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Yoshimi Egawa
Tokyo University of Science, disc_student_seminar@rs.tus.ac.jp

Michitaka Furuya
Tokyo University of Science, michitaka.furuya@gmail.com

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Path-factors involving paths of order seven and nine

Yoshimi Egawa[*], Michitaka Furuya[†]

Abstract

In this paper, we show the following two theorems (here $c_i(G-X)$ is the number of components C of $G-X$ with $|V(C)|=i$): (i) If a graph G satisfies $c_1(G-X)+\frac{1}{3}c_3(G-X)+\frac{1}{3}c_5(G-X)\leq\frac{2}{3}|X|$ for all $X\subseteq V(G)$, then G has a $\{P_2,P_7\}$-factor. (ii) If a graph G satisfies $c_1(G-X)+c_3(G-X)+\frac{2}{3}c_5(G-X)+\frac{1}{3}c_7(G-X)\leq\frac{2}{3}|X|$ for all $X\subseteq V(G)$, then G has a $\{P_2,P_9\}$-factor.

1 Introduction

In this paper, all graphs are finite and simple. Let G be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $u\in V(G)$, we let $N_G(u)$ and $d_G(u)$ denote the *neighborhood* and the *degree* of u, respectively. For $U\subseteq V(G)$, we let $N_G(U)=(\bigcup_{u\in U}N_G(u))-U$. For disjoint sets $X,Y\subseteq V(G)$, we let $E_G(X,Y)$ denote the set of edges of G joining a vertex in X and a vertex in Y. For $X\subseteq V(G)$, we let $G[X]$ denote the subgraph of G induced by X. For two graphs H_1 and H_2, we let $H_1\cup H_2$ and H_1+H_2 denote the *union* and the *join* of H_1 and H_2, respectively. For a graph H and an integer $s\geq 2$, we let sH denote the disjoint union of s copies of H. Let K_n and P_n denote the *complete graph* and the *path* of order n, respectively. For terms and symbols not defined here, we refer the reader to [3].

Let again G be a graph. A subset M of $E(G)$ is a *matching* if no two distinct edges in M have a common endvertex. If there is no fear of confusion, we often identify a matching M

[*]Department of Mathematical Information Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

[†]Department of Mathematical Information Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan michitaka.furuya@gmail.com

of G with the subgraph of G induced by M. A matching M of G is *perfect* if $V(M) = V(G)$. For a set $\mathcal{H}$ of connected graphs, a spanning subgraph F of G is called an *$\mathcal{H}$-factor* if each component of F is isomorphic to a graph in $\mathcal{H}$. Note that a perfect matching can be regarded as a $\{P_2\}$-factor. A *path-factor* of G is a spanning subgraph whose components are paths of order at least 2. Since every path of order at least 2 can be partitioned into paths of orders 2 and 3, a graph has a path-factor if and only if it has a $\{P_2, P_3\}$-factor. Akiyama, Avis and Era [1] gave a necessary and sufficient condition for the existence of a path-factor (here $i(G)$ denotes the number of isolated vertices of a graph G).

Theorem A (Akiyama, Avis and Era [1]). *A graph G has a $\{P_2, P_3\}$-factor if and only if $i(G - X) \leq 2|X|$ for all $X \subseteq V(G)$.*

On the other hand, it follows from a result of Loebal and Poljak [4] that for $k \geq 2$, the existence problem of a $\{P_2, P_{2k+1}\}$-factor is **NP**-complete. However, in general, the fact that a problem is **NP**-complete in terms of algorithm does not mean that one cannot obtain a theoretical result concerning the problem. In this paper, we discuss sufficient conditions for the existence of a $\{P_2, P_{2k+1}\}$-factor (for detailed historical background and motivations, we refer the reader to [2]).

In order to state our results, we need some more preparations. For a graph H, we let $\mathcal{C}(H)$ be the set of components of H, and for $i \geq 1$, let $\mathcal{C}_i(H) = \{C \in \mathcal{C}(H) \mid |V(C)| = i\}$ and $c_i(H) = |\mathcal{C}_i(H)|$. Note that $c_1(H)$ is the number of isolated vertices of H (i.e., $c_1(H) = i(H)$). For $k \geq 1$, if a graph G has a $\{P_2, P_{2k+1}\}$-factor, then $\sum_{0 \leq i \leq k-1}(k-i)c_{2i+1}(G - X) \leq (k+1)|X|$ for all $X \subseteq V(G)$ (see Section 2). Thus if a condition concerning $c_{2i+1}(G-X)$ $(0 \leq i \leq k-1)$ for $X \subseteq V(G)$ assures us the existence of a $\{P_2, P_{2k+1}\}$-factor, then it will make a useful sufficient condition.

Recently, in [2], the authors proved the following theorem, and showed that the bound $\frac{4}{3}|X| + \frac{1}{3}$ in the theorem is best possible.

Theorem B (Egawa and Furuya [2]). *Let G be a graph. If $c_1(G-X)+\frac{2}{3}c_3(G-X) \leq \frac{4}{3}|X|+\frac{1}{3}$ for all $X \subseteq V(G)$, then G has a $\{P_2, P_5\}$-factor.*

In [2], the authors also constructed examples which show that for $k \geq 3$ with $k \equiv 0 \pmod 3$, there exist infinitely many graphs G having no $\{P_2, P_{2k+1}\}$-factor such that $\sum_{0 \leq i \leq k-1} c_{2i+1}(G - X) \leq \frac{4k+6}{8k+3}|X| + \frac{2k+3}{8k+3}$ for all $X \subseteq V(G)$, and proposed a conjecture that, for an integer $k \geq 3$ and a graph G, if $\sum_{0 \leq i \leq k-1} c_{2i+1}(G - X) \leq \frac{4k+6}{8k+3}|X|$ for all $X \subseteq V(G)$, then G has a $\{P_2, P_{2k+1}\}$-factor.

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In this paper, we settle the above conjecture for the case where $k \in \{3, 4\}$ as follows (note that Theorem 1.2 implies that the coefficient $\frac{4k+6}{8k+3}$ of $|X|$ in the conjecture is not best possible for $k = 4$).

Theorem 1.1. *Let G be a graph. If $c_1(G-X) + \frac{1}{3}c_3(G-X) + \frac{1}{3}c_5(G-X) \leq \frac{2}{3}|X|$ for all $X \subseteq V(G)$, then G has a $\{P_2, P_7\}$-factor.*

Theorem 1.2. *Let G be a graph. If $c_1(G-X)+c_3(G-X)+\frac{2}{3}c_5(G-X)+\frac{1}{3}c_7(G-X) \leq \frac{2}{3}|X|$ for all $X \subseteq V(G)$, then G has a $\{P_2, P_9\}$-factor.*

We prove Theorems 1.1 and 1.2 in Sections 3–5. We remark that hypomatchable graphs play an important role in the proof, though P_7 and P_9 are not hypomatchable (see Section 4 for the definition of a hypomatchable graph). In Section 6, we discuss the sharpness of coefficients in Theorems 1.1 and 1.2.

In our proof of Theorems 1.1 and 1.2, we make use of the following fact.

Fact 1.1. *Let $k \geq 2$ be an integer, and let G be a graph. Then G has a $\{P_2, P_{2k+1}\}$-factor if and only if G has a path-factor F such that $\mathcal{C}_{2i+1}(F) = \emptyset$ for every i $(1 \leq i \leq k-1)$.*

2 A necessary condition for $\{P_2, P_{2k+1}\}$-factor

In this section, we give a necessary condition for the existence of a $\{P_2, P_{2k+1}\}$-factor in terms of invariants c_{2i+1} $(0 \leq i \leq k-1)$. We show the following proposition.

Proposition 2.1. *For an integer $k \geq 1$, if a graph G has a $\{P_2, P_{2k+1}\}$-factor, then $\sum_{0 \leq i \leq k-1}(k-i)c_{2i+1}(G-X) \leq (k+1)|X|$ for all $X \subseteq V(G)$.*

Proof. Let F be a $\{P_2, P_{2k+1}\}$-factor of G, and let $X \subseteq V(G)$. Observe that

$$\sum_{0 \leq i \leq k-1}(k-i)c_{2i+1}(G-X) = \sum_{C \in \bigcup_{0 \leq i \leq k-1} \mathcal{C}_{2i+1}(G-X)} \left(k + \frac{1}{2} - \frac{|V(C)|}{2}\right).$$

With this observation in mind, we first prove the following claim.

Claim 2.1. *Let $P \in \mathcal{C}(F)$. Then $\sum_{H \in \bigcup_{0 \leq i \leq k-1} \mathcal{C}_{2i+1}(P-Y)}(k + \frac{1}{2} - \frac{|V(H)|}{2}) \leq (k+1)|Y|$ for all $Y \subseteq V(P)$.*

Proof. We proceed by induction on $|Y|$. If $Y = \emptyset$, the desired inequality clearly holds. Thus let $Y \neq \emptyset$, and assume that the desired inequality holds for subsets of $V(P)$ with cardinality $|Y|-1$. Take $x \in Y$, and set $Y' = Y - \{x\}$. Then $\sum_{H \in \bigcup_{0 \leq i \leq k-1} \mathcal{C}_{2i+1}(P-Y')}(k + \frac{1}{2} - \frac{|V(H)|}{2}) \leq$

$(k+1)|Y'|$. Let H_0 be the component of $P-Y'$ containing x, and let H_1 and H_2 denote the two segments of H_0 obtained by deleting x from H_0. Note that H_1 or H_2 (or both) may be empty. If H_0 has even order, then precisely one of H_1 and H_2, say H_1, has odd order, and hence

$$
\begin{aligned}
&\sum_{H\in\bigcup_{0\le i\le k-1}\mathcal{C}_{2i+1}(P-Y)}\left(k+\frac{1}{2}-\frac{|V(H)|}{2}\right)\\
&\qquad=\sum_{H\in\bigcup_{0\le i\le k-1}\mathcal{C}_{2i+1}(P-Y')}\left(k+\frac{1}{2}-\frac{|V(H)|}{2}\right)+\left(k+\frac{1}{2}-\frac{|V(H_1)|}{2}\right)\\
&\qquad\le (k+1)|Y'|+k\\
&\qquad< (k+1)|Y|.
\end{aligned}
$$

Thus we may assume that H_0 has odd order. Note that $-(k+\frac{1}{2}-\frac{|V(H_0)|}{2})+(k+\frac{1}{2}-\frac{|V(H_1)|}{2})+(k+\frac{1}{2}-\frac{|V(H_2)|}{2})=k+\frac{1}{2}+\frac{|V(H_0)|-|V(H_1)|-|V(H_2)|}{2}=k+1$. Consequently

$$
\begin{aligned}
&\sum_{H\in\bigcup_{0\le i\le k-1}\mathcal{C}_{2i+1}(P-Y)}\left(k+\frac{1}{2}-\frac{|V(H)|}{2}\right)\\
&\qquad\le\sum_{H\in\bigcup_{0\le i\le k-1}\mathcal{C}_{2i+1}(P-Y')}\left(k+\frac{1}{2}-\frac{|V(H)|}{2}\right)-\left(k+\frac{1}{2}-\frac{|V(H_0)|}{2}\right)\\
&\qquad\quad+\left(k+\frac{1}{2}-\frac{|V(H_1)|}{2}\right)+\left(k+\frac{1}{2}-\frac{|V(H_2)|}{2}\right)\\
&\qquad\le (k+1)|Y'|+(k+1)\\
&\qquad= (k+1)|Y|,
\end{aligned}
$$

as desired (note that this argument works even if $Y'=\emptyset$ and $H_0=P$). $\square$

Let $C\in\bigcup_{0\le i\le k-1}\mathcal{C}_{2i+1}(G-X)$. Since $|V(C)|$ is odd, $F[V(C)]$ has a component H_C of odd order. We have $|V(H_C)|\le|V(C)|$ and $H_C\in\bigcup_{0\le i\le k-1}\mathcal{C}_{2i+1}(F-X)$. Now let $\mathcal{H}=\{H_C\mid C\in\bigcup_{0\le i\le k-1}\mathcal{C}_{2i+1}(G-X)\}$. Clearly we have $H_C\ne H_{C'}$ for any $C,C'\in$

$\bigcup_{0\le i\le k-1}\mathcal{C}_{2i+1}(G-X)$ with $C\neq C'$. Consequently

$$\begin{aligned}\sum_{0\le i\le k-1}(k-i)c_{2i+1}(G-X) &= \sum_{C\in\bigcup_{0\le i\le k-1}\mathcal{C}_{2i+1}(G-X)}\left(k+\frac{1}{2}-\frac{|V(C)|}{2}\right)\\ &\le \sum_{C\in\bigcup_{0\le i\le k-1}\mathcal{C}_{2i+1}(G-X)}\left(k+\frac{1}{2}-\frac{|V(H_C)|}{2}\right)\\ &= \sum_{H\in\mathcal{H}}\left(k+\frac{1}{2}-\frac{|V(H)|}{2}\right)\\ &\le \sum_{H\in\bigcup_{0\le i\le k-1}\mathcal{C}_{2i+1}(F-X)}\left(k+\frac{1}{2}-\frac{|V(H)|}{2}\right)\\ &= \sum_{P\in\mathcal{C}(F)}\left(\sum_{H\in\bigcup_{0\le i\le k-1}\mathcal{C}_{2i+1}(P-X)}\left(k+\frac{1}{2}-\frac{|V(H)|}{2}\right)\right).\end{aligned}$$

Therefore it follows from Claim 2.1 that

$$\begin{aligned}\sum_{0\le i\le k-1}(k-i)c_{2i+1}(G-X) &\le \sum_{P\in\mathcal{C}(F)}(k+1)|V(P)\cap X|\\ &= (k+1)|X|,\end{aligned}$$

as desired. $\square$

3 Linear forests in bipartite graphs

In this this section, we show the following proposition, which plays a key role in the proof of our main theorems.

