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Eternal Independent Sets in Graphs

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Abstract

The use of mobile guards to protect a graph has received much attention in the literature of late in the form of eternal dominating sets, eternal vertex covers and other models of graph protection. In this paper, eternal independent sets are introduced. These are independent sets such that the following can be iterated forever: a vertex in the independent set can be replaced with a neighboring vertex and the resulting set is independent.

1 Graph Protection

Let $G = (V, E)$ denote a finite, undirected graph with vertex set V and edge set E. The problem of protecting a graph with mobile guards has been studied in a number of recent papers. We shall begin with a review of some of these models before introducing the eternal independent set problem, which can be viewed in the same light.

A *dominating set* of a graph $G = (V, E)$ is a set $D \subseteq V$ such that each vertex in $V - D$ is adjacent to a vertex in D. The minimum cardinality amongst all dominating sets of G is the *domination number* $\gamma(G)$.

Let $\{D_i\}$, $D_i \subseteq V$, $i \geq 1$, be a collection of sets of vertices of the same cardinality, with one guard located on each vertex of D_i. Each protection problem can be modeled as a two-player game between a *defender* and an *attacker*: the defender chooses D_1 as well as each D_i, $i > 1$, while the attacker chooses the locations of the attacks $r_1, r_2, \ldots$ (which are sometimes called *requests*). Each attack is dealt with by the defender by choosing the next D_i in response to the attack r_i, subject to some constraints that depend on the particular game. The defender wins the game if they can successfully defend any sequence of attacks, subject to the constraints of the game described below; the attacker wins otherwise. We note that the sequence of attacks may be infinite in length.

We say that a vertex (edge) is *protected* if there is a guard on the vertex or on an adjacent (incident) vertex. A vertex v is *occupied* if there is a guard on v, otherwise v is *unoccupied*. An attack is *defended* if a guard moves to the attacked vertex (across one edge, i.e., in one "step").

1.1 Eternal Protection Problems

For the **eternal domination problem**, each D_i, $i \geq 1$, is required to be a dominating set, $r_i \in V$ (assume without loss of generality $r_i \notin D_i$), and D_{i+1} is obtained from D_i by moving one guard to r_i from an adjacent vertex $v \in D_i$. If the defender can win the game with the sets $\{D_i\}$, then each D_i is an *eternal dominating set*. The size of a smallest eternal dominating set of G is the *eternal domination number* $\gamma^\infty(G)$. This problem was first studied by Burger et al. in [1] and will be referred to as the *one-guard moves* model.

For the m-**eternal dominating set problem**, each D_i, $i \geq 1$, is required to be a dominating set, $r_i \in V$ (assume without loss of generality $r_i \notin D_i$), and D_{i+1} is obtained from D_i by allowing each guard to move to a neighboring vertex (if it so chooses). That is,

each guard in D_i may move to an adjacent vertex, as long as one guard moves to r_i. Thus it is required that $r_i \in D_{i+1}$. The size of a smallest *m-eternal dominating set* (defined similar to an eternal dominating set) of G is the *m-eternal domination number* $\gamma_{\mathrm{m}}^{\infty}(G)$. This "all guards move" version of the problem was introduced by Goddard, Hedetniemi and Hedetniemi [3]. The m in m-eternal denotes that multiple guards may move in response to an attack.

In the **eviction model**, each configuration D_i, $i \geq 1$, of guards is required to be a dominating set. An attack occurs at a vertex $r_i \in D_i$ such that there exists at least one $v \in N(r_i)$ with $v \notin D_i$. The next guard configuration D_{i+1} is obtained from D_i by moving the guard from r_i to a vertex $v \in N(r_i)$, $v \notin D_i$ (i.e., this is the "one-guard moves" model). The size of a smallest eternal dominating set in the eviction model for G is denoted $e^{\infty}(G)$. That is, attacks occur at vertices with guards and we must move that guard to an unoccupied neighboring vertex. This problem was introduced in [6].

