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Group-antimagic Labelings of Multi-cyclic Graphs

Dan Roberts
Illinois Wesleyan University, drobert1@iwu.edu

Richard M. Low
San Jose State University, richard.low@sjsu.edu

Abstract

Let A be a non-trivial abelian group. A connected simple graph $G = (V, E)$ is A-*antimagic* if there exists an edge labeling $f : E(G) \to A \backslash \{0\}$ such that the induced vertex labeling $f^+ : V(G) \to A$, defined by $f^+(v) = \Sigma\ \{f(u, v) : (u, v) \in E(G)\}$, is a one-to-one map. The *integer-antimagic spectrum* of a graph G is the set $\mathrm{IAM}(G) = \{k : G \text{ is } \mathbb{Z}_k\text{-antimagic and } k \geq 2\}$. In this paper, we analyze the integer-antimagic spectra for various classes of multi-cyclic graphs.

1 Introduction

A labeling of a graph is defined to be an assignment of values to the vertices and/or edges of the graph. Graph labeling is a very diverse and active field of study. A dynamic survey [6] maintained by Gallian contains over 1400 references to research papers and books on the topic.

Let G be a connected simple graph. For any non-trivial abelian group A (written additively), let $A^* = A\backslash\{0\}$, where 0 is the additive identity of A (sometimes denoted by 0_A). Let a function $f : E(G) \to A^*$ be an edge labeling of G. Any such labeling induces a map $f^+ : V(G) \to A$, defined by $f^+(v) = \sum_{uv \in E(G)} f(uv)$. If there exists such an edge labeling f whose induced map f^+ on $V(G)$ is one-to-one, we say that f is an A-*antimagic labeling* and that G is an A-*antimagic* graph. The *integer-antimagic spectrum* of a graph G is the set $\mathrm{IAM}(G) = \{k : G \text{ is } \mathbb{Z}_k\text{-antimagic and } k \geq 2\}$.

The concept of the A-antimagicness property for a graph G (introduced in [1]) naturally arises as a variation of the A-magic labeling problem (where the induced vertex labeling is a constant map). $\mathbb{Z}$-magic (or $\mathbb{Z}_1$-magic) graphs were considered by Stanley [32, 33], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [2, 3, 4] and others [11, 13, 19, 20, 27, 31] have studied A-magic graphs and $\mathbb{Z}_k$-magic graphs were investigated in [8, 10, 12, 14, 15, 16, 17, 18, 21, 24, 25, 26, 28, 30]. For other types of magic graph labelings, the interested reader is directed to Marr and Wallis' monograph [22].

A trivial lower bound for the least element of $\mathrm{IAM}(G)$ is the order of G; however, this is not always achieved, as seen in the following result from [1].

Lemma 1.1 (Chan et al.). *A graph of order $4m+2$, for all $m \in \mathbb{N}$, is not $\mathbb{Z}_{4m+2}$-antimagic.*

Motivation for our current work is found in the following conjecture.

Conjecture 1.1. *Let G be a connected simple graph. If t is the least positive integer such that G is Z_t-antimagic, then* $\mathrm{IAM}(G) = \{k : k \geq t\}$.

A result of Jones and Zhang [7] finds the minimum element of $\mathrm{IAM}(G)$ for all connected graphs on 3 or more vertices. In their paper, a $\mathbb{Z}_n$-antimagic labeling of a graph on n vertices is referred to as a *nowhere-zero modular edge-graceful labeling.* This is a variation of a *graceful* labeling (originally called a β-valuation) which was introduced by Rosa [23] in 1967. The result is as follows, where the terminology has been adapted to better suit this paper.

Theorem 1.2 (Jones and Zhang). *If G is a connected simple graph of order $n \geq 3$, then* $min\{t : t \in \mathrm{IAM}(G)\} \in \{n, n+1, n+2\}$. *Furthermore,*

1. *min*$\{t : t \in \mathrm{IAM}(G)\} = n$ *if and only if* $n \not\equiv 2 \pmod 4$, $G \neq K_3$, *and* G *is not a star of even order,*

2. *min*$\{t : t \in \mathrm{IAM}(G)\} = n+1$ *if and only if* $G = K_3$ *or* $n \equiv 2 \pmod 4$ *and* G *is not a star of even order, and*

3. *min*$\{t : t \in \mathrm{IAM}(G)\} = n+2$ *if and only if* G *is a star of even order.*

In [1], Conjecture 1.1 was shown to be true for various classes of graphs. The purpose of this paper is to provide additional evidence for Conjecture 1.1 by verifying it for various classes of multi-cyclic graphs. We use constructive methods to determine integer-antimagic spectra of the graph classes in question.

2 Some Known Results

In this section, we include some known results [1] for reference. In particular, theorems (with an included proof) are used in the construction of new $\mathbb{Z}_k$-antimagic labelings in this paper.

Theorem 2.1. *P_{4m+r} and C_{4m+r}, for all $m \in \mathbb{N}$, are $\mathbb{Z}_k$-antimagic, for all $k \geq 4m+r$ if $r = 0, 1, 3$. P_{4m+2} and C_{4m+2}, for all $m \in \mathbb{N}$, are $\mathbb{Z}_k$-antimagic, for all $k \geq 4m+3$.*

Proof. Let $e_1, e_2, \ldots, e_{n-1}$ be edges of P_n, from left to right. A $\mathbb{Z}_k$-antimagic labeling of P_n can be obtained as follows.