Proposition 3.1. *Let S and T be disjoint sets, and let T_1 and T_2 be disjoint subsets of T. Let G be a bipartite graph with bipartition (S,T), and let $L\subseteq E(G)$. Suppose that*

(i) $|N_G(X)|\ge|X|$ *for every* $X\subseteq S$, *and*

(ii) $|N_{G-L}(Y)|\ge|Y\cap T_1|+\frac{1}{2}|Y\cap T_2|$ *for every* $Y\subseteq T_1\cup T_2$.

Then G has a subgraph F with $V(F)\supseteq S\cup T_1\cup T_2$ such that each $A\in\mathcal{C}(F)$ is a path satisfying one of the following two conditions:

(I) $|V(A)|=2$; *or*

(II) $E(A) \subseteq E(G) - L$, $V(A) \cap T \subseteq T_1 \cup T_2$, $|V(A) \cap T_2| = 2$ *and the two vertices in* $V(A) \cap T_2$ *are the endvertices of* A.

As a preparation for the proof of Proposition 3.1, we first show the following lemma.

Lemma 3.1. *Let* S *and* T *be disjoint sets, and let* T_1 *and* T_2 *be disjoint subsets of* T *such that* $T_1 \cup T_2 = T$. *Let* H *be a bipartite graph with bipartition* (S, T), *and suppose that* $|N_H(Y)| \geq |Y \cap T_1| + \frac{1}{2}|Y \cap T_2|$ *for every* $Y \subseteq T$. *Then* H *has a subgraph* F *with* $V(F) \supseteq T_1 \cup T_2$ *such that each* $A \in \mathcal{C}(F)$ *is a path satisfying one of the following two conditions:*

(I') $|V(A)| = 2$*; or*

(II') $|V(A) \cap T_2| = 2$ *and the two vertices in* $V(A) \cap T_2$ *are the endvertices of* A.

Proof. By the assumption of the lemma, $|N_H(Y)| \geq |Y \cap T_1| + \frac{1}{2}|Y \cap T_2| = |Y|$ for every $Y \subseteq T_1$. Hence by Hall's marriage theorem, there exists a matching F of H such that $V(F) \cap T = T_1$. In particular, H has a subgraph F with $V(F) \supseteq T_1$ such that each $A \in \mathcal{C}(F)$ is a path satisfying (I') or (II'). Choose such a subgraph F so that $|(S \cup T_2) - V(F)|$ is as small as possible.

It suffices to show that $T_2 - V(F) = \emptyset$. By way of contradiction, suppose that $T_2 - V(F) \neq \emptyset$. Now we define the set $\mathcal{A}$ of paths of H as follows: Let $\mathcal{A}_0$ be the set of paths of H consisting of one vertex in $T_2 - V(F)$. For each $i \geq 1$, let $\mathcal{A}_i$ be the set of components A of F with $A \notin \bigcup_{0 \leq j \leq i-1} \mathcal{A}_j$ and $E_H(V(A) \cap S, \bigcup_{A' \in \mathcal{A}_{i-1}} (V(A') \cap T)) \neq \emptyset$. Let $\mathcal{A} = \bigcup_{i \geq 0} \mathcal{A}_i$.

Claim 3.1. *Every path* $A \in \mathcal{A}$ *with* $|V(A)| = 2$ *satisfies that* $V(A) \cap T \subseteq T_1$.

Proof. Suppose that $\mathcal{A}$ contains a path A such that $|V(A)| = 2$ and $V(A) \cap T \not\subseteq T_1$ (i.e., $V(A) \cap T \subseteq T_2$). Let i be the minimum integer such that $\mathcal{A}_i$ contains a path A_i such that $|V(A_i)| = 2$ and $V(A_i) \cap T \subseteq T_2$. Write $A_i = v_1^{(i)} v_2^{(i)}$, where $v_1^{(i)} \in S$ and $v_2^{(i)} \in T_2$, and set $l_i = 2$. By the minimality of i, every path A belonging to $\bigcup_{1 \leq j \leq i-1} \mathcal{A}_j$ with $|V(A)| = 2$ satisfies $V(A) \cap T \subseteq T_1$. By the definition of $\mathcal{A}_j$, there exist paths $A_j = v_1^{(j)} \cdots v_{l_j}^{(j)} \in \mathcal{A}_j$ ($0 \leq j \leq i-1$) such that $E_H(V(A_{j+1}) \cap S, V(A_j) \cap T) \neq \emptyset$ for every j ($0 \leq j \leq i-1$). For each j ($0 \leq j \leq i-1$), we fix an edge $e_j \in E_H(V(A_{j+1}) \cap S, V(A_j) \cap T)$, and write $e_j = v_{s_{j+1}}^{(j+1)} v_{t_j}^{(j)}$. By renumbering the vertices $v_1^{(j)}, \ldots, v_{l_j}^{(j)}$ of A_j backward (i.e., by tracing the path $v_1^{(j)} \cdots v_{l_j}^{(j)}$ backward and numbering the vertices accordingly) if necessary, we may assume that $t_j < s_j$ for each j ($1 \leq j \leq i-1$). For each j ($0 \leq j \leq i-1$), let Q'_j be the path on A_j from $v_1^{(j)}$ to $v_{t_j}^{(j)}$. For each j ($1 \leq j \leq i$), let Q''_j be the path on A_j from $v_{s_j}^{(j)}$ to $v_{l_j}^{(j)}$ (see Figure 1). Note that if A_j satisfies (II'), then $|V(Q'_j)|$ is odd and $|V(Q''_j)|$ is even.

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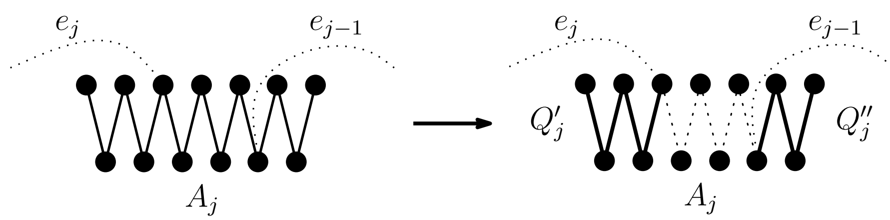

Figure 1: Paths Q'_j and Q''_j

Write $\{j \mid 1 \le j \le i-1, |V(A_j)| \ge 3\} = \{k_1, k_2, \dots, k_{m-1}\}$ with $1 \le k_1 < k_2 < \cdots < k_{m-1} \le i-1$, and let $k_0 = 0$ and $k_m = i$ (it is possible that $m = 1$). Write $\{j \mid 1 \le j \le i-1, |V(A_j)| \ge 3\} = \{k_1, k_2, \dots, k_{m-1}\}$ with $1 \le k_1 < k_2 < \cdots < k_{m-1} \le i-1$, and let $k_0 = 0$ and $k_m = i$ (it is possible that $m = 1$).

Recall that every $A \in \bigcup_{1 \le j \le i-1} \mathcal{A}_j$ with $|V(A)| = 2$ satisfies $V(A) \cap T \subseteq T_1$. Hence for each h ($1 \le h \le m$), the graph

$$B_h = \left(\bigcup_{k_{h-1}+1 \le j \le k_h - 1} A_j \right) + \{e_j \mid k_{h-1} + 1 \le j \le k_h - 2\}$$

is a path of H with $V(B_h) \cap T \subseteq T_1$ (here B_h may be an empty graph). Therefore for each h ($1 \le h \le m$), the graph

$$Q_h = (Q'_{k_{h-1}} \cup B_h \cup Q''_{k_h}) + \{e_{k_{h-1}}, e_{k_h - 1}\}$$

is a path of H satisfying (II') (see Figure 2). Note that when $h = m$, we here use the assumption that $V(A_i) \cap T \subseteq T_2$. Further, for $1 \le h \le m-1$, since $|V(A_{k_h})|$ and $|V(Q'_{k_h})|$ are odd and $|V(Q''_{k_h})|$ is even, $A_{k_h} - (V(Q'_{k_h}) \cup V(Q''_{k_h}))$ is a path of even order, and hence it has a perfect matching M_h.

Let

$$F' = \left(F - \bigcup_{1 \le j \le i} V(A_j) \right) \cup \left(\bigcup_{1 \le h \le m} Q_h \right) \cup \left(\bigcup_{1 \le h \le m-1} M_h \right).$$

Then F' is a subgraph of H such that $V(F') = V(F) \cup V(A_0)$ $(= V(F) \cup \{v_1^{(0)}\})$ and each $A \in \mathcal{C}(F')$ is a path satisfying (I') or (II'), which contradicts the minimality of $|(S \cup T_2) - V(F)|$, completing the proof of Claim 3.1. $\square$

Let $Y_0 = (\bigcup_{A \in \mathcal{A}} V(A)) \cap T$.

Claim 3.2. *We have $N_H(Y_0) = (\bigcup_{A \in \mathcal{A}} V(A)) \cap S$.*

Proof. Suppose that $N_H(Y_0) \ne (\bigcup_{A \in \mathcal{A}} V(A)) \cap S$. Then there exists an integer i and there exists a vertex $v \in S - (\bigcup_{A \in \mathcal{A}} V(A))$ such that $N_H(v) \cap (\bigcup_{A \in \mathcal{A}_i} V(A)) \ne \emptyset$. Let A_{i+1} be the

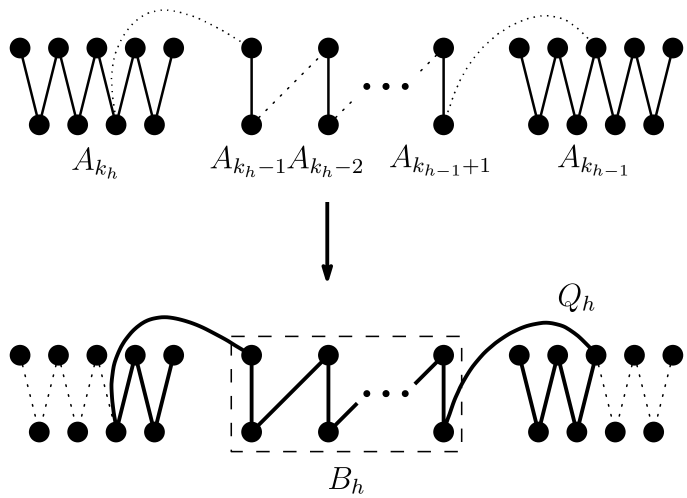

Figure 2: Paths B_h and Q_h

path of H consisting of v. By the definition of $\mathcal{A}_j$, there exist paths $A_j \in \mathcal{A}_j$ $(0 \leq j \leq i)$ such that $E_H(V(A_{j+1}) \cap S, V(A_j) \cap T) \neq \emptyset$ for every j $(0 \leq j \leq i)$. For each j $(0 \leq j \leq i)$, we fix an edge $u_j v_{j+1} \in E_H(V(A_{j+1}) \cap S, V(A_j) \cap T)$ with $u_j \in V(A_j) \cap T$ and $v_{j+1} \in V(A_{j+1}) \cap S$.