A *vertex cover* of G is a set $C \subseteq V$ such that each edge of G is incident with a vertex in C. The minimum cardinality of a vertex cover of G is the *vertex cover number* $\tau(G)$ of G. An *independent set* of G is a set $I \subseteq V$ such that no two vertices in I are adjacent. The maximum cardinality amongst all independent sets is the *independence number* $\alpha(G)$. It is well known that $\alpha(G) + \tau(G) = n$ for all graphs G of order n (see e.g. [2, p. 241]).

The *clique covering number* $\theta(G)$ is the minimum number k of sets in a partition $V = V_1 \cup \cdots \cup V_k$ of V such that each $G[V_i]$ is complete. Hence, as is well-known, $\theta(G)$ equals the chromatic number $\chi(\overline{G})$ of the complement $\overline{G}$ of G. Thus for every graph G, $\alpha(G) \leq \theta(G)$. It is known that $\gamma^{\infty}(G) \leq \theta(G)$ for all G [3].

A *matching* in G is a set of edges, no two of which have a common endvertex. The *matching number* $m(G)$ is the maximum cardinality of a matching of G. It is also well known that $\tau(G) \geq m(G)$ for all graphs, and that equality holds for bipartite graphs (see e.g. [2, Theorem 9.13]). An *induced matching* in G is a set of edges M, such that the subgraph induced by the endvertices of M contains no edges other than M. The size of a maximum induced matching in G is denoted as $m_i(G)$. A matching $M = \{e_i = (v_i, u_i) : i = 1, \ldots, k\}$ is called a *free matching* (sometimes called a bipartite matching) if $\{v_1, \ldots, v_k\}$ and $\{u_1, \ldots, u_k\}$ are both independent sets. The cardinality of largest free matching in G denoted $m_f(G)$.

For the *m*-**eternal vertex covering problem,** each D_i, $i \geq 1$, is required to be a vertex cover, $r_i \in E$, and D_{i+1} is obtained from D_i by moving one or more guards to neighboring vertices; i.e., each guard in D_i may move to an adjacent vertex provided that one guard moves across edge r_i (we assume without loss of generality that one end-vertex of r_i is not in D_i, otherwise the two guards on the endvertices of r_i simply interchange positions). If the defender can win the game with the sets $\{D_i\}$, then each D_i is an *eternal vertex cover*. The size of a smallest eternal vertex cover of G is the *eternal covering number* $\tau_{\mathrm{m}}^{\infty}(G)$. This problem was introduced in [7].

A survey on eternal protection problems can be found in [8].

1.2   Eternal Independent Sets

For the **eternal independent set problem** each D_i, $i \geq 1$, is required to be an independent set, $r_i \in D_i$, and D_{i+1} is obtained from D_i by moving the guard on r_i to an adjacent vertex. (We say the vertex r_i is *attacked*.) If the defender can win the game with the sets $\{D_i\}$, then each D_i is an *eternal independent set*. The size of a largest eternal independent set of G is the *eternal independence number* $\alpha^\infty(G)$. This will sometimes be referred to as the *one-guard moves* model.

For the m-**eternal independent set problem** each D_i, $i \geq 1$, is required to be an independent set, $r_i \in D_i$, and D_{i+1} is obtained from D_i by moving the guard on r_i to an adjacent vertex while the remaining guards in D_i may also move to neighboring vertices (so long as $r_i \notin D_{i+1}$). If the defender can win the game with the sets $\{D_i\}$, then each D_i is an m-*eternal independent set*. The size of a largest m-eternal independent set of G is the m-*eternal independence number* $\alpha_m^\infty(G)$. This will sometimes be referred to as the *all-guards move* model.