Case 1. $n = 4m$:

$$f(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd;} \\ \frac{i}{2} & \text{if } i \text{ is even and } 2 \leq i \leq 2m-2; \\ \frac{i+2}{2} & \text{if } i \text{ is even and } 2m \leq i \leq 4m-2. \end{cases}$$

Case 2. $n = 4m+1$:

$$f(e_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even;} \\ \frac{i+3}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq 2m-3; \\ \frac{i+5}{2} & \text{if } i \text{ is odd and } 2m-1 \leq i \leq 4m-1. \end{cases}$$

Case 3. $n = 4m+2$:

$$f(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd;} \\ \frac{i+2}{2} & \text{if } i \text{ is even and } 2 \leq i \leq 2m-2; \\ \frac{i+4}{2} & \text{if } i \text{ is even and } 2m \leq i \leq 4m. \end{cases}$$

Case 4. $n = 4m+3$:

$$f(e_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even;} \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq 2m-1; \\ \frac{i+3}{2} & \text{if } i \text{ is odd and } 2m+1 \leq i \leq 4m+1. \end{cases}$$

Let $e_1, e_2, \ldots, e_n$ be edges of C_n arranged in counter-clockwise direction. A $\mathbb{Z}_k$-antimagic labeling of C_n can be obtained as follows.

Case 1. $n = 4m$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2m; \\ 3 + 2(2m - \lceil \frac{i}{2} \rceil) & \text{if } 2m+1 \leq i \leq 4m. \end{cases}$$

Case 2. $n = 4m + 1$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2m; \\ 3 + 2(2m - \lceil \frac{i}{2} \rceil) & \text{if } 2m + 1 \leq i \leq 4m + 1. \end{cases}$$

Case 3. $n = 4m + 2$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2m + 3; \\ 3 + 2(2m - \lceil \frac{i-2}{2} \rceil) & \text{if } 2m + 4 \leq i \leq 4m + 2. \end{cases}$$

Case 4. $n = 4m + 3$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2m + 3; \\ 3 + 2(2m - \lceil \frac{i-3}{2} \rceil) & \text{if } 2m + 4 \leq i \leq 4m + 3. \end{cases}$$

□

Theorem 2.2. *Let $n \geq 4$ and S_n denote the star graph having $n-1$ leaves. If n is odd, then S_n is $\mathbb{Z}_k$-antimagic, for all $k \geq n$. Otherwise, S_n is $\mathbb{Z}_k$-antimagic, for all $k \geq n + 2$; but not $\mathbb{Z}_n$-antimagic nor $\mathbb{Z}_{n+1}$-antimagic.*

3 $\mathbb{Z}_k$-antimagic Labelings of Wheels and Wheel-like Graphs

Let W_n denote the *wheel on n spokes*, which is the graph containing a cycle of length n with another special vertex not on the cycle, called the *central vertex*, that is adjacent to every vertex on the cycle. Name the vertices of W_n as follows: the central vertex is named v_0 and the other vertices are named counter-clockwise as $v_1, \ldots, v_n$. We will refer to edges of the form $v_0 v_i$ for $1 \leq i \leq n$ as *spokes* and edges of the form $v_i v_{i+1}$ for $1 \leq i \leq n-1$ or $v_n v_1$ as *outer-cycle edges*. The subgraph of W_n formed by the outer-cycle edges will be referred to as the *outer-cycle*. Following the naming for the edges of a cycle found in the proof of Theorem 2.1, an outer-cycle edge $v_i v_{i+1}$ receives the name e_{i+1}, and the edge $v_n v_1$ is named e_1. Furthermore, for every $i \neq 0$ the spoke with end-vertex v_i receives the name e'_i.

First, we note that $W_2 \cong C_3$ is clearly not $\mathbb{Z}_3$-antimagic. Figure 1 illustrates $\mathbb{Z}_k$-antimagic labelings ($k \geq 4$), for W_2 and W_3.

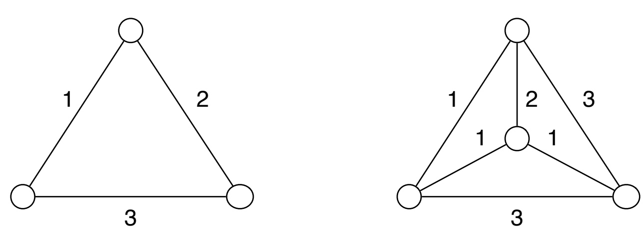

Figure 1: $\mathbb{Z}_k$-antimagic labelings ($k \geq 4$) of W_2 and W_3, respectively.