Let k $(0 \leq k \leq i)$ be the maximum integer such that $|V(A_k)|$ is odd (the fact that $|V(A_0)| = 1$ assures us the existence of k). Then for each j $(k+1 \leq j \leq i)$, we have $|V(A_j)| = 2$ (i.e., $A_j = u_j v_j$). Furthermore, since A_k is a path with $|V(A_k) \cap T| = |V(A_k) \cap S| + 1$ and $u_k \in T$, $A_k - u_k$ has a perfect matching M. Hence $M^* = \{u_j v_{j+1} \mid k \leq j \leq i\} \cup M$ is a perfect matching of the subgraph of H induced by $\bigcup_{k \leq j \leq i+1} V(A_j)$. Therefore $F' = (F - \bigcup_{k \leq j \leq i} V(A_j)) \cup M^*$ is a subgraph of H such that $V(F') \supseteq V(F) \cup \{v\}$ and each $A \in \mathcal{C}(F')$ is a path satisfying (I') or (II'), which contradicts the minimality of $|(S \cup T_2) - V(F)|$. $\square$

We continue with the proof of the lemma. By the definition of $\mathcal{A}$, we have

$$Y_0 \cap T_1 = \left(\bigcup_{A \in \mathcal{A} - \mathcal{A}_0} V(A) \right) \cap T_1 \tag{1}$$

and

$$Y_0 \cap T_2 = \left(\left(\bigcup_{A \in \mathcal{A} - \mathcal{A}_0} V(A) \right) \cap T_2 \right) \cup (T_2 - V(F)). \tag{2}$$

If $A \in \mathcal{A}$ satisfies (I'), then $|V(A) \cap S| = 1 = |V(A) \cap T_1|$ and $V(A) \cap T_2 = \emptyset$ by Claim 3.1. Thus

$$|V(A) \cap S| = |V(A) \cap T_1| + \frac{1}{2}|V(A) \cap T_2| \quad \text{for each } A \in \mathcal{A} \text{ satisfying (I')}. \tag{3}$$

If $A \in \mathcal{A}$ satisfies (II'), then $|V(A) \cap T_1| = |V(A) \cap S| - 1$ and $|V(A) \cap T_2| = 2$ by (II'). Thus

$$|V(A) \cap S| = |V(A) \cap T_1| + \frac{1}{2}|V(A) \cap T_2| \quad \text{for each } A \in \mathcal{A} \text{ satisfying (II').} \qquad (4)$$

Recall that $T_2 - V(F) \neq \emptyset$. Hence by Claim 3.2 and (1)–(4),

$$\begin{aligned}
|N_H(Y_0)| &= \sum_{A \in \mathcal{A}} |V(A) \cap S| \\
&= \sum_{A \in \mathcal{A} - \mathcal{A}_0} |V(A) \cap S| \\
&= \sum_{A \in \mathcal{A} - \mathcal{A}_0} \left(|V(A) \cap T_1| + \frac{1}{2}|V(A) \cap T_2| \right) \\
&= \sum_{A \in \mathcal{A} - \mathcal{A}_0} |V(A) \cap T_1| + \frac{1}{2} \sum_{A \in \mathcal{A} - \mathcal{A}_0} |V(A) \cap T_2| \\
&= |Y_0 \cap T_1| + \frac{1}{2}(|Y_0 \cap T_2| - |T_2 - V(F)|) \\
&< |Y_0 \cap T_1| + \frac{1}{2}|Y_0 \cap T_2|,
\end{aligned}$$

which contradicts the assumption of the lemma.

This completes the proof of Lemma 3.1. $\square$

Proof of Proposition 3.1. Applying Lemma 3.1 to $(G - L)[S \cup T_1 \cup T_2]$, we see that $G - L$ has a subgraph F' with $V(F') \cap T = T_1 \cup T_2$ such that each $A \in \mathcal{C}(F')$ is a path with $V(A) \cap T \subseteq T_1 \cup T_2$ satisfying (I) or (II). In particular, G has a subgraph F with $V(F) \supseteq T_1 \cup T_2$ such that each $A \in \mathcal{C}(F)$ is a path satisfying (I) or (II). Choose F so that $|S - V(F)|$ is as small as possible.

It suffices to show that $S - V(F) = \emptyset$. By way of contradiction, suppose that $S - V(F) \neq \emptyset$. Now we define the set $\mathcal{A}$ of paths of G as follows: Let $\mathcal{A}_0$ be the set of paths of G consisting of one vertex in $S - V(F)$. Let $\mathcal{D}$ be the set of paths of G consisting of one vertex in $T - V(F)$. For each $i \geq 1$, let $\mathcal{A}_i$ be the set of those members A of $\mathcal{C}(F) \cup \mathcal{D}$ such that $A \notin \bigcup_{0 \leq j \leq i-1} \mathcal{A}_j$ and $E_G(V(A) \cap T, \bigcup_{A' \in \mathcal{A}_{i-1}} (V(A') \cap S)) \neq \emptyset$. Set $\mathcal{A} = \bigcup_{i \geq 0} \mathcal{A}_i$.

Suppose that $\mathcal{A} - \mathcal{A}_0$ contains a path of odd order. Let i be the minimum integer such that $\mathcal{A}_i$ contains a path A_i of odd order. By the definition of $\mathcal{A}_j$, there exist paths $A_j \in \mathcal{A}_j$ $(0 \leq j \leq i-1)$ such that $E_G(V(A_{j+1}) \cap T, V(A_j) \cap S) \neq \emptyset$ for every $0 \leq j \leq i-1$. Write $V(A_0) = \{v_0\}$. By the minimality of i, for each j $(1 \leq j \leq i-1)$, we have $|V(A_j)| = 2$. For each j $(1 \leq j \leq i-1)$, write $A_j = u_j v_j$, where $V(A_j) \cap T = \{u_j\}$ and $V(A_j) \cap S = \{v_j\}$. Let $u_i \in N_G(v_{i-1}) \cap V(A_i)$. Since A_i is a path with $|V(A_i) \cap T| = |V(A_i) \cap S| + 1$ and $u_i \in T$, $A_i - u_i$ has a perfect matching M. Hence $M^* = \{v_j u_{j+1} \mid 0 \leq j \leq i-1\} \cup M$ is a perfect matching of the subgraph of G induced by $\bigcup_{0\leq j\leq i} V(A_j)$. Therefore $F' = (F - \bigcup_{1\leq j\leq i} V(A_j)) \cup M^*$ is a subgraph of G such that $V(F') \supseteq V(F) \cup \{v_0\}$ and each $A \in \mathcal{C}(F')$ is a path satisfying (I) or (II), which contradicts the minimality of $|S - V(F)|$. Thus every element of $\mathcal{A} - \mathcal{A}_0$ is a path of order 2. In particular, $\mathcal{A} \cap \mathcal{D} = \emptyset$.

Let $X_0 = (\bigcup_{A\in\mathcal{A}} V(A)) \cap S$. Since $\mathcal{A} \cap \mathcal{D} = \emptyset$, $N_G(X_0) = (\bigcup_{A\in\mathcal{A}-\mathcal{A}_0} V(A)) \cap T$. Since every element of $\mathcal{A}-\mathcal{A}_0$ is a path of order 2, $|(\bigcup_{A\in\mathcal{A}-\mathcal{A}_0} V(A)) \cap T| = |(\bigcup_{A\in\mathcal{A}-\mathcal{A}_0} V(A)) \cap S|$. Consequently

$$\begin{aligned}
|N_G(X_0)| &= \sum_{A\in\mathcal{A}-\mathcal{A}_0} |V(A) \cap T| \\
&= \sum_{A\in\mathcal{A}-\mathcal{A}_0} |V(A) \cap S| \\
&= \sum_{A\in\mathcal{A}} |V(A) \cap S| - |S - V(F)| \\
&< \sum_{A\in\mathcal{A}} |V(A) \cap S| \\
&= |X_0|,
\end{aligned}$$

which contradicts the assumption of the proposition. □

4 Hypomatchable graphs having no $\{P_2, P_{2k+1}\}$-factor

A graph G is *hypomatchable* if $G - x$ has a perfect matching for every $x \in V(G)$. In this section, we characterize hypomatchable graphs having no $\{P_2, P_{2k+1}\}$-factor for $k \in \{3, 4\}$.

4.1 Fundamental properties of hypomatchable graphs

We start with a structure theorem for hypomatchable graphs. Let G be a graph. A sequence $(H_1, \dots, H_m)$ of edge-disjoint subgraphs of G is an *ear decomposition* if

(E1) $V(G) = \bigcup_{1\leq i\leq m} V(H_i)$;

(E2) for each $1 \leq i \leq m$, $|E(H_i)|$ is odd and $|E(H_i)| \geq 3$;

(E3) H_1 is a cycle; and

(E4) for each $2 \leq i \leq m$, either

(E4-1) H_i is a path and only the endvertices of H_i belong to $\bigcup_{1\leq j\leq i-1} V(H_j)$, or

(E4-2) H_i is a cycle with $|V(H_i) \cap (\bigcup_{1 \leq j \leq i-1} V(H_j))| = 1$.

Lovász [5] proved the following theorem.

Theorem C (Lovász [5])**.** *Let G be a graph with $|V(G)| \geq 3$.*

(i) *If G has an ear decomposition, then G is hypomatchable.*

(ii) *If G is hypomatchable, then for each $e \in E(G)$, G has an ear decomposition $(H_1, \ldots, H_m)$ such that $e \in E(H_1)$.*

In the remainder of this subsection, we let G be a hypomatchable graph, and let $\mathcal{H} = (H_1, \ldots, H_m)$ be an ear decomposition of G. We start with lemmas which hold for an ear decomposition of a hypomatchable graph in general.

Lemma 4.1. *For each i $(2 \leq i \leq m)$, there exists an ear decomposition $(H'_1, \ldots, H'_{m'})$ of G such that $H_i \subseteq H'_1$.*

Proof. Set $H = H_1 \cup \cdots \cup H_i$. Then $(H_1, \ldots, H_i)$ is an ear decomposition of H, and hence H is hypomatchable by Theorem C(i). Take $e \in E(H_i)$. By Theorem C(ii), H has an ear decomposition $(H'_1, \ldots, H'_n)$ such that $e \in E(H'_1)$. Since H_i satisfies (E4), we have $d_H(v) = 2$ for all $v \in V(H_i) - (\bigcup_{1 \leq j \leq i-1} V(H_j))$. Since H'_1 satisfies (E3), this implies $H_i \subseteq H'_1$. Since $\bigcup_{1 \leq j \leq n} V(H'_j) = \bigcup_{1 \leq j \leq i} V(H_j)$, it follows that $(H'_1, \ldots, H'_n, H_{i+1}, \ldots, H_m)$ is an ear decomposition of G with the desired property. □

Lemma 4.2. *Suppose that each H_i $(1 \leq i \leq m)$ is a cycle, and let $i_1, \ldots, i_m$ be a permutation of $1, \ldots, m$ such that $V(H_{i_l}) \cap (\bigcup_{1 \leq j \leq l-1} V(H_{i_j})) \neq \emptyset$ for each l $(2 \leq l \leq m)$. Then $(H_{i_1}, \ldots, H_{i_m})$ is an ear decomposition of G.*

Proof. Since each H_i is a cycle, it follows from the definition of an ear decomposition that H_i is a block of G for each i. Thus for each l $(2 \leq l \leq m)$, the assumption that $V(H_{i_l}) \cap (\bigcup_{1 \leq j \leq l-1} V(H_{i_j})) \neq \emptyset$ implies that $|V(H_{i_l}) \cap (\bigcup_{1 \leq j \leq l-1} V(H_{i_j}))| = 1$. Hence by the definition of an ear decomposition, $(H_{i_1}, \ldots, H_{i_m})$ is also an ear decomposition. □

Our next result is concerned with a hypomatchable graph with no $\{P_2, P_{2k+1}\}$-factor. In order to state the result, we need some more definitions. For each i $(1 \leq i \leq m)$, let $P_{\mathcal{H}}(i) = H_i - \bigcup_{1 \leq j \leq i-1} V(H_j)$. Note that $V(P_{\mathcal{H}}(i)) \cap V(H_j) = \emptyset$ for any i, j with $i > j$, and $\bigcup_{1 \leq j \leq i} V(H_j) = \bigcup_{1 \leq j \leq i} V(P_{\mathcal{H}}(j))$ for each i. We have $P_{\mathcal{H}}(1) = H_1$ and, by (E2) and (E4), $P_{\mathcal{H}}(i)$ is a path of even order for $2 \leq i \leq m$. For an odd integer $s \geq 5$, a set $I \subseteq \{1, 2, \ldots, m\}$ of indices with $1 \in I$ is *s-large* with respect to $\mathcal{H}$ if $\sum_{i \in I} |V(P_{\mathcal{H}}(i))| \geq s$ and the subgraph of G induced by $\bigcup_{i \in I} V(P_{\mathcal{H}}(i))$ has a spanning path.