We shall sometimes say that if a guard at u moves to v, that vertices u and v are *switched*. A *total-switch* of an independent set D into an independent set Z is a simultaneous replacement of all vertices in D where each vertex $v_i \in D$ is replaced by a neighbor z_i such that $|D| = |Z|$. Note that $D \cap Z = \emptyset$, since D is an independent set. For the **total-eternal independent set problem** each D_i, $i \geq 1$, is required to be an independent set, $r_i \in D_i$, and D_{i+1} is obtained from D_i by a total-switch. If the defender can win the game with the sets $\{D_i\}$, then each D_i is an *total-eternal independent set*. The largest cardinality of a total-eternal independent set is the total-eternal independence number of G denoted $\alpha_t^\infty(G)$. Clearly $\alpha_t^\infty(G) \leq \alpha_m^\infty(G)$. Observe that for the total-eternal independent set problem, the actual sequence of attacks does not matter, since all the guards must move upon each attack.

These eternal independent set problems are analogous to the eviction model of eternal domination. Related concepts for independent sets have been considered in [4, 5, 9], but the exact parameters defined here have not been studied prior to this, as far as we know.

1.3   Examples

We give a few small examples to illustrate the various definition. Observe that $\alpha^\infty(C_4) = 1$, $\alpha^\infty(C_5) = 2$, $\alpha_m^\infty(C_4) = 2$, and $\alpha_m^\infty(C_5) = 2$. We alert the reader to the fact that C_5 is an example that will be used several more times throughout the paper and illustrated in Figure 1. In Figure 1, a guard on a shaded vertex can move to an unshaded neighbor (the left guard must move clockwise, the right guard must move counterclockwise from this initial configuration) and the resulting guard configuration induces an independent set (and is isomorphic to the initial configuration). Also $\alpha^\infty(K_{n,n}) = 1$ and $\alpha_m^\infty(K_{n,n}) = n = \alpha_t^\infty(K_{n,n})$.

The *corona* of a graph G, denoted $cor(G)$, is the graph obtained from G by adding a pendant vertex to every vertex of G. $cor(K_3)$ is an example with $\alpha_m^\infty(cor(K_3)) = \theta(cor(K_3)) = 3 > m_i(cor(K_3)) = 1$ as well as $\alpha_m^\infty(cor(K_3)) > m_f(cor K_3) = 2$. Furthermore, $\alpha_t^\infty(cor(K_3)) = 2 < \alpha_m^\infty(cor(K_3))$ and $\alpha^\infty(cor(K_3)) = \alpha(K_3) = m_i(K_3) = 1$. More generally, it is easy to see that $\alpha^\infty(cor(G)) = \alpha(G)$ and the simple proof is omitted.
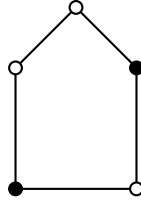
Figure 1: C_5, eternal independent set shaded

2   Chain of Inequalities

Theorem 2.1 *Let G be a graph without isolated vertices. Then $m(G) \geq \alpha_m^\infty(G) \geq m_f(G) = \alpha_t^\infty(G) \geq \alpha^\infty(G) \geq m_i(G)$.*

Proof.   1. That $m(G) \geq \alpha_m^\infty(G)$ follows from the second part of the proof of Theorem 4.2, below.

2. Clearly $m(G) \geq m_f(G)$.

3. That $\alpha_m^\infty(G) \geq m_f(G)$ is true follows since we can perform a total-switch along the edges of a free-matching.

4. Suppose $M = \{e_i = (v_i, u_i) : i = 1, \ldots, k\}$ is a maximum cardinality free matching. Take $D = \{v_1, \ldots, v_k\}$ and $Z = \{u_1, \ldots, u_k\}$ with $D \cap Z = \emptyset$. Observe that $|D| = |Z|$ and both are independent sets as M is a free matching.

In a total-switch request for D, we replace D by Z where v_i moves to u_i . If another request is done (now on Z) we switch back to D. Hence D is total-eternal and $\alpha_t^\infty(G) \geq |D| = m_f(G)$.