Theorem 3.1. *Let $m \in \mathbb{N}$. Then, W_{4m+r} is $\mathbb{Z}_k$-antimagic for all $k \geq 4m+r+1$ if $r = 0, 2, 3$ and W_{4m+1} is $\mathbb{Z}_k$-antimagic for all $k \geq 4m + 3$.*

Proof. Case 1. Labeling W_{4m+r} for $r = 0, 2$:

Let $k \geq 4m + r + 1$ be fixed. The outer-cycle is of even length, and hence admits a one-factorization into two one-factors, say M_1 and M_2. We will first define our labeling, f, on the outer-cycle edges as follows:

$$f(e) = \begin{cases} 1 & \text{if } e \in M_1; \\ k-1 & \text{if } e \in M_2. \end{cases}$$

The subgraph formed by the spokes is a star with $4m+r+1$ vertices, and hence has a $\mathbb{Z}_k$-antimagic labeling, g, by Theorem 2.2. Define the labeling on the spokes as $f(e) = g(e)$ for all spokes e. This induces a labeling, $f^+ : V(W_{4m+r}) \to \mathbb{Z}_k$, on the vertices where $f^+(v_0) \equiv 0 \pmod k$ and $f^+(v_i) = g(e_i')$ for each $1 \leq i \leq 4m+r$. For $1 \leq i \neq j \leq 4m + r$ we have $g(e_i') \neq g(e_j') \neq 0$. Thus, f is the desired $\mathbb{Z}_k$-antimagic labeling.

Case 2. Labeling W_{4m+3}:

Let $k \geq 4m + 4$ be fixed. We will first define a labeling, g, on the outer-cycle edges to be the labeling defined for a cycle of length $4m + 3$ in Theorem 2.1. Notice that the labels induced by g on the vertices in the outer-cycle, denoted by g^+, form the set $\{3, 4, \ldots, 4m + 4, 4m + 5\}$. Now, define $f : E(W_{4m+3}) \to \mathbb{Z}_k^*$ such that for every outer-cycle edge, e_i, we have $f(e_i) = g(e_i)$, and for the spokes:

$$f(e_i') = \begin{cases} 3 & \text{if } g^+(v_i) = 4m+4; \\ 1 & \text{if } g^+(v_i) \neq 4m+4. \end{cases}$$

Thus,

$$f^+(v_i) = \begin{cases} g^+(v_i) + 1 & \text{if } g^+(v_i) \neq 4m+4; \\ (4m+4) + 3 & \text{if } g^+(v_i) = 4m+4; \\ 1(4m+2) + 3 & \text{if } i = 0. \end{cases}$$

The labels on the vertices induced by f form the set $\{4, 5, \ldots, 4m+6, 4m+7\}$; thus, f is the desired $\mathbb{Z}_k$-antimagic labeling.

Case 3. Labeling W_{4m+1}:

Let $k \geq 4m + 3$ be fixed. We will first define a labeling, g, on the outer-cycle edges to be the labeling defined for a cycle of length $4m + 1$ in Theorem 2.1. Notice that the labels induced by g on the vertices in the outer-cycle, denoted by g^+, form the set $\{2, 3, \ldots, 4m + 1, 4m + 2\}$. Now, define $f : E(W_{4m+1}) \to \mathbb{Z}_k^*$ such that for every outer-cycle edge, e_i, we have $f(e_i) = g(e_i)$, and for the spokes:

$$f(e_i') = \begin{cases} 3 & \text{if } g^+(v_i) = 4m+2; \\ 1 & \text{if } g^+(v_i) \neq 4m+2. \end{cases}$$

Thus,

$$f^+(v_i) = \begin{cases} g^+(v_i) + 1 & \text{if } g^+(v_i) \neq 4m+2; \\ (4m+2) + 3 & \text{if } g^+(v_i) = 4m+2; \\ 1(4m) + 3 & \text{if } i = 0. \end{cases}$$

The labels on the vertices induced by f form the set $\{3, 4, \ldots, 4m+2, 4m+3, 4m+5\}$; thus, f is the desired $\mathbb{Z}_k$-antimagic labeling.

□

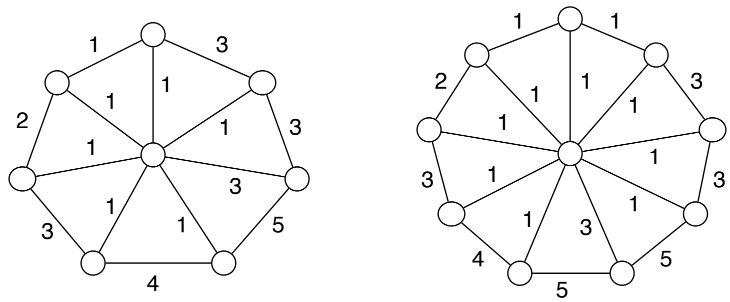

Figure 2: $\mathbb{Z}_k$-antimagic labelings of W_7 for $k \geq 8$, and W_9 for $k \geq 11$, respectively.

Let F_n denote the *fan* on $n+1$ vertices, which is defined to be the graph obtained from W_n by deleting the edge e_1. We have the same vertex and edge names for F_n as we did for W_n (omitting the edge name e_1, of course). Concerning F_n, the subgraph formed by the edges of the form e_i will be referred to as the *outer-path*, and its edges will be referred to as *outer-path edges*. Edges of the form e'_i will still be referred to as *spokes*.