Lemma 4.3. *Let $k \geq 3$, and suppose that G has no $\{P_2, P_{2k+1}\}$-factor. Then there is no $(2k+1)$-large set with respect to $\mathcal{H}$.*

Proof. Suppose that there exists a $(2k+1)$-large set I with respect to $\mathcal{H}$. Then by Fact 1.1, the subgraph of G induced by $\bigcup_{i \in I} V(P_{\mathcal{H}}(i))$ has a $\{P_2, P_{2k+1}\}$-factor F. On the other hand, for each i with $2 \leq i \leq m$ and $i \notin I$, from the fact that $P_{\mathcal{H}}(i)$ is a path of even order, we see that $P_{\mathcal{H}}(i)$ has a perfect matching M_i. Since $\{V(P_{\mathcal{H}}(i)) \mid i \notin I\}$ is a partition of $V(G) - (\bigcup_{i \in I} V(P_{\mathcal{H}}(i)))$, $F \cup (\bigcup_{i \notin I} M_i)$ is a $\{P_2, P_{2k+1}\}$-factor of G, which is a contradiction. □

Hereafter we consider the following condition:

(D1) $\mathcal{H} = (H_1, \ldots, H_m)$ is chosen so that $|E(H_1)|$ is as large as possible.

Lemma 4.4. *Suppose that $\mathcal{H} = (H_1, \ldots, H_m)$ satisfies (D1). Let $2 \leq i \leq m$, and let v, v' be the endvertices of $P_{\mathcal{H}}(i)$. Then no two vertices w, w' with $w \in N_G(v) \cap V(H_1)$ and $w' \in N_G(v') \cap V(H_1)$ are adjacent in H_1.*

Proof. Suppose that there exist $w \in N_G(v) \cap V(H_1)$ and $w' \in N_G(v') \cap V(H_1)$ such that w and w' are adjacent in H_1. Then $G[V(H_1) \cup V(P_{\mathcal{H}}(i))]$ contains a spanning cycle C. Since $|E(C)| = |V(C)| = |V(H_1)| + |V(P_{\mathcal{H}}(i))|$, $|E(C)|$ is odd and $|E(C)| > |E(H_1)|$. Since $V(H_j) \cap V(P_{\mathcal{H}}(i)) = \emptyset$ for every j with $2 \leq j \leq i-1$, $(C, H_2, \ldots, H_{i-1})$ is an ear decomposition of $G[V(H_1) \cup \cdots \cup V(H_i)]$, and hence $(C, H_2, \ldots, H_{i-1}, H_{i+1}, \ldots, H_m)$ is an ear decomposition of G, which contradicts (D1). □

Lemma 4.5. *Suppose that (D1) holds, and suppose further that $|E(H_1)| = 3$. Then each H_i is a cycle of order 3, and $G = H_1 \cup \cdots \cup H_m$.*

Proof. Let $2 \leq i \leq m$. By Lemma 4.1, there is an ear decomposition $(H'_1, \ldots, H'_{m'})$ such that $H_i \subseteq H'_1$. If $|E(H_i)| > 3$ or H_i is a path, then we get $|E(H'_1)| > 3$, which contradicts (D1). Thus each H_i is a cycle of order 3.

Now suppose that there exists $e = ab \in E(G)$ such that $e \notin E(H_1 \cup \cdots \cup H_m)$. Since $(H_1 \cup \cdots \cup H_m) + e$ is hypomatchable by Theorem C(i), it follows from Theorem C(ii) that there is an ear decomposition $(H'_1, \ldots, H'_{m'})$ of $(H_1 \cup \cdots \cup H_m) + e$ such that $e \in E(H'_1)$. By (D1), $|E(H'_1)| = 3$. Write $H'_1 = abva$. Let i, j be the indices such that $av \in E(H_i)$ and $bv \in E(H_j)$. Then $i \neq j$, $v \in V(H_i) \cap V(H_j)$, and $(H_i \cup H_j) + e$ has a spanning cycle C. By Lemma 4.2, G has an ear decomposition $(H''_1, \ldots, H''_m)$ with $H''_1 = H_i$ and $H''_2 = H_j$. This implies that $(C, H''_3, \ldots, H''_m)$ is an ear decomposition of G, which contradicts (D1). Thus $G = H_1 \cup \cdots \cup H_m$. □

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4.2 Constructions of hypomatchable graphs

In this subsection, we construct five families $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ of hypomatchable graphs (see Figure 3).

(G0) Let $\mathcal{G}_0^* = \{K_1 + sK_2 \mid s \geq 2\}$ and $\mathcal{G}_0 = \{K_1 + sK_2 \mid s \geq 3\}$. Note that for each $H \in \mathcal{G}_0^*$, H is hypomatchable and has no $\{P_2, P_7\}$-factor.

Let s_1, s_2, s_3 be nonnegative integers. Let $Q = u_1u_2u_3$ be a path of order 3 and, for $i \in \{1, 2, 3\}$ and $1 \leq j \leq s_i$, let $L_{i,j}$ be a path of order 2. For each $1 \leq j \leq s_2$, write $L_{2,j} = v_{1,j}v_{3,j}$.

(G1) Let $A_1(s_1, s_2, s_3)$ be the graph obtained from $Q \cup (\bigcup_{i \in \{1,2,3\}} (\bigcup_{1 \leq j \leq s_i} L_{i,j}))$ by adding the edge u_1u_3 and joining u_i to all vertices in $\bigcup_{1 \leq j \leq s_i} V(L_{i,j})$ for each $i \in \{1, 2, 3\}$. Note that $A_1(s_1, 0, 0) \simeq K_1 + (s_1 + 1)K_2$. Let $\mathcal{G}_1^* = \{A_1(s_1, s_2, s_3) \mid s_1 + s_2 + s_3 \geq 1\}$ and $\mathcal{G}_1 = \{A_1(s_1, s_2, s_3) \mid s_1 + s_2 + s_3 \geq 3\}$.

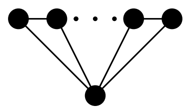
We divide the set $\mathcal{G}_1$ into three sets. Let $\mathcal{G}_1^{(1)} = \{A_1(s_1, s_2, s_3) \in \mathcal{G}_1 \mid \min\{s_1, s_2, s_3\} \leq 1\}$, $\mathcal{G}_1^{(2)} = \{A_1(s_1, s_2, s_3) \in \mathcal{G}_1 \mid \min\{s_1, s_2, s_3\} = 2\}$ and $\mathcal{G}_1^{(3)} = \{A_1(s_1, s_2, s_3) \in \mathcal{G}_1 \mid \min\{s_1, s_2, s_3\} \geq 3\}$.

(G2) Let $A'_2(s_1, s_2, s_3)$ be the graph obtained from $Q \cup (\bigcup_{i \in \{1,2,3\}} (\bigcup_{1 \leq j \leq s_i} L_{i,j}))$ by joining u_i to all vertices in $(\bigcup_{1 \leq j \leq s_i} V(L_{i,j})) \cup \{v_{i,j} \mid 1 \leq j \leq s_2\}$ for each $i \in \{1, 3\}$. Let $A''_2(s_1, s_2, s_3)$ be the graph obtained from $Q \cup (\bigcup_{i \in \{1,2,3\}} (\bigcup_{1 \leq j \leq s_i} L_{i,j}))$ by adding the edge u_1u_3 and joining u_i to all vertices in $(\bigcup_{1 \leq j \leq s_i} V(L_{i,j})) \cup (\bigcup_{1 \leq j \leq s_2} V(L_{2,j}))$ for each $i \in \{1, 3\}$. Let $\mathcal{G}_2 = \{H \mid A'_2(s_1, s_2, s_3) \subseteq H \subseteq A''_2(s_1, s_2, s_3)$ with $s_2 \geq 1$, and $s_1 + s_2 + s_3 \geq 3$, and either $s_1 \geq 1$ and $s_3 \geq 1$ or $s_2 \geq 2\}$.

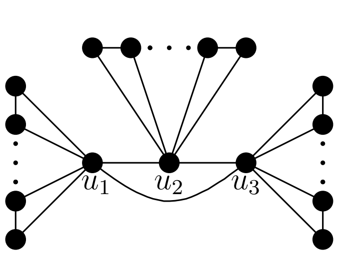
(G3) Assume $s_2 = 1$ and $s_3 = 0$. Let $A'_3(s_1) = A'_2(s_1, 1, 0)$. Let $A''_3(s_1)$ be the graph obtained from $A'_3(s_1)$ by joining all possible pairs of vertices in $V(Q) \cup L_{2,1}$. Let $\mathcal{G}_3 = \{H \mid A'_3(s_1) \subseteq H \subseteq A''_3(s_1)$ with $s_1 \geq 2\}$.

(G4) Assume that $s_2 = 2$ and $s_3 = 0$. Let $A'_4(s_1)$ be the graph obtained from $A'_2(s_1, 2, 0)$ by adding the edge $v_{3,1}v_{3,2}$. Let $A''_4(s_1)$ be the graph obtained from $A'_4(s_1)$ by adding the edges $u_1u_3, u_1v_{3,1}, u_1v_{3,2}$. Let $\mathcal{G}_4 = \{H \mid A'_4(s_1) \subseteq H \subseteq A''_4(s_1)$ with $s_1 \geq 1\}$.

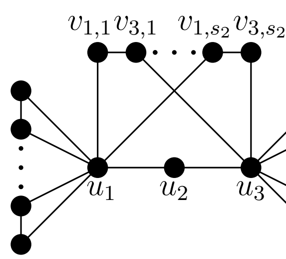
We can verify that for each $H \in \mathcal{G}_1^* \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$, H is hypomatchable and has no $\{P_2, P_9\}$-factor.