Conversely, let D be a maximum total eternal independent set. A total-switch sends D to Z such that Z is independent, $|D| = |Z|$ and every vertex v_i in D moved to a neighbor z_i in Z. Since $|Z| = |D|$, it follows that every v_i moved to a distinct neighbor z_i in Z hence $M = \{e_i = (v_i, z_i) : i = 1, \ldots, |D|\}$ is a free matching. Hence $m_f(G) \geq |M| = |D| = \alpha_t^\infty(G)$.

5. Let D be a maximum eternal independent set, $D = \{v_1, \ldots, v_k\}$ .Suppose the sequence of requests is $v_1, v_2, \ldots, v_k$. Then v_1 moves to a neighbor z_1, v_2 to a neighbor z_2, and so on so $Z = \{z_1, \ldots, z_k\}$ is an independent set. Now a total-switch of D sends it to Z via the same edges $e_i = (v_i, z_i)$ and a total switch on Z send it back to D, hence D is also total eternal independent set, and $\alpha_t^\infty(G) \geq |D| = \alpha^\infty(G)$.

6. Suppose $M = \{e_i = (v_i, u_i) : i = 1, \ldots, k\}$ is an induced matching of maximum cardinality . Take $D = \{v_1, \ldots, v_k\}$ and $Z = \{u_1, \ldots, u_k\}$ and observe $|D| = |Z|$ and both are independent sets as M is an induced matching. Observe also that any vertex v_i is independent of $Z \setminus \{u_i\}$ and any vertex u_i is independent of $D \setminus \{v_i\}$ as M is an induced matching.

In any infinite sequence of switchings imposed on D, we always keep moving v_i to u_i and u_i to v_i . So all these requests keep us with independent set T with none/some/all vertices in D and none/some/all vertices in Z such that $|D| = |T| = |Z|$; hence D is an eternal independent set. Hence $\alpha^\infty(G) \geq |D| = m_i(G)$. ∎

There exist graphs for which equality in the chain give in Theorem 2.1 does not necessarily hold. Consider C_5, where $2 = \alpha^\infty(C_5) > m_i(C_5) = 1$. There are also graphs for which $\alpha_m^\infty(G) < m(G)$ such as K_3 with a pendant vertex attached to one of the vertices or a K_5 with a pendant vertex attached to one of the vertices.

It seems that a graph with large matching and with low chromatic number should force a large free matching and hence a large total-eternal independence number. We detail this relationship in the next proposition.

Proposition 2.2 *Let G be a graph with $\chi(G) = k$ and $m(G) = m$. Then $\alpha_m^\infty(G) \geq \alpha_t^\infty(G) = m_f(G) \geq 2m/k(k-1)$.*

Proof. Let M be a maximum matching of cardinality m. The subgraph of G induced by M, denoted G^*, has $\chi(G^*) = t \leq k$. Let $A_1, \dots, A_t$ be the color classes of G^*.

Now the m edges of M are divided into $t(t-1)/2$ pairs (A_i, A_j). Hence, by averaging, for some pair (i, j), the pair (A_i, A_j) contains at least $m/(t(t-1)/2 \geq m/(k(k-1)/2 = 2m/k(k-1)$ edges from M and these edges form a free matching. ∎

3 Clique Coverings

Observe that $\alpha^\infty(G) \leq \theta(G)$, for all graphs G, since $\alpha(G) \leq \theta(G)$, for all G (since no clique in a clique cover can contain more than one vertex from any independent set).