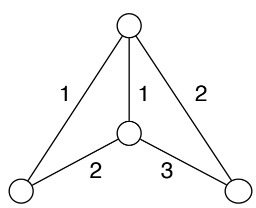

Figure 3: A $\mathbb{Z}_k$-antimagic labeling of F_3, for $k \geq 4$.

Theorem 3.2. *Let $m \in \mathbb{N}$. Then, F_{4m+r} is $\mathbb{Z}_k$-antimagic for all $k \geq 4m+r+1$ if $r = 0, 2, 3$ and F_{4m+1} is $\mathbb{Z}_k$-antimagic for all $k \geq 4m+3$.*

Proof. Case 1. Labeling F_{4m+1}:

Let $k \geq 4m+3$ be fixed. Let g be the labeling on the outer-path edges obtained from the labeling of a cycle of length $4m+1$ in Theorem 2.1 with the edge e_1 omitted. Notice that the labels induced by g on the vertices in the outer-path, denoted by g^+, form the set $\{1, 2, 4, 5, \ldots, 4m+2\}$. Now, define $f : E(F_{4m+1}) \to \mathbb{Z}_k^*$ such that for every outer-path edge, e_i, we have $f(e_i) = g(e_i)$, and for the spokes:

$$f(e'_i) = \begin{cases} 2 & \text{if } g^+(v_i) = 2, 4m+2; \\ 1 & \text{if } g^+(v_i) \neq 2, 4m+2. \end{cases}$$

Thus,

$$f^+(v_i) = \begin{cases} g^+(v_i)+1 & \text{if } g^+(v_i) \neq 2, 4m+2; \\ g^+(v_i)+2 & \text{if } g^+(v_i) = 2, 4m+2; \\ 1(4m-1)+2(2) & \text{if } i = 0. \end{cases}$$

The labels on the vertices induced by f form the set $\{2, 4, 5, \ldots, 4m+4\}$; thus, f is the desired $\mathbb{Z}_k$-antimagic labeling.

Case 2. Labeling F_{4m+3}:

Let $k \geq 4m+4$ be fixed. Let g be the labeling on the outer-path edges obtained from the labeling of a cycle of length $4m+3$ in Theorem 2.1 with the edge e_1 omitted. Notice that the labels induced by g on the vertices in the outer-path, denoted by g^+, form the set $\{2, 3, 5, 6, \ldots, 4m+5\}$. Now, define $f : E(F_{4m+3}) \to \mathbb{Z}_k^*$ such that for every outer-path edge, e_i, we have $f(e_i) = g(e_i)$, and for the spokes:

$$f(e_i') = \begin{cases} 3 & \text{if } g^+(v_i) = 2; \\ 2 & \text{if } g^+(v_i) = 4m+5; \\ 1 & \text{if } g^+(v_i) \neq 2, 4m+5. \end{cases}$$

Thus,

$$f^+(v_i) = \begin{cases} g^+(v_i)+1 & \text{if } g^+(v_i) \neq 2, 4m+5; \\ g^+(v_i)+3 & \text{if } g^+(v_i) = 2; \\ g^+(v_i)+2 & \text{if } g^+(v_i) = 4m+5; \\ 1(4m+1)+3+2 & \text{if } i = 0. \end{cases}$$

The labels on the vertices induced by f form the set $\{4, 5, \ldots, 4m+7\}$; thus, f is the desired $\mathbb{Z}_k$-antimagic labeling.

Case 3. Labeling F_{4m}:

Let $k \geq 4m+1$ be fixed. Let g be the labeling on the outer-path edges obtained from the labeling of a cycle of length $4m$ in Theorem 2.1 with the edge e_1 omitted. Notice that the labels induced by g on the vertices in the outer-path, denoted by g^+, form the set $\{2, 3, 5, 6, \ldots, 4m+2\}$. Now, define $f : E(F_{4m}) \to \mathbb{Z}_k^*$ such that for every outer-path edge, e_i, we have $f(e_i) = g(e_i)$, and for the spokes:

$$f(e_i') = \begin{cases} 2 & \text{if } g^+(v_i) = 2, 3, 4m+2; \\ 1 & \text{if } g^+(v_i) \neq 2, 3, 4m+2. \end{cases}$$

Thus,

$$f^+(v_i) = \begin{cases} g^+(v_i)+1 & \text{if } g^+(v_i) \neq 2, 3, 4m+2; \\ g^+(v_i)+2 & \text{if } g^+(v_i) = 2, 3, 4m+2; \\ 1(4m-3)+2(3) & \text{if } i = 0. \end{cases}$$

The labels on the vertices induced by f form the set $\{4, 5, \ldots, 4m+4\}$; thus, f is the desired $\mathbb{Z}_k$-antimagic labeling.