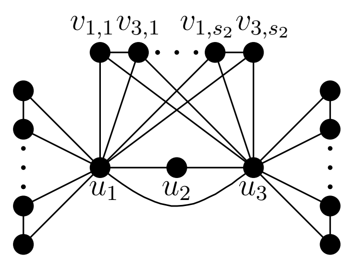
$K_1 + sK_2$ defined in (G0)



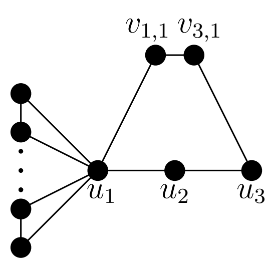
$A_1(s_1, s_2, s_3)$ defined in (G1)



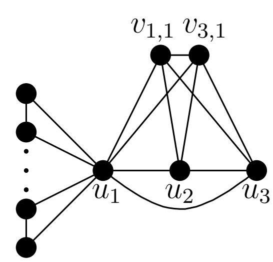
$A_2'(s_1, s_2, s_3)$ defined in (G2)



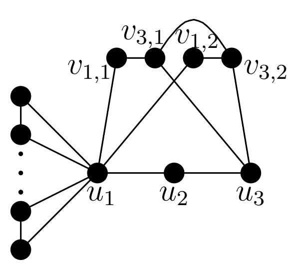
$A_2''(s_1, s_2, s_3)$ defined in (G2)



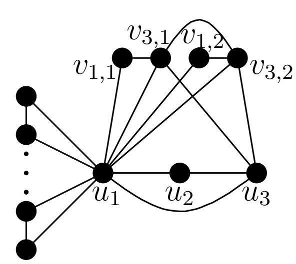
$A_3'(s_1)$ defined in (G3)



$A_3''(s_1)$ defined in (G3)



$A_4'(s_1)$ defined in (G4)



$A_4''(s_1)$ defined in (G4)

Figure 3: Graphs in $\mathcal{G}_i^*$ or $\mathcal{G}_i$

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Now we define crush sets of graphs belonging to $\bigcup_{0\le i\le 4}\mathcal{G}_i$. For $H\in\mathcal{G}_0$, a set $X\subseteq V(H)$ is a *crush set* of H if $x\in X$ and $|X\cap V(C)|=1$ for each $C\in\mathcal{C}(H-x)$, where x is the unique cutvertex of H. Let $H\in\mathcal{G}_1$, and write $H=A_1(s_1,s_2,s_3)$. We may assume that $\min\{s_1,s_2,s_3\}=s_3$. If $H\in\mathcal{G}_1^{(1)}\cup\mathcal{G}_1^{(2)}$, a *crush set* of H is a set $X\subseteq V(G)$ such that $X\cap V(Q)=\{u_1,u_2\}$, $X\cap(\bigcup_{1\le j\le s_3}V(L_{3,j}))=\emptyset$ and $|X\cap V(L_{i,j})|=1$ for each $i\in\{1,2\}$ and each $1\le j\le s_i$ (note that if $s_3=0$ and s_1 or s_2 is zero, then this definition is consistent with the definition of a crush set for a graph in $\mathcal{G}_0$). If $H\in\mathcal{G}_1^{(3)}$, a *crush set* of H is a set $X\subseteq V(G)$ such that $V(Q)\subseteq X$ and $|X\cap V(L_{i,j})|=1$ for each $i\in\{1,2,3\}$ and each $1\le j\le s_i$. For $H\in\mathcal{G}_2$, a set $X\subseteq V(H)$ is a *crush set* of H if $X\cap V(Q)=\{u_1,u_3\}$ and $|X\cap V(L_{i,j})|=1$ for each $i\in\{1,2,3\}$ and each $1\le j\le s_i$. For $H\in\mathcal{G}_3$, a set $X\subseteq V(H)$ is a *crush set* of H if $X\cap V(Q)=\{u_1,u_3\}$, $X\cap V(L_{2,1})=\emptyset$ and $|X\cap V(L_{1,j})|=1$ for each $1\le j\le s_1$. For $H\in\mathcal{G}_4$, a set $X\subseteq V(H)$ is a *crush set* of H if $X\cap V(Q)=\{u_1,u_3\}$, $X\cap V(L_{2,1})=\{v_{3,1}\}$, $X\cap V(L_{2,2})=\{v_{3,2}\}$ and $|X\cap V(L_{1,j})|=1$ for each $1\le j\le s_1$.

By inspection, we get the following lemma, which will be used in Section 5.

Lemma 4.6. *Let $H\in\bigcup_{0\le i\le 4}\mathcal{G}_i$, and let X be a crush set of H. Then the following hold.*

(i) *If $H\in\mathcal{G}_0$, then $c_1(H-X)=c_1(H-X)+c_3(H-X)+\frac{2}{3}c_5(H-X)=|X|-1$ and $|X|\ge 4$.*

(ii) *If $H\in\mathcal{G}_1^{(1)}\cup\mathcal{G}_2\cup\mathcal{G}_3\cup\mathcal{G}_4$, then $c_1(H-X)+c_3(H-X)+\frac{2}{3}c_5(H-X)=|X|-1$ and $|X|\ge 4$.*

(iii) *If $H\in\mathcal{G}_1^{(2)}$, then $c_1(H-X)+c_3(H-X)+\frac{2}{3}c_5(H-X)=|X|-\frac{4}{3}$ and $|X|\ge 6$.*

(iv) *If $H\in\mathcal{G}_1^{(3)}$, then $c_1(H-X)+c_3(H-X)+\frac{2}{3}c_5(H-X)=|X|-3$ and $|X|\ge 12$.*

4.3 Hypomatchable graphs having no $\{P_2,P_7\}$-factor

In this subsection, we prove the following proposition, Proposition 4.1, which characterizes hypomatchable graphs with no $\{P_2,P_7\}$-factor. The proposition can be derived as a corollary of Proposition 4.2, which will be proved in Subsection 4.4, but we here give a proof which does not depend on Proposition 4.2 because the proof is not too long.

Proposition 4.1. *Let G be a hypomatchable graph of order at least 7 having no $\{P_2,P_7\}$-factor. Then $G\in\mathcal{G}_0$.*

Proof. By Lemma C, G has an ear decomposition $\mathcal{H}=(H_1,\dots,H_m)$. Choose $\mathcal{H}$ so that (D1) holds. We use the notation introduced in Subsection 4.1.

By Lemma 4.3, $\{1\}$ is not a 7-large set. Hence $|V(H_1)| \le 5$. Since $|V(H)| \ge 7$ by assumption, this implies $m \ge 2$. By the definition of an ear decomposition, $H_1 \cup H_2$ contains a spanning path. Since $\{1, 2\}$ is not 7-large by Lemma 4.3, we get $|V(H_1)| + |V(P_{\mathcal{H}}(2))| \le 5$. Hence $|V(H_1)| = 3$. We also have $m \ge 3$.

By Lemma 4.5, each H_i $(1 \le i \le m)$ is a cycle of order 3, and $G = H_1 \cup \cdots \cup H_m$. Since $|V(H)| \ge 7$ by assumption, it suffices to show that $G \in \mathcal{G}_0^*$. We actually prove that for each i $(2 \le i \le m)$, we have $H_1 \cup \cdots \cup H_i \in \mathcal{G}_0^*$, i.e., $H_1 \cup \cdots \cup H_i \simeq K_1 + iK_2$. We proceed by induction on i. We clearly have $H_1 \cup H_2 \simeq K_1 + 2K_2$. Thus let $i \ge 3$, and assume that $H_1 \cup \cdots \cup H_{i-1} \simeq K_1 + (i-1)K_2$. Write $V(H_1) \cap \cdots \cap V(H_{i-1}) = \{u\}$. Suppose that $V(H_i) \cap (V(H_1) \cup \cdots \cup V(H_{i-1})) \ne \{u\}$. In view of Lemma 4.2, by relabeling $H_1, \ldots, H_{i-1}$ if necessary, we may assume that $V(H_i) \cap V(H_1) \ne \emptyset$. Then $H_2 \cup H_1 \cup H_i$ contains a spanning path, and hence $\{1, 2, i\}$ is 7-large, which contradicts Lemma 4.3. Thus $V(H_i) \cap (V(H_1) \cup \cdots \cup V(H_{i-1})) = \{u\}$, and hence $H_1 \cup \cdots \cup H_{i-1} \cup H_i \simeq K_1 + iK_2$, as desired. □

4.4 Hypomatchable graphs having no $\{P_2, P_9\}$-factor

Proposition 4.2. *Let G be a hypomatchable graph of order at least 9 having no $\{P_2, P_9\}$-factor. Then $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$.*

Proof. By Lemma C, G has an ear decomposition $\mathcal{H} = (H_1, \ldots, H_m)$. Choose $\mathcal{H}$ so that (D1) holds.

By Lemma 4.3, $\{1\}$ is not a 9-large set. Hence $|V(H_1)| \le 7$. Since $|V(G)| \ge 9$, this implies $m \ge 2$. By the definition of an ear decomposition, $H_1 \cup H_2$ contains a spanning path. Since $\{1, 2\}$ is not 9-large by Lemma 4.3, we get $|V(H_1)| + |V(P_{\mathcal{H}}(2))| \le 7$. Hence $|V(H_1)| = 3$ or 5. We also have $m \ge 3$.

Case 1: $|V(H_1)| = 3$.

By Lemma 4.5, each H_i $(1 \le i \le 3)$ is a cycle of order 3, and $G = H_1 \cup \cdots \cup H_m$. We show that $G \in \mathcal{G}_1$. We actually prove that for each i $(2 \le i \le m)$, we have $H_1 \cup \cdots \cup H_i \in \mathcal{G}_1^*$, i.e., $H_1 \cup \cdots \cup H_i \simeq A_1(s_1, s_2, s_3)$ for some s_1, s_2, s_3 with $s_1 + s_2 + s_3 = i - 1$. We proceed by induction on i. Note that $H_1 \cup H_2 \simeq A_1(1, 0, 0)$. Thus let $i \ge 3$, and assume that $H_1 \cup \cdots \cup H_{i-1} \simeq A_1(s'_1, s'_2, s'_3)$ with $s'_1 + s'_2 + s'_3 = i - 2$. If only one of s'_1, s'_2 and s'_3 is nonzero, i.e., $H_1 \cup \cdots \cup H_{i-1} \simeq A_1(i-2, 0, 0)$, then $H_1 \cup \cdots \cup H_{i-1} \cup H_i \simeq A_1(i-1, 0, 0)$ or $A_1(i-2, 1, 0)$. Thus we may assume that at least two of s'_1, s'_2 and s'_3 are nonzero. In view of Lemma 4.2, by relabeling $H_1, \ldots, H_{i-1}$ if necessary, we may assume that H_1

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intersects with all of $H_2, \dots, H_{i-1}$. Write $H_1 = w_1w_2w_3w_1$. We may assume that $s'_h = |\{j \mid 2 \leq j \leq i-1, V(H_j) \cap V(H_1) = \{w_h\}\}|$ for each $h = 1,2,3$. Suppose that there exists j $(2 \leq j \leq i-1)$ such that $V(H_i) \cap V(P_{\mathcal{H}}(j)) \neq \emptyset$. Since at least two of s'_1, s'_2, s'_3 are nonzero, there exists j' $(2 \leq j' \leq i-1)$ with $j' \neq j$ such that $V(H_{j'}) \cap V(H_1) \neq V(H_j) \cap V(H_1)$. Then $H_{j'} \cup H_1 \cup H_j \cup H_i$ contains a spanning path, and hence $\{1, j, j', i\}$ is 9-large, which contradicts Lemma 4.3. Consequently $V(H_i) \cap V(P_{\mathcal{H}}(j)) = \emptyset$ for every j $(2 \leq j \leq i-1)$, which implies $V(H_1) \cap (\bigcup_{1 \leq j \leq i-1} V(H_j)) = V(H_i) \cap V(H_1)$. We may assume $V(H_i) \cap V(H_1) = \{w_1\}$. Thus $H_1 \cup \cdots \cup H_{i-1} \cup H_i \simeq A_1(s'_1 + 1, s'_2, s'_3)$, as desired.

Case 2: $|V(H_1)| = 5$.

We first prove two claims.

Claim 4.1. *For each i $(2 \leq i \leq m)$, $|V(P_{\mathcal{H}}(i))| = 2$ and*

$$N_G(V(P_{\mathcal{H}}(i))) \cap \left(\bigcup_{2 \leq j \leq i-1} V(P_{\mathcal{H}}(j)) \right) = \emptyset.$$

Proof. We proceed by induction on i. Since $|V(H_1)| + |V(P_{\mathcal{H}}(2))| \leq 7$, the desired conclusion clearly holds for $i = 2$. Thus let $i \geq 3$, and assume that for each i' with $2 \leq i' \leq i-1$, we have $|V(P_{\mathcal{H}}(i'))| = 2$ and $N_G(V(P_{\mathcal{H}}(i'))) \cap (\bigcup_{2 \leq j \leq i'-1} V(P_{\mathcal{H}}(j))) = \emptyset$. It follows from (E4) that for each i' $(2 \leq i' \leq i-1)$ and for each $v \in V(P_{\mathcal{H}}(i'))$, $H_1 \cup H_{i'}$ contains a spanning path having v as one of its endvertices. Let U be the set of the endvertices of $P_{\mathcal{H}}(i)$. Suppose that $N_G(U) \cap (\bigcup_{2 \leq j \leq i-1} V(P_{\mathcal{H}}(j))) \neq \emptyset$, and take $v \in N_G(U) \cap (\bigcup_{2 \leq j \leq i-1} V(P_{\mathcal{H}}(j)))$. Let i' denote the index such that $v \in V(P_{\mathcal{H}}(i'))$. Then since $H_1 \cup H_{i'}$ contains a spanning path having endvertex v, $G[V(H_1) \cup V(P_{\mathcal{H}}(i')) \cup V(P_{\mathcal{H}}(i))]$ contains a spanning path. Since $|V(H_1)| + |V(P_{\mathcal{H}}(i'))| + |V(P_{\mathcal{H}}(i))| = 7 + |V(P_{\mathcal{H}}(i))| \geq 9$, this implies that $\{1, i, i'\}$ is 9-large, which contradicts Lemma 4.3. Thus $N_G(U) \cap (\bigcup_{2 \leq j \leq i-1} V(P_{\mathcal{H}}(j))) = \emptyset$. It now follows from (E4) that $H_1 \cup H_i$ contains a spanning path. Hence by Lemma 4.3, $|V(P_{\mathcal{H}}(i))| \leq 7 - |V(H_1)| = 2$. This implies $U = V(P_{\mathcal{H}}(i))$, and thus the claim is proved. $\square$