Proposition 3.1 *Let G be a connected triangle–free graph with $\theta(G) \geq 2$ and no isolated vertices. Then $\alpha^\infty(G) < \theta(G)$.*

Proof. Suppose to the contrary that $\alpha^\infty(G) = \theta(G)$. Let $C = C_1, C_2, \dots, C_\theta$ be a minimum clique cover. Note that $|C_i| \leq 2$ since G is triangle-free. Then an eternal independent set D must contain exactly one vertex from each C_i. If we request a vertex $v \in C_i, v \in D$, that vertex must switch to another vertex in C_i (since every $C_j, j \neq i$ contains another vertex in D). Thus each C_i must be a K_2. Let u, v be two vertices of minimum distance in D and such that the cliques from C in which they are contained are connected by an edge. Clearly $2 \leq dist(u, v) \leq 3$. If $dist(u, v) = 2$, then a request to one of them (which one depends on their locations) will switch one of them so that u and v are adjacent. If $dist(\text{u, v}) = 3$, then consecutive requests to both u and v will cause two switches resulting in u and v being adjacent. ∎

Proposition 3.1 is sharp for infinitely many graphs. Let G consist of n paths of length three having a common vertex w, i.e., a star $K_{1,n}$ where each edge is subdivided once. G is

K_3 free with $\theta(G) = n + 1$ and $\alpha^\infty(G) = n$ (because of the induced matching, see Theorem 4.1 below). We leave open the problem of characterizing the triangle-free graphs for which $\alpha^\infty(G) = \theta(G) - 1$.

As another example, for a triangle–free graph G on n vertices, $cor(G)$ (which is again triangle–free) has the following properties: $\theta(cor(G)) = n > \alpha^\infty(cor(G)) \geq c\sqrt{n \log n}$. The left hand-side come from Proposition 3.1, while the right side come from the Ramsey number $R(K_3, K_n)$ and the fact that $\alpha^\infty(cor(G)) = \alpha(G)$, since it is well-known that if G is triangle–free, it has an independent set of cardinality at least $c\sqrt{n \log n}$ and this is sharp.

We can ask for which connected graphs is $\alpha^\infty(G) = \alpha(G) = \theta(G)$? It seems difficult to structurally describe these graphs but some observations are in order. If $\theta = 1$, then $\alpha^\infty(G) = \alpha(G) = \theta(G)$. Now let $\theta(G) > 1$ and $C = \{C_1, C_2, \ldots, C_k\}$ be a minimum clique covering. Supposing $\alpha(G) = \theta(G)$, we get that there is an independent set consisting of one vertex from each C_i. In order for $\alpha^\infty(G) = \alpha(G)$, clearly each C_i must contain at least two vertices and no two C_i's that are both K_2's can be joined by an edge. This leads us to the following.

Theorem 3.2 *Let G be a connected graph. Then $\alpha^\infty(G) = \theta(G)$ if and only if $m_i(G) = \theta(G)$.*

Proof.   First suppose $m_i(G) = \theta(G)$. Recall from Theorem 2.1 that $\alpha^\infty(G) \geq m_i(G)$. If $\alpha^\infty(G) > \theta(G)$, then by the pigeonhole principle there must simultaneously exist two guards within the same clique from some minimum clique-covering. But two such guards cannot be on independent vertices. Thus $\alpha^\infty(G) = \theta(G)$.

For the other direction, let us assume $\alpha^\infty(G) = \theta(G)$. From the observation above, we may assume that $\theta(G) > 1$.

Using the notation from above, each clique C_i from clique cover C can contain at most one edge from any matching. Further, each C_i is a clique with at least two vertices, because if any C_i is a K_1, then we can easily force a switch that destroys independence. Any eternal independent set D of cardinality $\theta(G)$ contains exactly one vertex from each clique of clique cover C, since $\alpha^\infty(G) = \alpha(G) = \theta(G)$. Denote the vertices in D as $D = \{v_1, v_2, \ldots, v_k\}$, $k = \theta(G)$ and let $v_i \in C_i$. Obviously D is an independent set. If C_i is a $K_i, i > 1$, let $\{u_i^1, u_i^2, \ldots\}$ be the other vertices in the clique C_i along with v_i. For simplicity in what follows, we shall omit the superscript on a u_i^k vertex when it is clear from the context and refer to these as u_i type vertices.

