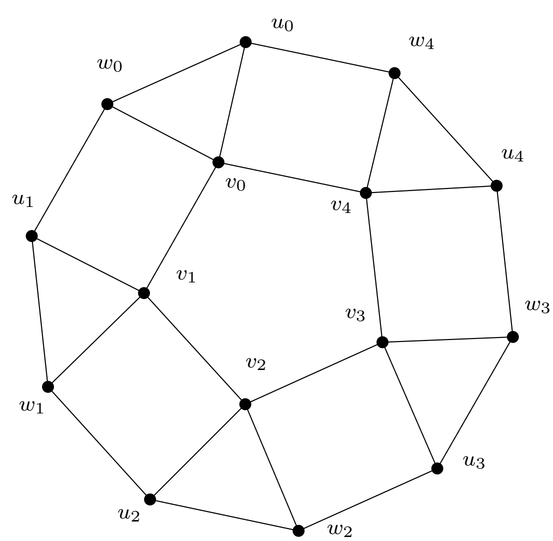

Figure 6: The graph $G[5]$, which is induced C_7-saturated

Claim 3.2. *The graph $G[k]$ is induced C_{k+2}-saturated.*

Proof. First we show that $G[k]$ does not contain an induced cycle of length $k+2$. Note that every copy of C_{k+2} in $G[k]$ contains vertices from both the internal and external cycles. Any induced cycle C in $G[k]$ contains paths on the internal cycle of the form $v_i v_{i+1} v_{i+2} \dots v_j$ and/or paths on the outer cycle, and edges joining these paths into a cycle. The cycle C therefore has length either k (if $i = j$) or at least $(k+3)$.

Now we show that the addition of any edge e to $G[k]$ results in an induced C_{k+2}. Note that there are three potential forms that e can take: an edge among the vertices of the internal cycle, an external cycle edge, or an edge between these cycles. If $e = v_i v_j$ then we need only consider $k > 3$. There is a newly created induced cycle of length $l \geq (\lceil \frac{k}{2} \rceil + 1)$ along the internal cycle. This can be extended by considering an edge from v_i to one of its neighbors on the external cycle, and traversing an appropriate number of edges before rejoining the internal cycle. In this way we create an induced cycle of every length between $(\lceil \frac{k}{2} \rceil + 4)$ and $(k+3)$, inclusive.

If instead e joins vertices between the internal and external cycles, then we create an induced C_{k+2} in the following way. Without loss of generality we assume that $e = v_i u_0$. We get an induced C_{k-i+2} by proceeding around the internal cycle from v_i to v_{k-1} then to w_{k-1}. Other induced cycles result from returning to the outer cycle sooner, creating cycles of length $(k-i+3)$ through $(2k-2i+1)$. We can also find induced cycles proceeding in the other direction along the internal cycle from v_i down to v_1 (or v_0 if $i \neq (k-1)$), then back to u_1, yielding induced cycles with lengths from $(i+3)$ through $(2i+2)$. Therefore, an induced cycle of length $(k+2)$ can be found in $G[k]$ with the added edge for $1 \leq i \leq \frac{k-1}{2}$ in the latter case and $\frac{k}{2} \leq i \leq (k-1)$ in the former.

Finally, if the new edge joins vertices on the external cycle of $G[k]$ then an induced cycle of length $(k+2)$ can be formed by utilizing an edge from v_i to the internal cycle, continuing along a sufficiently long path, then rejoining the described induced path along the external cycle. □

Theorem 3.3. *For any $k \geq 3$ and $n \geq 3(k-2)$ there is a non-complete induced C_k-saturated graph of order n.*

Proof. By Claim 3.2 there is an n and a graph $G[k-2]$ on n vertices that is induced C_k-saturated. We can extend Construction 3.1 to a larger number of vertices by joining it to a clique, since any vertex in the joined clique is adjacent to all other vertices so cannot be in the induced cycle. □

The graph $G[k]$ can also be extended to more vertices by replacing each vertex with a clique. If the vertices are distributed in a balanced way the resulting graph has size approximately $3k\binom{n/(3k)}{2} + 5k\frac{n^2}{9k^2}$.