Claim 4.2. *If $N_G(\bigcup_{2 \leq i \leq m} V(P_{\mathcal{H}}(i))) \cap V(H_1)$ contains two vertices w, w' which are adjacent in H_1, then $|N_G(w) \cap (\bigcup_{2 \leq i \leq m} V(P_{\mathcal{H}}(i)))| = |N_G(w') \cap (\bigcup_{2 \leq i \leq m} V(P_{\mathcal{H}}(i)))| = 1$ and $N_G(w) \cap (\bigcup_{2 \leq i \leq m} V(P_{\mathcal{H}}(i))) = N_G(w') \cap (\bigcup_{2 \leq i \leq m} V(P_{\mathcal{H}}(i)))$.*

Proof. Suppose that $|N_G(w) \cap (\bigcup_{2 \leq i \leq m} V(P_{\mathcal{H}}(i)))| \geq 2$ or $|N_G(w') \cap (\bigcup_{2 \leq i \leq m} V(P_{\mathcal{H}}(i)))| \geq 2$ or $N_G(w) \cap (\bigcup_{2 \leq i \leq m} V(P_{\mathcal{H}}(i))) \neq N_G(w') \cap (\bigcup_{2 \leq i \leq m} V(P_{\mathcal{H}}(i)))$. Then we can take $v \in N_G(w) \cap (\bigcup_{2 \leq i \leq m} V(P_{\mathcal{H}}(i)))$ and $v' \in N_G(w') \cap (\bigcup_{2 \leq i \leq m} V(P_{\mathcal{H}}(i)))$ so that $v \neq v'$. Let i and i' be the indices such that $v \in V(P_{\mathcal{H}}(i))$ and $v' \in V(P_{\mathcal{H}}(i'))$. By Claim 4.1, $|V(P_{\mathcal{H}}(i))| =$

$|V(P_{\mathcal{H}}(i'))| = 2$. Hence by Lemma 4.4, $i \neq i'$. Note that $G[V(H_1) \cup V(P_{\mathcal{H}}(i)) \cup V(P_{\mathcal{H}}(i'))]$ contains a spanning path. Since $|V(H_1)| + |V(P_{\mathcal{H}}(i))| + |V(P_{\mathcal{H}}(i'))| = 9$, this contradicts Lemma 4.3. $\square$

We return to the proof of Proposition 4.2. Write $H_1 = w_1w_2w_3w_4w_5w_1$. We first consider the case where $N_G(\bigcup_{1\le i\le m} V(P_{\mathcal{H}}(i))) \cap V(H_1)$ contains two vertices w, w' which are adjacent in H_1. We may assume $w = w_3$ and $w' = w_4$. By Claim 4.2, there exists $b \in \bigcup_{1\le i\le m} V(P_{\mathcal{H}}(i))$ such that $N_G(w_3) \cap (\bigcup_{1\le i\le m} V(P_{\mathcal{H}}(i))) = N_G(w_4) \cap (\bigcup_{1\le i\le m} V(P_{\mathcal{H}}(i))) = \{b\}$. Note that Claim 4.1 in particular implies that for any permutation $i_2, \dots, i_m$ of $2, \dots, m$, $(H_1, H_{i_2}, \dots, H_{i_m})$ is an ear decomposition. Thus we may assume $b \in V(P_{\mathcal{H}}(2))$. Write $P_{\mathcal{H}}(2) = bb'$. By Claim 4.2 and (E4), $N_G(b') \cap V(H_1) = \{w_1\}$, $\{w_3, w_4\} \subseteq N_G(b) \cap V(H_1) \subseteq \{w_1, w_3, w_4\}$, and $N_G(v) \cap V(H_1) = \{w_1\}$ for all $v \in \bigcup_{3\le i\le m} V(P_{\mathcal{H}}(i))$. Consequently $A_4'(m-2) \subseteq G$. By Lemma 4.3, $\{1, 2, 3\}$ is not 9-large. Hence $G[V(H_1) \cup V(P_{\mathcal{H}}(2))]$ does not contain a spanning path with endvertex w_1. This implies $w_2w_4, w_2w_5, w_3w_5 \notin E(G)$, and hence it follows from Claim 4.1 that $G \subseteq A_4''(m-2)$. Therefore $G \in \mathcal{G}_4$.

We now consider the case where $N_G(\bigcup_{2\le i\le m} V(P_{\mathcal{H}}(i))) \cap V(H_1)$ does not contain two vertices which are adjacent in H_1. In this case, $|N_G(\bigcup_{2\le i\le m} V(P_{\mathcal{H}}(i))) \cap V(H_1)| \le 2$. We may assume $N_G(\bigcup_{2\le i\le m} V(P_{\mathcal{H}}(i))) \cap V(H_1) \subseteq \{w_1, w_3\}$. Let $s_h = |\{i \mid 2 \le i \le m, N_G(V(P_{\mathcal{H}}(i))) \cap V(H_1) = \{w_h\}\}|$ for each $h \in \{1, 3\}$, and $s_2 = |\{i \mid 2 \le i \le m, N_G(V(P_{\mathcal{H}}(i))) \cap V(H_1) = \{w_1, w_3\}\}|$. Since $m \ge 3$, $s_1 + s_2 + s_3 \ge 2$. If $s_2 = 0$ and s_1 or s_3 (say s_3) is zero, then it follows from Claim 4.1 that $A_3'(s_1) \subseteq G \subseteq A_3''(s_1)$, and hence $G \in \mathcal{G}_3$. Thus we may assume that we have $s_2 \neq 0$, or $s_1 \neq 0$ and $s_3 \neq 0$. Since $s_1 + s_2 + s_3 \ge 2$, it follows from Lemma 4.3 that $G[V(H_1)]$ does not contain a spanning path connecting w_1 and w_3. Hence $w_2w_4, w_2w_5 \notin E(G)$, which together with Claim 4.1 implies that $A_2'(s_1, s_2 + 1, s_3) \subseteq G \subseteq A_2''(s_1, s_2 + 1, s_3)$. Therefore $G \in \mathcal{G}_2$.

This completes the proof of Proposition 4.2. $\square$

4.5 Alternating paths

In this appendant subsection, we prove two lemmas about hypomatchable graphs, which we use in the proof of Theorem 1.2. Throughout this subsection, we let G denote a hypomatchable graph, let $v \in V(G)$, and let M be a perfect matching of $G - v$. A path $v_1v_2 \cdots v_l$ with $v_1 = v$ is called an *alternating path* if $v_{2i}v_{2i+1} \in M$ for each i with $1 \le i \le \frac{l-1}{2}$.

Lemma 4.7. *For each $w \in V(G)$, G contains an alternating path Q of odd order connecting v and w such that $M - E(Q)$ is a perfect matching of $G - V(Q)$.*

Proof. If $w = v$, then it suffices simply to let $Q = v$. Thus we may assume $w \neq v$. Let M' be a perfect matching of $G - w$, and let H denote the subgraph induced by the symmetric difference of M and M'. Then $d_H(v) = d_H(w) = 1$, and $d_H(x) = 2$ for all $x \in V(H) - \{v, w\}$. This implies that the component Q of H containing v is an alternating path connecting v and w. Since the edge of Q incident with v does not belong to M and the edge of Q incident with w belongs to M, Q has odd order, and $M - E(Q)$ is a perfect matching of $G - V(Q)$. □

Lemma 4.8. *Suppose that $|V(G)| \geq 5$ and, in the case where G is isomorphic to $K_1 + sK_2$ for some $s \geq 2$, suppose further that v is not the unique cutvertex of G. Then G contains an alternating path Q of odd order having v as one of its endvertices such that $|V(Q)| \geq 5$ and $M - E(Q)$ is a perfect matching of $G - V(Q)$.*

Proof. If $vu \in E(G)$ for all $u \in V(G) - \{v\}$, then the assumption of the lemma implies that G contains an edge xy joining endvertices of two distinct edges xx', yy' in M, and hence $vx'xyy'$ is a path with the desired properties. Thus we may assume that there exists $u \in V(G) - \{v\}$ such that $vu \notin E(G)$. Let $uw \in M$. By Lemma 4.7, G contains an alternating path Q of odd order connecting v and w such that $M - E(Q)$ is a perfect matching of $G - V(Q)$. Since Q is an alternating path of odd order and $vu \notin E(G)$, we get $|V(Q)| \geq 5$, as desired. □

5 Proof of main theorems

For a graph H, we let $\mathcal{C}_{\mathrm{odd}}(H)$ denote the set of those components of H having odd order, and set $c_{\mathrm{odd}}(H) = |\mathcal{C}_{\mathrm{odd}}(H)|$.

The following theorem is known as Tutte's 1-factor theorem.

Theorem D (Tutte [6])**.** *If a graph G of even order has no perfect matching, then there exists $S \subseteq V(G)$ such that $c_{\mathrm{odd}}(G - S) \geq |S| + 2$.*

In this section, we often choose a set S of vertices of a given graph G so that

(S1) $c_{\mathrm{odd}}(G - S) - |S|$ is as large as possible, and

(S1) subject to (S1), $|S|$ is as large as possible.

Note that $c_{\mathrm{odd}}(G - S) - |S| \geq c_{\mathrm{odd}}(G) - |\emptyset| \geq 0$ (it is possible that $S = \emptyset$, but our argument in this section works even if $S = \emptyset$).

We first give a fundamental lemma.

Lemma 5.1. *Let G be a graph, and let S be a subset of $V(G)$ satisfying (S1) and (S2). Then the following hold.*

(i) *We have $\mathcal{C}(G-S) = \mathcal{C}_{\text{odd}}(G-S)$.*

(ii) *For each $C \in \mathcal{C}_{\text{odd}}(G-S)$, C is hypomatchable.*

(iii) *Let H be the bipartite graph with bipartition $(S, \mathcal{C}_{\text{odd}}(G-S))$ defined by letting $uC \in E(H)$ $(u \in S, C \in \mathcal{C}_{\text{odd}}(G-S))$ if and only if $N_G(u) \cap V(C) \neq \emptyset$. Then for every $X \subseteq S$, $|N_H(X)| \geq |X|$.*

Proof. (i) Suppose that there exists $C \in \mathcal{C}(G-S)$ such that $|V(C)|$ is even, and take $v \in V(C)$. Then $c_{\text{odd}}(C-v) \geq 1$. Let $S_1 = S \cup \{v\}$. Then $c_{\text{odd}}(G-S_1) - |S_1| = (c_{\text{odd}}(G-S) + c_{\text{odd}}(C-v)) - (|S|+1) \geq c_{\text{odd}}(G-S) - |S|$ and $|S_1| > |S|$, which contradicts (S1) or (S2).

(ii) Suppose that C is not hypomatchable. Then there exists $v \in V(C)$ such that $C-v$ has no perfect matching. Note that $C-v$ has even order. Applying Theorem D to $C-v$, we see that there exists $S'' \subseteq V(C) - \{v\}$ such that $c_{\text{odd}}((C-v)-S'') \geq |S''|+2$. Hence $S''_0 = S'' \cup \{v\}$ $(\subseteq V(C))$ satisfies $c_{\text{odd}}(C-S''_0) \geq |S''_0|+1$. Let $S_2 = S \cup S''_0$. Then $c_{\text{odd}}(G-S_2) - |S_2| = (c_{\text{odd}}(G-S) - 1 + c_{\text{odd}}(C-S''_0)) - (|S|+|S''_0|) \geq c_{\text{odd}}(G-S) - |S|$ and $|S_2| = |S| + |S''_0| > |S|$, which contradicts (S1) or (S2).