We construct a modified graph G' as follows. If any u_i type vertex is adjacent to any $v_j, j \neq i$, delete that u_i vertex (since if v_i were attacked first in G, the guard could not switch to that u_i vertex without destroying independence). Then if any K_2's in the resulting graph have a u_i type vertex adjacent to any u_j type vertices, $j \neq i$, delete all such u_j vertices (since such vertices cannot be switched to without destroying independence). In the resulting graph G', what must remain are K_2 components and other cliques with more than two vertices (any two such cliques with more than two vertices may be connected via a limited number of edges). If there are any K_1 components in G', then D is not an eternal independent set, since we could force a switch in G that destroys independence. The K_2 components can be

removed and placed into the induced matching, M, that we are building. So only cliques with more than two vertices remain in the reduced graph G'. Observe that neither $(u_i v_j)$ nor $(u_j v_i)$ are edges for any distinct cliques C_i, C_j, in the clique cover C when restricted to G'.

Let $D' \subseteq D$ be the vertices of D that are in G'. Let $D' = \{v_1, v_2, \ldots, v_t\}$ and let $D^* = D \setminus D'$. Considering G, start with guards on the vertices of D and attack all the vertices of D^* and then attack each of the vertices in D', with $v_i \in D'$ switching to a vertex u_i^a, for some a. The set of u_i^a vertices are an independent set. Either the edges switched across form an induced matching or some u_i^a is adjacent to some v_b. But there are no such adjacencies in the graph G'. Hence we can add these edges switched across to the K_2 components above to form an induced matching of G. ∎

It seems interesting to find graphs classes for which $\alpha_m^\infty(G) = \theta(G)$; C_4 and P_4 are two examples where equality holds, but which have $\alpha_m^\infty(G) > m_i(G)$.

4   Bipartite Graphs

Theorem 4.1 *Let $G = (A, B, E)$ be a bipartite graph. Then $\alpha^\infty(G) = m_i(G)$.*

Proof.   If there is a maximum induced matching with t edges, then a vertex can be switched along each of these edges eternally; therefore there exists an eternal independent set with t vertices.

Suppose there exists an eternal independent set D with k vertices. We can request a set vertices be attacked such that all these vertices are in A, say on $a_1, a_2, \ldots, a_k$. Because if some vertex $v \in D$ is not in A (i.e., $v \notin A$), then it must be that $V \in B$; so therefore we can attack v. Thus the guard on v cannot stay in B and must move to A , so we can repeat requesting until all vertices are in A, say $a_1, \ldots, a_k$, $|D| = k$.

Let b_i be the vertex in B such that if a_i is attacked next, a_i is switched to b_i. Then b_i cannot be adjacent to any a_k, $k \neq i$, otherwise independence would be destroyed, contradicting that the set of vertices was an eternal independent set.

So consider the set of edges $e_i = (a_i, b_i)$ which is a matching. There are no edges between the a_i being all in A , there are no edges between the b_i being all in B. If there is an edge between e_i and e_j $(j \neq i)$ then it is either (a_i, b_j) or (a_j, b_i) which is impossible. As all b_i are independent from all a_j $(j \neq i)$, this is an induced matching. ∎

This property does not hold for all non-bipartite graphs: C_5 is an example of a graph with $\alpha^\infty(C_5) = 2$ and $m_i(C_5) = 1$. Furthermore, observe that any tree T with $\theta(T) = \alpha(T) > 2$ has $\theta(T) > m_i(T)$. This is because in order for $\theta(T) = m_i(T)$, T would have to all the edges in the tree in some minimum clique covering. But two edges that are joined by an edge cannot be in the same induced matching.

A linear-time algorithm for finding a maximum induced matching in a tree is given in [10, 11]. Thus, using Theorem 4.1, one can find give an algorithm that computes the order of the maximum-eternal independent set in a tree in linear time. We give here an alternative

linear time algorithm which is simpler and directly finds a maximum eternal independent set in a tree.

A *stem* in a tree is a vertex adjacent to a leaf and the *height* of a tree with specified root vertex r is the maximum distance from r to any leaf.