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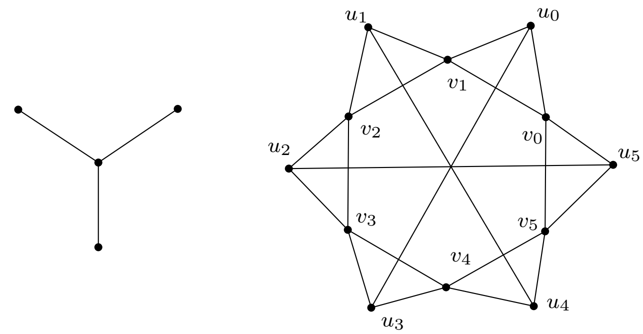

Figure 7: A claw $K_{1,3}$ and the induced $K_{1,3}$-saturated graph G

4 Claws

We now turn our attention to the claw graph $K_{1,3}$ (Figure 7). We build a graph that is induced $K_{1,3}$-saturated.

Construction 4.1. *Let G be a 6-cycle on the vertices $\{v_0, \ldots, v_5\}$. To this set, we add the vertices $\{u_0, \ldots, u_5\}$ and for each i join u_i to v_i, v_{i+1}, and u_{i+3} with addition taken modulo 6 (Figure 7).*

It is easy to see that the graph in Construction 4.1 is claw-free. We can now prove the following theorem.

Theorem 4.2. *For all $n \geq 12$, there is a graph on n vertices that is induced $K_{1,3}$-saturated and is non-complete graph.*

Proof. First we demonstrate that the graph G in Construction 4.1 is induced $K_{1,3}$-saturated. If we join u_i to u_j then u_i is the center of a claw with neighbors u_j, u_{i+3}, and v_i. If we join u_i to v_j then v_j has pairwise nonadjacent neighbors u_i, v_{j-1} or v_{j+1}, and either u_j or u_{j-1}. Finally, if edge v_iv_j is added to G then v_i is the center of a claw along with v_j, u_i, and u_{i-1}.

Since the disjoint union of induced $K_{1,3}$-saturated graphs is itself induced $K_{1,3}$-saturated we can generate a graph on n vertices with disjoint copies of G and possibly a complete connected component. □

5 Future Work

It would be interesting to find a smallest construction $G(m)$ that is induced P_m-saturated for all $m > 1$, or determine that no such construction exists. It is suspected that $G(m)$ has size $\frac{3}{2}n(G(m))$, but the largest such graph, in the spirit of Turán's Theorem [12], would also be worth investigating. Induced P_m-saturated graphs with pendant edges also remain to be studied, as these graphs may be smaller than those in Theorem 2.11. Indeed, it is not hard to construct such a graph by joining $K_1 \cup G_k$ to a single vertex, but this graph is quite large.

Further, as we have considered paths and claw graphs in this paper the study of induced saturation could be furthered by considering the family of trees.

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References

[1] M.S. Chung, T. Jiang, D.B. West "Induced Turán problems: Largest P_m-free graphs with bounded degree", *Preprint*.

[2] H.S.M. Coxeter, R. Frucht, D.L. Powers "Zero-symmetric graphs: Trivalent graphical regular representations of groups", *New York: Academic Press* (1981).

[3] P. Erdős, A. Hajnal, J.W. Moon "A problem in graph theory", *Amer. Math. Monthly 71* (1964), **1107-1110**.

[4] J.R. Faudree, R.J. Faudree, J.R. Schmitt "A survey of minimum saturated graphs", *Elec. J. of Comb. DS19* (2011), **36 pp**.

[5] M. Ferrara, M. Jacobson, K. Milans, C. Tennenhouse, P. Wenger "Saturation numbers for families of graph subdivisions", *J. Graph Theory 71,4* (2012), **416-434**.

[6] R. Frucht "A canonical representation of trivalent Hamiltonian graphs", *J. Graph Theory 1,1* (1976), **45-60**.

[7] M.S. Jacobson, C. Tennenhouse "Oriented graph saturation", *J. Comb. Math. Comb. Comp. 80* (2012), **157-169**.

[8] J. Lederberg "DENDRAL-64: A System for Computer Construction, Enumeration and Notation of Organic Molecules as Tree Structures and Cyclic Graphs. Part II. Topology of Cyclic Graphs", *Interim Report to the National Aeronautics and Space Administration* (1965).

[9] W. Mantel, "Problem 28", solution by H. Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh, W.A. Wythoff, *Wiskundige Opgaven 10* (1907), **60-61**.

[10] R.R. Martin, J.J. Smith "Induced saturation number", *Disc. Math. 312 21* (2012), **3096-3106**.

[11] W. Stein et al. "Sage Mathematics Software (Version 5.8)", *The SAGE Development Team* (2013), http://www.sagemath.org.

[12] P. Turán "On the theory of graphs", *Colloq. Math 3* (1954), **19-30**, MR 15,976b.

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