(iii) Suppose that there exists $X \subseteq S$ such that $|N_H(X)| < |X|$. Set $S_3 = S - X$. Then every component in $\mathcal{C}_{\text{odd}}(G-S) - N_H(X)$ belongs to $\mathcal{C}_{\text{odd}}(G-S_3)$. Hence

$$\begin{aligned} c_{\text{odd}}(G-S_3) - |S_3| &\geq (c_{\text{odd}}(G-S) - |N_H(X)|) - |S_3| \\ &> c_{\text{odd}}(G-S) - |X| - |S_3| \\ &= c_{\text{odd}}(G-S) - |S|, \end{aligned}$$

which contradicts (S1).

□

5.1 Proof of Theorem 1.1

For a graph H, we let $\mathcal{C}'(H)$ denote the set of those components $C \in \mathcal{C}_{\text{odd}}(H)$ such that $|V(C)| \geq 3$ and C is a hypomatchable graph having no $\{P_2, P_7\}$-factor, and set $c'(H) = |\mathcal{C}'(H)|$.

We first give a sufficient condition for the existence of a $\{P_2, P_7\}$-factor in terms of c_1 and c'.

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Theorem 5.2. *Let G be a graph. If $c_1(G-X)+\frac{1}{2}c'(G-X)\le |X|$ for all $X\subseteq V(G)$, then G has a $\{P_2,P_7\}$-factor.*

Proof. Choose $S\subseteq V(G)$ so that (S1) and (S2) hold.

Set $T=\mathcal{C}_{\mathrm{odd}}(G-S)$ $(=\mathcal{C}(G-S))$, $T_1=\mathcal{C}_1(G-S)$ and $T_2=\mathcal{C}'(G-S)$. Then $T_1\cap T_2=\emptyset$ and $T_1\cup T_2\subseteq T$. We construct a bipartite graph H with bipartition (S,T) by letting $uC\in E(H)$ $(u\in S, C\in T)$ if and only if $N_G(u)\cap V(C)\ne\emptyset$.

Claim 5.1. *For every $Y\subseteq T_1\cup T_2$, $|N_H(Y)|\ge |Y\cap T_1|+\frac{1}{2}|Y\cap T_2|$.*

Proof. Suppose that there exists $Y\subseteq T_1\cup T_2$ such that $|N_H(Y)|<|Y\cap T_1|+\frac{1}{2}|Y\cap T_2|$. Set $X'=N_H(Y)$. Then each element of $Y\cap T_1$ belongs to $\mathcal{C}_1(G-X')$, and each element of $Y\cap T_2$ belongs to $\mathcal{C}'(G-X')$. Hence $|Y\cap T_1|\le c_1(G-X')$ and $|Y\cap T_2|\le c'(G-X')$. Consequently $|X'|=|N_H(Y)|<|Y\cap T_1|+\frac{1}{2}|Y\cap T_2|\le c_1(G-X')+\frac{1}{2}c'(G-X')$, which contradicts the assumption of the theorem. □

Now we apply Proposition 3.1 with G and L replaced by H and $\emptyset$, respectively. Then by Lemma 5.1(iii) and Claim 5.1, H has a subgraph F with $V(F)\supseteq S\cup T_1\cup T_2$ such that each $A\in\mathcal{C}(F)$ is a path satisfying one of (I) and (II) in Proposition 3.1. For $A\in\mathcal{C}(F)$, let $U_A=V(A)\cap S$ and $\mathcal{L}_A=V(A)\cap T$, and let $G_A=G[U_A\cup(\bigcup_{C\in\mathcal{L}_A}V(C))]$.

Claim 5.2. *For each $A\in\mathcal{C}(F)$, G_A has a $\{P_2,P_7\}$-factor.*

Proof. We first assume that A satisfies (I) in Proposition 3.1. Then $|U_A|=|\mathcal{L}_A|=1$. Write $U_A=\{u\}$ and $\mathcal{L}_A=\{D\}$, and let $v\in V(D)$ be a vertex with $uv\in E(G)$. Since D is hypomatchable by Lemma 5.1(ii), $D-v$ has a perfect matching M. Hence $M\cup\{uv\}$ is a perfect matching of G_A. In particular, G_A has a $\{P_2,P_7\}$-factor.

Next we assume that A satisfies (II). Note that $|V(A)|$ is odd and $|V(A)|\ge 3$. Write $A=D_1u_1D_2u_2\cdots D_lu_lD_{l+1}$ $(u_i\in U_A, D_i\in\mathcal{L}_A)$. Let $v_i\in N_G(u_i)\cap V(D_i)$ for $1\le i\le l$, and let $v_{l+1}\in N_G(u_l)\cap V(D_{l+1})$. Since A satisfies (II), $|V(D_1)-\{v_1\}|\ge 2$, $|V(D_{l+1})-\{v_{l+1}\}|\ge 2$ and $V(D_i)=\{v_i\}$ $(2\le i\le l)$. Fix $i\in\{1,l+1\}$. Since D_i is hypomatchable by the definition of T_2, D_i-v_i has a perfect matching M_i. Since $|V(D_i)|\ge 3$, v_i is adjacent to a vertex $u'_i\in V(D_i)$. Let $v'_i\in V(D_i)$ be the vertex with $u'_iv'_i\in M_i$. Then $P=v'_1u'_1v_1u_1v_2u_2\cdots v_lu_lv_{l+1}u'_{l+1}v'_{l+1}$ is a path of order at least 7. Since $M_i-\{u'_iv'_i\}$ is a matching for each $i\in\{1,l+1\}$, $F_A=P\cup(M_1-\{u'_1v'_1\})\cup(M_{l+1}-\{u'_{l+1}v'_{l+1}\})$ is a path-factor of G_A with $\mathcal{C}_3(F_A)=\mathcal{C}_5(F_A)=\emptyset$. By Fact 1.1, G_A has a $\{P_2,P_7\}$-factor. □

By Lemma 5.1(i)(ii), each component in $\mathcal{C}(G-S)-\mathcal{C}_1(G-S)-\mathcal{C}'(G-S)$ has a $\{P_2,P_7\}$-factor. This together with Claim 5.2 implies that G has a $\{P_2,P_7\}$-factor.

This completes the proof of Theorem 5.2. □

Proof of Theorem 1.1. Let G be as in Theorem 1.1. Suppose that G has no $\{P_2, P_7\}$-factor. Then by Theorem 5.2, there exists $X \subseteq V(G)$ such that $c_1(G-X) + \frac{1}{2}c'(G-X) > |X|$. Write $\mathcal{C}'(G-X) - (\mathcal{C}_3(G-X) \cup \mathcal{C}_5(G-X)) = \{D_1, \ldots, D_q\}$. For each i $(1 \leq i \leq q)$, since D_i is a hypomatchable graph of order at least 7 with no $\{P_2, P_7\}$-factor, it follows from Proposition 4.1 that $D_i \in \mathcal{G}_0$. For each i $(1 \leq i \leq q)$, let X_i be a crush set of D_i. By Lemma 4.6, $c_1(D_i - X_i) = |X_i| - 1$ and $|X_i| \geq 4$, and hence $c_1(D_i - X_i) \geq \frac{3}{4}|X_i|$. Let $X_0 = X \cup (\bigcup_{1 \leq i \leq q} X_i)$.

Then $c_1(G - X_0) = c_1(G-X) + \sum_{1 \leq i \leq q} c_1(D_i - X_i) \geq c_1(G-X) + \frac{3}{4}\sum_{1 \leq i \leq q} |X_i|$. Consequently

$$\begin{aligned}
c_1(G-X_0) &- \frac{2}{3}c_1(G-X) - \frac{1}{3}q - \frac{2}{3}\sum_{1\leq i\leq q}|X_i| \\
&\geq c_1(G-X) + \frac{3}{4}\sum_{1\leq i\leq q}|X_i| - \frac{2}{3}c_1(G-X) - \frac{1}{3}q - \frac{2}{3}\sum_{1\leq i\leq q}|X_i| \\
&= \frac{1}{3}c_1(G-X) + \sum_{1\leq i\leq q}\left(\frac{3}{4}|X_i| - \frac{1}{3} - \frac{2}{3}|X_i|\right) \\
&= \frac{1}{3}c_1(G-X) + \sum_{1\leq i\leq q}\left(\frac{1}{12}|X_i| - \frac{1}{3}\right) \\
&\geq 0,
\end{aligned}$$

and hence

$$\frac{2}{3}c_1(G-X) + \frac{1}{3}q + \frac{2}{3}\sum_{1\leq i\leq q}|X_i| \leq c_1(G-X_0).$$

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This leads to

$$\begin{aligned}
\frac{2}{3}|X_0| &= \frac{2}{3}\left(|X| + \sum_{1\le i\le q}|X_i|\right)\\
&< \frac{2}{3}\left(c_1(G-X) + \frac{1}{2}c'(G-X) + \sum_{1\le i\le q}|X_i|\right)\\
&= \frac{2}{3}c_1(G-X) + \frac{1}{3}|\mathcal{C}'(G-X)\cap\mathcal{C}_3(G-X)| + \frac{1}{3}|\mathcal{C}'(G-X)\cap\mathcal{C}_5(G-X)|\\
&\quad + \frac{1}{3}|\mathcal{C}'(G-X) - (\mathcal{C}_3(G-X)\cup\mathcal{C}_5(G-X))| + \frac{2}{3}\sum_{1\le i\le q}|X_i|\\
&\le \frac{2}{3}c_1(G-X) + \frac{1}{3}c_3(G-X) + \frac{1}{3}c_5(G-X) + \frac{1}{3}q + \frac{2}{3}\sum_{1\le i\le q}|X_i|\\
&= \frac{2}{3}c_1(G-X) + \frac{1}{3}c_3(G-X_0) + \frac{1}{3}c_5(G-X_0) + \frac{1}{3}q + \frac{2}{3}\sum_{1\le i\le q}|X_i|\\
&\le c_1(G-X_0) + \frac{1}{3}c_3(G-X_0) + \frac{1}{3}c_5(G-X_0),
\end{aligned}$$

which contradicts the assumption of the theorem.

This completes the proof of Theorem 1.1. □

5.2 Proof of Theorem 1.2

Let H be a graph. We let $\mathcal{C}^*(H)$ denote the set of those components $C \in \mathcal{C}_{\text{odd}}(H)$ such that C is a hypomatchable graph having no $\{P_2, P_9\}$-factor, and let $\mathcal{C}^*_{\le 5}(H) = \{C \in \mathcal{C}^*(H) \mid |V(C)| \le 5\}$, $\mathcal{C}^*_{\ge 7}(H) = \{C \in \mathcal{C}^*(H) \mid |V(C)| \ge 7\}$ and $\mathcal{C}^{**}_{\ge 7}(H) = \{C \in \mathcal{C}^*_{\ge 7}(H) \mid C$ is isomorphic to $K_1 + sK_2$ for some $s \ge 3\}$.

Proof of Theorem 1.2. Let G be as in Theorem 1.2. Choose $S \subseteq V(G)$ so that (S1) and (S2) hold.

Set $T = \mathcal{C}_{\text{odd}}(G-S)$ $(= \mathcal{C}(G-S))$, $T_1 = \mathcal{C}^*_{\le 5}(G-S)$ and $T_2 = \mathcal{C}^*_{\ge 7}(G-S)$. Then $T_1 \cap T_2 = \emptyset$ and $T_1 \cup T_2 \subseteq T$. Now we construct a bipartite graph H with bipartition (S, T) by letting $uC \in E(H)$ $(u \in S, C \in T)$ if and only if $N_G(u) \cap V(C) \ne \emptyset$. Let L be the set of those edges $uC \in E(H)$ such that $u \in S$, $C \in \mathcal{C}^{**}_{\ge 7}(G-S)$ and $N_G(u) \cap V(C)$ consists only of the unique cutvertex of C.

Claim 5.3. *For every* $Y \subseteq T_1 \cup T_2$, $|N_{H-L}(Y)| \ge |Y \cap T_1| + \frac{1}{2}|Y \cap T_2|$.