Case 4. Labeling F_{4m+2}:

Let $k \geq 4m+3$ be fixed. First, re-label the edges of the outer-path with the assignment $e_i \to e_{i-1}$. Let g be the labeling on the outer-path edges obtained from the labeling of a path on $4m+2$ vertices in Theorem 2.1. Notice that the labels induced by g on the vertices in the outer-path, denoted by g^+, form the set $\{1, 3, 4, \ldots, 4m+3\}$. Now, define $f : E(F_{4m+2}) \to \mathbb{Z}_k^*$ such that for every outer-path edge, e_i, we have $f(e_i) = g(e_i)$, and for the spokes:

$$f(e_i') = \begin{cases} 2 & \text{if } g^+(v_i) = 1, 4m+3; \\ 1 & \text{if } g^+(v_i) \neq 1, 4m+3. \end{cases}$$

Thus,

$$f^+(v_i) = \begin{cases} g^+(v_i) + 1 & \text{if } g^+(v_i) \neq 1, 4m+3; \\ g^+(v_i) + 2 & \text{if } g^+(v_i) = 1, 4m+3; \\ 1(4m) + 2(2) & \text{if } i = 0. \end{cases}$$

The labels on the vertices induced by f form the set $\{3, 4, \ldots, 4m+5\}$; thus, f is the desired $\mathbb{Z}_k$-antimagic labeling.

□

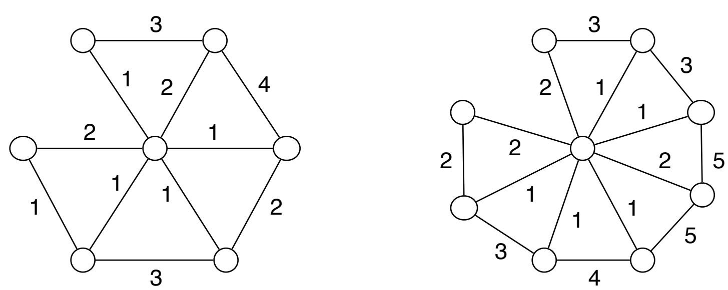

Figure 4: $\mathbb{Z}_k$-antimagic labelings of F_6 for $k \geq 7$, and F_8 for $k \geq 9$, respectively.

A *friendship* graph is a simple graph in which any two distinct vertices have exactly one common neighbor. A result of Erdős et al. [5] shows that all friendship graphs are isomorphic to some W_n with a 1-factor deleted from the outer-cycle (thus, n must be even). Let FG_n denote the friendship graph on $n+1$ vertices. We name the vertices of FG_n in the same way that we named the vertices of W_n. We will refer to edges of the form $v_0 v_i$ for $1 \leq i \leq n$, named e_i', as *spokes* and edges of the form $v_i v_{i+1}$ for $i = 1, 3, 5, \ldots, n-1$, named $e_{(i+1)/2}$, as *outer 1-factor edges*. The subgraph of FG_n formed by the outer 1-factor edges will be referred to as the *outer 1-factor*.

Theorem 3.3. *Let $n \in \{4, 6, 8, 10, \ldots\}$. Then, FG_n is $\mathbb{Z}_k$-antimagic for all $k \geq n+1$.*

Proof. First note that $FG_2 \cong C_3$, which is $\mathbb{Z}_k$-antimagic if and only if $k \geq 4$. For the remainder of the proof, we assume that $n \geq 4$ and n is even.

Let $k \geq n+2$ be fixed. The subgraph formed by the spokes is a star with n edges, and therefore admits a $\mathbb{Z}_k$-antimagic labeling g (with central vertex having induced label 0 (mod k)), by Theorem 2.2. There must be some element $x \in \mathbb{Z}_k^*$ that g doesn't assign to any spoke.

Now, define $f : E(FG_n) \to \mathbb{Z}_k^*$ such that for every spoke, e_i', we have $f(e_i') = g(e_i')$, and for every outer 1-factor edge, e_i, we have $f(e_i) = k - x$. Let f^+ denote the labels induced by f on the vertices of FG_n. Notice that $\{f^+(v_i) : 1 \le i \le n\} = \{g(e_i') + k - x : 1 \le i \le n\}$. Since all of the $g(e_i')$'s are distinct, so are the labels induced by f on the vertices of the outer 1-factor. Furthermore, we have that for all $1 \le i \le n$, $f^+(v_i) \not\equiv 0 \pmod{k}$, otherwise there would be some i for which $g(e_i') = x$. Since $f^+(v_0) \equiv 0 \pmod{k}$, f is the desired $\mathbb{Z}_k$-antimagic labeling.

Now, let $k = n + 1$. Define the labeling $f : E(FG_n) \to \mathbb{Z}_k^*$ such that for every spoke, e_i', we have $f(e_i') = i$, and for the outer 1-factor edges:

$$f(e_i) = \begin{cases} 2 & \text{for } i = 1, 2, 3, \ldots, \frac{n-2}{2}; \\ 3 & \text{for } i = \frac{n}{2}. \end{cases}$$

The labels induced on the vertices of the outer 1-factor by f form the set $\{3, 4, 5, \ldots, n\} \cup \{n+2, n+3\}$, and $f^+(v_0) \equiv 0 \pmod{n+1}$. Thus, f is the desired $\mathbb{Z}_k$-antimagic labeling. $\square$

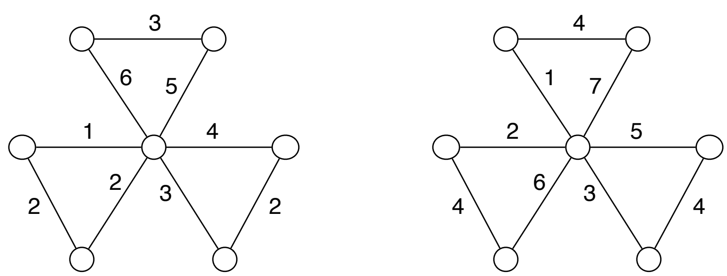

Figure 5: $\mathbb{Z}_k$-antimagic labelings of FG_6 for $k = 7$ and 8, respectively.