If the height of a tree T with at least two vertices is one, then the maximum eternal independent set is of size 1. Otherwise, suppose the height of tree T is more than one. In this case, we find a root vertex r of T that is not a stem, which necessarily exists as T is not a $K_{1,m}$ for any $m \geq 1$. The root may be a leaf.

We shall build a set D that will eventually contain the vertices of a maximum eternal independent set. Pick a stem v_1 of maximum distance from r. Let w be parent of v_1 (possibly, $w = r$). Let $v_1, \ldots, v_k$ be all the stems that are children of w. Place each v_i in D. Remove all children and grandchildren of w from T, letting the resulting tree be T'. Proceed recursively on T', terminating when the tree T' has height at most one. If T' has height at most one, then no more vertices will be added to D.

We now prove the algorithm finds a maximum eternal independent set.

Proof. When the height of T is one, $\alpha^\infty(T) = m_i(T) = 1$.

Now assume the height of T is $h > 1$. When $h = 2$, D consists of the children of r. In this case $|D| = m_i(G)$. Let us suppose $h > 2$. Consider the tree T' as described in the algorithm. Clearly the maximum eternal independent set of $T - T'$ consists of k vertices: $v_1, \ldots, v_k$ (none of which are leaves), since a guard on v_i can move to one of its children. Note that w is a leaf in T'. Then the eternal independent set D' found by the algorithm in T' is a largest eternal independent set not containing w. Therefore, $D' \cup \{v_1, \ldots, v_k\}$ is a maximum eternal independent set of T, since a guard in v_i can move to its child when attacked.

In other words, D consists of vertices labeled v_i at any time in the algorithm. These vertices form an independent set; therefore no vertex ever labeled w can be part of this same independent set. No vertex labeled w can be subsequently labeled as v_i, as w becomes a leaf in the tree T'. If we think of the edges that guards move across in this scheme as a matching, then D consists of one endvertex from each edge in this matching. Each neighbor of a w vertex is an endvertex of an edge in this matching.

Furthermore, the root, r, cannot be part of this independent set unless it is labeled as v_1 at some point in the algorithm, otherwise a guard on r would have to move to one of its children when r is attacked, but this child was once a w vertex (and thus is adjacent to some v_i vertex that may have a guard on it). ∎

Theorem 4.2 *Let $G = (A, B, E)$ be a bipartite graph. Then $\alpha_m^\infty(G) = m(G)$.*

Proof. Recall that in the m-eternal independent set problem, we may move as many guards as needed (including the possibility of a total-switch), as long as we move the guard from the attacked vertex.

Let $M = \{e_i = (v_i, u_i) : i = 1, \ldots, k\}$ be a matching of maximum cardinality, so $k = m(G)$. Since we can do total-switches from the endvertices of M in A to the endvertices of

M in B, it follows that $\alpha_m^\infty(G) \geq m(G)$.

We claim that $\alpha_m^\infty(G) \leq m(G)$. Suppose by way of contradiction that $\alpha_m^\infty(G) > m(G)$. Then there exists an m-eternal independent set $D = \{v_1, v_2, \ldots, v_j\}$ with $j > k$. For each $v_i \in D$, let u_i be the vertex that would be switched with v_i if v_i were requested first. Let U be the set of all the u_i's and choose U to be of maximum cardinality over all the possible choices of the u_i vertices. There are two cases.

Case 1. Suppose all the u_i are distinct. Then the set of edges $v_i u_i$ is a matching that is larger than M, a contradiction.

Case 2. Suppose $u_i = u_h = x$ for some i, h. Then if v_i is attacked first, it is switched to x and if v_h is attacked first, it is switched to x. Then if v_i is attacked first, v_h must also switch, else the resulting set of vertices is not independent. Say that v_h moves to $y \neq x$. If $y \notin U$, then the set $U \cup \{y\}$ is a larger set with the property described above, a contradiction. So suppose $y \in U$. Then y is adjacent to some $v_a \neq v_h, v_a \neq v_i$. Then by the same logic as before, there must be some z that v_a switches to when v_i and v_h are switched to x and y, respectively. Again, we can either use z to produce a larger set than U or continue to iterate the argument. Eventually, we must arrive at a similar contradiction or else reach a point where there is no legal switch for a vertex, which is also a contradiction. ∎

Summarizing the results for bipartite graphs, we have the following.