A *helm on* $2n+1$ *vertices*, denoted H_n, is the graph obtained from the wheel graph, W_n, by adjoining a pendant vertex to each vertex on the outer-cycle. In H_n, the names of all vertices and edges in the subgraph isomorphic to W_n have the same names as in W_n, except that the edges e_i' are referred to as *inner-spokes*. For each vertex v_i, we name the leaf that is adjacent to it w_i and refer to these vertices as *pendant vertices*. Each edge of the form v_iw_i is named e_i'', and these edges are referred to as *outer-spokes*.

If we delete the outer-cycle edges of H_n, then we are left with a tree rooted at v_0 with n vertex-disjoint paths of length two attached to it. Denote this graph by H_n'. It will be helpful to first define a $\mathbb{Z}_k$-antimagic labeling of H_n'. We adopt the same names for vertices and edges in this graph as we have already defined for the underlying helm.

Lemma 3.4. *Let $n \in \mathbb{N}$ be even. Then, H_n' is $\mathbb{Z}_k$-antimagic for all $k \ge 2n + 1$.*

Proof. Let n be an even positive integer, and let $k \ge 2n + 1$ be fixed. Define the function $f : E(H_n') \to \mathbb{Z}_k^*$ on the inner-spokes as follows.

$$f(e_i') = \begin{cases} 1 & \text{if } i \text{ is odd}; \\ k - 1 & \text{if } i \text{ is even}. \end{cases}$$

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Define f on the outer-spokes as follows.

$$f(e_i'') = \begin{cases} i & \text{if } i \text{ is odd;} \\ k-i+1 & \text{if } i \text{ is even.} \end{cases}$$

Now, we check that the induced vertex labels are all distinct. For vertices with odd subscripts, we get the sets of induced vertex labels $\{f^+(v_i) : i \text{ odd}\} = \{1+i : i = 1,3,5,\ldots,n-1\} = \{2,4,6,\ldots,n\}$ and $\{f^+(w_i) : i \text{ odd}\} = \{i : i = 1,3,5,\ldots,n-1\} = \{1,3,5,\ldots,n-1\}$. For vertices with even positive subscripts, we get the sets of induced vertex labels $\{f^+(v_i) : i \text{ even}\} = \{2k-i : i = 2,4,6,\ldots,n\} = \{2k-2, 2k-4, 2k-6, \ldots, 2k-n\}$ and $\{f^+(w_i) : i \text{ even}\} = \{k-i+1 : i = 2,4,6,\ldots,n\} = \{k-1, k-3, k-5, \ldots, k-(n-1)\}$. To evaluate the induced vertex label on v_0, let i range from 1 to n inclusive, and we have that $f^+(v_0) = \sum_{i \text{ odd}} 1 + \sum_{i \text{ even}}(k-1) = \sum_{i=1}^{n/2} k$. Considering all induced vertex labels modulo k, we find that $\{f^+(x) : x \in V(H_n')\} = \{0, \pm1, \pm2, \ldots, \pm(n-1), \pm n\}$. Since $k \geq 2n+1$, all of the vertex labels are distinct. Thus, f is the desired $\mathbb{Z}_k$-antimagic labeling. $\square$

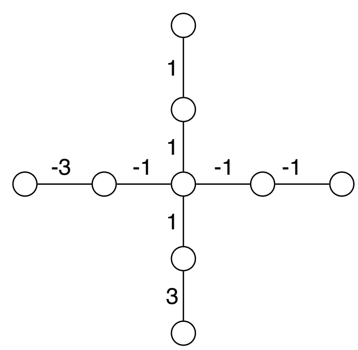


Figure 6: $\mathbb{Z}_k$-antimagic labeling of H_4', for all $k \geq 9$.

Theorem 3.5. *Let $n \in \mathbb{N}$ with $n \geq 2$. Then, H_n is $\mathbb{Z}_k$-antimagic for all $k \geq 2n+1$.*

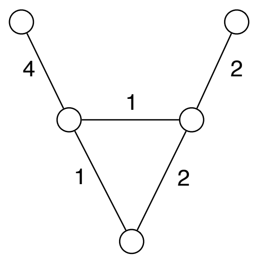
Proof. For the cases where $n = 2, 3$, see Figure 7.

Case 1. n even:

Let $k \geq 2n+1$ be fixed. Define the function $f : E(H_n) \to \mathbb{Z}_k^*$ as follows. For the edges of the subgraph H_n', we define f the same as the $\mathbb{Z}_k$-antimagic labeling given in the proof of Lemma 3.4. The edges in the set $E(H_n) \setminus E(H_n')$ form a cycle of length n. Since n is even, we can label the outer-cycle edges by alternating 1 and $k-1$. Thus, the labels on the outer-cycle edges contribute k to each induced vertex label $f^+(v_i)$. It follows that f is the desired $\mathbb{Z}_k$-antimagic labeling.

Case 2. n odd:

Let $k \geq 2n+1$ be fixed. Notice that the outer-cycle can be viewed as a path on $n+1$ vertices in which the first and last vertices are identified. In order to label the edges of the outer-cycle, we first consider a path on $n+1$ vertices. Label the edges of the



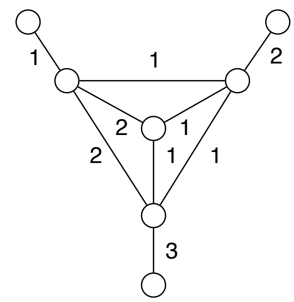


Figure 7: A $\mathbb{Z}_k$-antimagic labeling of H_2 for $k \geq 5$, and a $\mathbb{Z}_k$-antimagic labeling of H_3 for $k \geq 7$.

path by alternating the edge labels 1 and $k-1$, and making sure to begin with 1. Now, define the function $f : E(H_n) \to \mathbb{Z}_k^*$ on the outer-cycle edges by considering the outer-cycle as a path in which the first and last vertices are identified with $v_{\frac{n+3}{2}}$, and using the path labeling just described. Define f on the edges contained in the subgraph H'_n as follows.

$$f(e'_i) = \begin{cases} 1 & \text{if } 1 \leq i \leq \frac{n+1}{2}; \\ k-1 & \text{if } \frac{n+3}{2} \leq i \leq n-1; \\ k-2 & \text{if } i = n. \end{cases}$$

$$f(e''_i) = \begin{cases} 2i & \text{if } 1 \leq i \leq \frac{n+3}{2}; \\ k-2i+n+3 & \text{if } \frac{n+5}{2} \leq i \leq n-1\ ; \\ 1 & \text{if } i = n. \end{cases}$$

Now, we check that the induced vertex labels are all distinct. Notice that $\{f^+(v_i) : 1 \leq i \leq \frac{n+3}{2}\} \cup \{f^+(w_i) : 1 \leq i \leq \frac{n+3}{2}\} = \{2i+1 : 1 \leq i \leq \frac{n+3}{2}\} \cup \{2i : 1 \leq i \leq \frac{n+3}{2}\} = \{2, 3, \ldots, n+3, n+4\}$. We also have that $\{f^+(v_i) : \frac{n+5}{2} \leq i \leq n-1\} \cup \{f^+(w_i) : \frac{n+5}{2} \leq i \leq n-1\} = \{2k-2i+n+2 : \frac{n+5}{2} \leq i \leq n-1\} \cup \{k-2i+n+3 : \frac{n+5}{2} \leq i \leq n-1\} = \{2k-3, 2k-5, \ldots, 2k-(n-6), 2k-(n-4)\} \cup \{k-2, k-4, \ldots, k-(n-7), k-(n-5)\}$, and reducing all elements modulo k yields the set $\{-2, -3, \ldots, -(n-5), -(n-4)\}$. For the case where $i = n$ we have that $f^+(v_n) = k-1$ and $f^+(w_n) = 1$. It is easy to see that $f^+(v_0) \equiv 0 \pmod{k}$. Putting all of these sets of induced vertex labels together we have $\{f^+(x) : x \in V(H_n)\} = \{0, \pm 1, \pm 2, \ldots, \pm(n-5), \pm(n-4)\} \cup \{n-3, n-2, \ldots, n+4\}$. Since $k \geq 2n+1$, all of the induced vertex labels are distinct. Thus, f is the desired $\mathbb{Z}_k$-antimagic labeling.

□

4 $\mathbb{Z}_k$-antimagic Labelings of the Square of Paths

The *kth power* of a graph G, denoted G^k, is a graph with the same vertex set as G and two vertices are adjacent in G^k if and only if their distance in G is at most k.

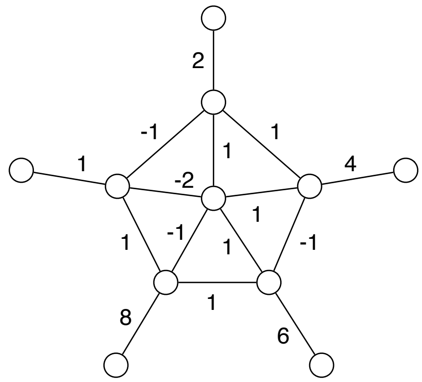


Figure 8: A $\mathbb{Z}_k$-antimagic labeling of H_5, for $k \geq 11$.

Theorem 4.1. *Let $n \geq 4$. If $n \equiv 2 \pmod 4$, then P_n^2 is $\mathbb{Z}_k$-antimagic for all $k \geq n+1$. Otherwise, P_n^2 is $\mathbb{Z}_k$-antimagic for all $k \geq n$.*

Proof. Let $G = P_n^2$. Suppose the vertices of P_n (from left to right) are $v_1, v_2, \dots, v_n$ and the edges of P_n (from left to right) are $e_1, e_2, \dots, e_{n-1}$.