Theorem 4.3 *Let G be a bipartite graph. Then $m(G) = \alpha_m^\infty(G) \geq m_f(G) = \alpha_t^\infty(G) \geq \alpha^\infty(G) = m_i(G)$.*

5 Open Problems

We list some future problems for consideration, most of which concern characterizing graphs for which the extremes are attained.

1. Characterize the graphs G having $\alpha_m^\infty(G) = m(G)$.

2. Characterize the graphs G having $\alpha_m^\infty(G) = m_f(G)$.

3. Find further graphs classes for which $\alpha^\infty(G) = m_i(G)$; in particular classes of triangle-free graphs.

4. Characterize the graphs G having $\alpha_m^\infty(G) = m_f(G)$.

5. Find graphs G with $\alpha_m^\infty(G) = n - \tau_m^\infty(G)$. $K_{1,n}$ for $n > 2$ is an example where equality does not hold. Equality holds for cycles, as $\tau_m^\infty(C_n) = \lceil \frac{n}{2} \rceil$, see [7].

6. Describe some graph classes for which $\alpha_m^\infty(G) = \alpha(G)$. Well-covered graphs (i.e., graphs in which all maximal independent sets have the same cardinality) have this property, since there exists a perfect matching between the vertices in the symmetric difference of any two maximal independent sets, c.f. [4]. When a vertex is attacked, there exists a maximal independent set containing a neighbor of the attacked vertex (since each vertex belongs to some maximal independent set) and there exists a perfect matching that can be switched across between the vertices in the symmetric difference of these two maximal independent sets.

Cayley graphs are another class of graphs that may have this property.

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References

[1] Burger, A. P., Cockayne, E. J., Gründlingh, W. R., Mynhardt, C. M., van Vuuren, J. H., Winterbach, W., Infinite order domination in graphs. *J. Combin. Math. Combin. Comput.* **50** (2004), 179–194.

[2] Chartrand, G., Lesniak, L., *Graphs Digraphs*. Fourth Edition, Chapman & Hall, London, 2005.

[3] Goddard, W., Hedetniemi, S. M., Hedetniemi, S. T., Eternal security in graphs. *J. Combin. Math. Combin. Comput.* **52** (2005), 169–180.

[4] Hartnell, B., Mynhardt, C. M., Independent protection in graphs. *Discrete Mathematics* **335** (2014), 100–109.

[5] Kaminski, M., Medvedev, P., Milanic, M., Complexity of independent set reconfigurability problems, *Theor Comput Sci*, **439** (2012), pp. 9-15.

[6] Klostermeyer, W., Lawrence, M., MacGillivray, G., A Dynamic Domination Problem: the Eviction Model for Eternal Domination. To appear in *J. Combin. Math. Combin. Comput.* (2016).

[7] Klostermeyer, W., Mynhardt, C. M., Edge protection in graphs. *Australas. J. Combin.* **45** (2009), 235–250.

[8] Klostermeyer, W., Mynhardt, C. M., Protecting a graph with mobile guards, to appear in *Appl. Anal. Discrete Math.* (2016), see also http://arxiv.org/abs/1407.5228

[9] Regan, F., *Dynamic variants of domination and independence in graphs*, graduate thesis, Rheinischen Friedrich-Wilhlems University, Bonn, 2007.

[10] Zito, M., Maximum induced matching on regular graphs and trees, manuscript (1999).

[11] Zito, M. Linear time maximum induced matching algorithm for trees, *Nordic Journal of Computing* **7** (2000), pp. 58–63.