Case 1. $n = 4m+1$:

Let $C_G(n)$ be the n-cycle $v_1v_3v_5\cdots v_nv_{n-1}v_{n-3}\cdots v_2v_1$ in G. Using the $\mathbb{Z}_k$-antimagic labeling ($k \geq n$) found in Theorem 2.1, we label P_n. We label all of the edges of $C_G(n)$ with 1, which gives it a **magic** labeling with magic-value 2. Now, overlay the labelings of P_n and $C_G(n)$, by identifying the vertices and edges (and adding their values). All of the vertex labels of G are distinct (mod k). We need to check if edges e_1 and e_{n-1} are still non-zero (mod k). Edge e_i was initially labeled with one of the following: $\frac{i+3}{2}$, if i is odd and $1 \leq i \leq 2m-3$; otherwise $\frac{i+5}{2}$, if i is odd and $2m-1 \leq i \leq 4m-1$. Adding 1 to either $\frac{i+3}{2}$ or $\frac{i+5}{2}$ yield non-zero values (mod k), for all $k \geq n = 4m+1$. Thus, P_n^2 is $\mathbb{Z}_k$-antimagic for all $k \geq n$.

Case 2. $n = 4m+3$:

Let $k \geq n$. Label $C_G(n)$ and P_n in the same way, as found in Case 1. Now, overlay the labelings of P_n and $C_G(n)$, by identifying the vertices and edges (and adding their values). All of the vertex labels of G are distinct (mod k). We need to check if edges e_1 and e_{n-1} are still non-zero (mod k). Edge e_i was initially labeled with one of the following: $\frac{i+1}{2}$, if i is odd and $1 \leq i \leq 2m-1$; otherwise $\frac{i+3}{2}$, if i is odd and $2m+1 \leq i \leq 4m+1$. Adding 1 to either $\frac{i+1}{2}$ or $\frac{i+3}{2}$ yield non-zero values (mod k), for all $k \geq n = 4m+3$. Thus, P_n^2 is $\mathbb{Z}_k$-antimagic for all $k \geq n$.

Case 3. $n = 4m$:

Let $C_G(n)$ be the n-cycle $v_1v_3v_5\cdots v_{n-1}v_nv_{n-2}v_{n-4}\cdots v_2v_1$ in G. Using the $\mathbb{Z}_k$-antimagic labeling ($k \geq n$) found in Theorem 2.1, we label P_n. Label the edges of $C_G(n)$ in the following way: $2 \mapsto v_1v_3$, $-2 \mapsto v_3v_5$, $2 \mapsto v_5v_7$, $\dots$, $2 \mapsto v_{n-3}v_{n-1}$, $-2 \mapsto v_{n-1}v_n$, $2 \mapsto v_nv_{n-2}$, $-2 \mapsto v_{n-2}v_{n-4}$, $\dots$, $2 \mapsto v_4v_2$ and $-2 \mapsto v_2v_1$. This is a **magic** labeling of $C_G(n)$ with magic-value 0. Now, overlay the labelings of P_n and

$C_G(n)$, by identifying the vertices and edges (and adding their values). All of the vertex labels of G are distinct (mod k). We need to check if edges e_1 and e_{n-1} are still non-zero (mod k). Edge e_i was initially labeled $\frac{i+1}{2}$, if i is odd. Adding -2 to $\frac{i+1}{2}$ yields a non-zero value (mod k), for all $k \geq n = 4m$. Thus, P_n^2 is $\mathbb{Z}_k$-antimagic for all $k \geq n$.

Case 4. $n = 4m + 2$:
Let $k \geq n + 1$. Using the $\mathbb{Z}_k$-antimagic labeling found in Theorem 2.1, we label P_n. We label the edges of $C_G(n)$ in the following way: $2 \mapsto v_1v_3$, $-2 \mapsto v_3v_5$, $2 \mapsto v_5v_7$, $\ldots$, $-2 \mapsto v_{n-3}v_{n-1}$, $2 \mapsto v_{n-1}v_n$, $-2 \mapsto v_nv_{n-2}$, $2 \mapsto v_{n-2}v_{n-4}$, $\ldots$, $2 \mapsto v_4v_2$ and $-2 \mapsto v_2v_1$. This is a **magic** labeling of $C_G(n)$ with magic-value 0. Now, overlay the labelings of P_n and $C_G(n)$, by identifying the vertices and edges (and adding their values). All of the vertex labels of G are distinct (mod k). We need to check if edges e_1 and e_{n-1} are still non-zero (mod k). Edge e_i was initially labeled $\frac{i+1}{2}$, if i is odd. In G, edge e_1 is labeled $\frac{1+1}{2} - 2$ and edge e_{n-1} is labeled $\frac{n-1+1}{2} + 2$, which are both non-zero (mod k), for all $k \geq n+1$. Thus, P_n^2 is $\mathbb{Z}_k$-antimagic for all $k \geq n+1$.

□

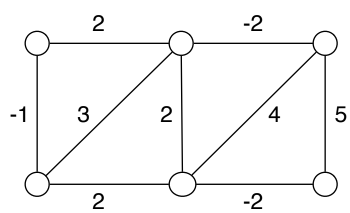


Figure 9: A $\mathbb{Z}_k$-antimagic labeling of P_6^2, for $k \geq 7$.

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