Proof. Suppose that there exists $Y \subseteq T_1 \cup T_2$ such that $|N_{H-L}(Y)| < |Y \cap T_1| + \frac{1}{2}|Y \cap T_2|$. Set $X' = N_{H-L}(Y)$. We divide $Y \cap T_2$ into two disjoint sets. Let Z_1 be the set of those elements

C of $Y \cap T_2$ such that $|V(C)| = 7$ and $C \notin \mathcal{C}^{**}_{\geq 7}(G-S)$, and let $Z_2 = (Y \cap T_2) - Z_1$. Note that Z_2 is the set of those elements C of $Y \cap T_2$ such that C is either isomorphic to $K_1 + 3K_2$ or a hypomatchable graph of order at least 9 with no $\{P_2, P_9\}$-factor. Hence by the definition of $\mathcal{G}_0$ and Proposition 4.2, each element of Z_2 belongs to $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$. Write $Z_2 = \{D_1, \ldots, D_q\}$. Let X_i be a crush set of D_i for each $1 \leq i \leq q$, and set $X_0 = X' \cup (\bigcup_{1 \leq i \leq q} X_i)$. Let $1 \leq i \leq q$. We show that $\bigcup_{0 \leq j \leq 2} \mathcal{C}_{2j+1}(D_i - X_i) \subseteq \bigcup_{0 \leq j \leq 2} \mathcal{C}_{2j+1}(G - X_0)$. This clearly holds if D_i is a component of $G - X'$. Thus we may assume that D_i is not a component of $G - X'$. By the definition of L, this means that $D_i \in \mathcal{C}^{**}_{\geq 7}(G - S)$ and the unique cutvertex of D_i is the only vertex of D_i that is adjacent to vertices in $S - X'$. On the other hand, the unique cutvertex of D_i is contained in X_i by the definition of a crush set. Hence $\bigcup_{0 \leq j \leq 2} \mathcal{C}_{2j+1}(D_i - X_i) \subseteq \bigcup_{0 \leq j \leq 2} \mathcal{C}_{2j+1}(G - X_0)$.

Since i is arbitrary, we see that $c_{2j+1}(G - X_0) = c_{2j+1}(G - X') + \sum_{1 \leq i \leq q} c_{2j+1}(D_i - X_i)$ for each $0 \leq j \leq 2$. By Lemma 4.6, $c_1(D_i - X_i) + c_3(D_i - X_i) + \frac{2}{3} c_5(D_i - X_i) \geq \frac{3}{4}|X_i|$ and $|X_i| \geq 4$ for every $1 \leq i \leq q$. Consequently

$$\begin{aligned} &c_1(G - X_0) + c_3(G - X_0) + \frac{2}{3} c_5(G - X_0) \\ &\qquad \geq c_1(G - X') + c_3(G - X') + \frac{2}{3} c_5(G - X') + \frac{3}{4} \sum_{1 \leq i \leq q} |X_i|. \end{aligned}$$

Hence

$$\begin{aligned} &c_1(G - X_0) + c_3(G - X_0) + \frac{2}{3} c_5(G - X_0) - \frac{2}{3} \sum_{0 \leq j \leq 2} c_{2j+1}(G - X') - \frac{1}{3} q - \frac{2}{3} \sum_{1 \leq i \leq q} |X_i| \\ &\geq c_1(G - X') + c_3(G - X') + \frac{2}{3} c_5(G - X') + \frac{3}{4} \sum_{1 \leq i \leq q} |X_i| \\ &\quad - \frac{2}{3} \sum_{0 \leq j \leq 2} c_{2j+1}(G - X') - \frac{1}{3} q - \frac{2}{3} \sum_{1 \leq i \leq q} |X_i| \\ &= \frac{1}{3} c_1(G - X_0) + \frac{1}{3} c_3(G - X_0) + \sum_{1 \leq i \leq q} \left(\frac{3}{4}|X_i| - \frac{1}{3} - \frac{2}{3}|X_i| \right) \\ &= \frac{1}{3} c_1(G - X_0) + \frac{1}{3} c_3(G - X_0) + \sum_{1 \leq i \leq q} \left(\frac{1}{12}|X_i| - \frac{1}{3} \right) \\ &\geq 0, \end{aligned}$$

which implies

$$\frac{2}{3} \sum_{0 \leq j \leq 2} c_{2j+1}(G - X') + \frac{1}{3} q + \frac{2}{3} \sum_{1 \leq i \leq q} |X_i| \leq c_1(G - X_0) + c_3(G - X_0) + \frac{2}{3} c_5(G - X_0).$$

Recall the definition of X', Z_1 and X_0. Since each element of $Y \cap T_1$ belongs to $\mathcal{C}^*_{\le 5}(G - X')$, we have $|Y \cap T_1| \le |\mathcal{C}^*_{\le 5}(G - X')| \le \sum_{0 \le j \le 2} c_{2j+1}(G - X')$. Since each element of Z_1 belongs to $\mathcal{C}_7(G - X_0)$, we have $|Z_1| \le c_7(G - X_0)$. Therefore

$$\begin{aligned}
\frac{2}{3}|X_0| &= \frac{2}{3}\left(|X'| + \sum_{1 \le i \le q} |X_i|\right) \\
&= \frac{2}{3}\left(|N_{H-L}(Y)| + \sum_{1 \le i \le q} |X_i|\right) \\
&< \frac{2}{3}\left(|Y \cap T_1| + \frac{1}{2}|Y \cap T_2| + \sum_{1 \le i \le q} |X_i|\right) \\
&\le \frac{2}{3}\left(\sum_{0 \le j \le 2} c_{2j+1}(G - X') + \frac{1}{2}(|Z_1| + |Z_2|) + \sum_{1 \le i \le q} |X_i|\right) \\
&\le \frac{2}{3}\left(\sum_{0 \le j \le 2} c_{2j+1}(G - X') + \frac{1}{2}(c_7(G - X_0) + q) + \sum_{1 \le i \le q} |X_i|\right) \\
&\le c_1(G - X_0) + c_3(G - X_0) + \frac{2}{3}c_5(G - X_0) + \frac{1}{3}c_7(G - X_0),
\end{aligned}$$

which contradicts the assumption of the theorem. □

Now we apply Proposition 3.1 with G replaced by H. Then by Lemma 5.1(iii) and Claim 5.3, H has a subgraph F with $V(F) \supseteq S \cup T_1 \cup T_2$ such that each $A \in \mathcal{C}(F)$ is a path satisfying one of (I) and (II) in Proposition 3.1. For $A \in \mathcal{C}(F)$, let $U_A = V(A) \cap S$ and $\mathcal{L}_A = V(A) \cap T$, and let $G_A = G[U_A \cup (\bigcup_{C \in \mathcal{L}_A} V(C))]$.

Claim 5.4. *For each $A \in \mathcal{C}(F)$, G_A has a $\{P_2, P_9\}$-factor.*

Proof. We first assume that A satisfies (I). Then $|U_A| = |\mathcal{L}_A| = 1$. Write $U_A = \{u\}$ and $\mathcal{L}_A = \{D\}$, and let $v \in V(D)$ be a vertex with $uv \in E(G)$. Since D is hypomatchable by Lemma 5.1(ii), $D - v$ has a perfect matching M. Hence $M \cup \{uv\}$ is a perfect matching of G_A. In particular, G_A has a $\{P_2, P_9\}$-factor.

Next we assume that A satisfies (II). Note that $|V(A)|$ is odd and $|V(A)| \ge 3$. Write $A = D_1u_1D_2u_2 \cdots D_lu_lD_{l+1}$ ($u_i \in U_A, D_i \in \mathcal{L}_A$). For $1 \le i \le l$, let $v_i \in N_G(u_i) \cap V(D_i)$ and $w_{i+1} \in N_G(u_i) \cap V(D_{i+1})$. Since u_1D_1 and u_lD_{l+1} are edges of $H - L$, we may assume that v_1 is not the unique cutvertex of D_1 if $D_1 \simeq K_1 + sK_2$ for some $s \ge 3$, and w_{l+1} is not the unique cutvertex of D_{l+1} if $D_{l+1} \simeq K_1 + s'K_2$ for some $s' \ge 3$. Since D_1 and D_{l+1} are hypomatchable graphs of order at least 7 by the definition of T_2, it follows from Lemma 4.8 that D_1 contains a path Q_1 with endvertex v_1 such that $|V(Q_1)| \ge 5$ and

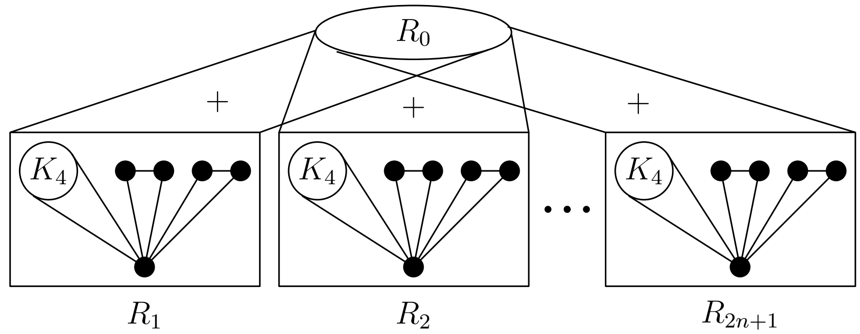

Figure 4: Graph H_n

$D_1 - V(Q_1)$ has a perfect matching M_1, and D_{l+1} contains a path Q_{l+1} with endvertex w_{l+1} such that $|V(Q_{l+1})| \geq 5$ and $D_{l+1} - V(Q_{l+1})$ has a perfect matching M_{l+1}. We regard v_1 as the terminal vertex of Q_1, and w_{l+1} as the initial vertex of Q_{l+1}. For each i $(2 \leq i \leq l)$, since D_i is hypomatchable by the definition of T_1, it follows from Lemma 4.7 that D_i contains a path Q_i connecting w_i to v_i such that $D_i - V(Q_i)$ has a perfect matching M_i. Hence $P = Q_1u_1Q_2u_2 \cdots Q_lu_lQ_{l+1}$ is a path of G_A having order at least 11. Consequently $F_A = P \cup (\bigcup_{1 \leq i \leq l+1} M_i)$ is a path-factor of G_A with $\mathcal{C}_3(F_A) = \mathcal{C}_5(F_A) = \mathcal{C}_7(F_A) = \emptyset$ (and $\mathcal{C}_9(F_A) = \emptyset$). By Fact 1.1, G_A has a $\{P_2, P_9\}$-factor. □

By Lemma 5.1(i)(ii), each component in $\mathcal{C}(G-S) - \mathcal{C}^*_{\leq 5}(G-S) - \mathcal{C}^*_{\geq 7}(G-S)$ has a $\{P_2, P_9\}$-factor. This together with Claim 5.4 implies that G has a $\{P_2, P_9\}$-factor.

This completes the proof of Theorem 1.2. □

6 Sharpness of Theorems 1.1 and 1.2

We first consider the coefficient of $|X|$ in Theorem 1.2. Let $n \geq 1$ be an integer. Let R_0 be a complete graph of order n. For each i $(1 \leq i \leq 2n+1)$, let R_i be a graph isomorphic to $K_1 + (K_4 \cup 2K_2)$. Let $H_n = R_0 + (\bigcup_{1 \leq i \leq 2n+1} R_i)$ (see Figure 4).

For $1 \leq i \leq 2n+1$, since $|V(R_i)| = 9$ and R_i does not contain a path of order 9, R_i has no $\{P_2, P_9\}$-factor. Suppose that H_n has a $\{P_2, P_9\}$-factor F. Then for each i $(1 \leq i \leq 2n+1)$, F contains an edge joining $V(R_i)$ and $V(R_0)$. Since $2n+1 > 2|V(R_0)|$, this implies that there exists $x \in V(R_0)$ such that $d_F(x) \geq 3$, which is a contradiction. Thus H_n has no $\{P_2, P_9\}$-factor.

Lemma 6.1. *For all $X \subseteq V(H_n)$, $\sum_{0 \leq j \leq 3} c_{2j+1}(H_n - X) \leq \frac{2}{3}|X| + \frac{1}{3}$.*

Proof. Let $X \subseteq V(H_n)$.

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Claim 6.1. *For each i $(1 \leq i \leq 2n+1)$, $\sum_{0 \leq j \leq 3} c_{2j+1}(R_i - X) \leq \frac{2}{3}|V(R_i) \cap X| + \frac{1}{3}$.*

Proof. Let u be the unique cutvertex of R_i.

We first assume that $u \notin X$. Then $R_i - X$ is connected. Clearly we may assume that $\sum_{0 \leq j \leq 3} c_{2j+1}(R_i - X) = 1$. Then $|V(R_i) \cap X| \geq 2$ because $|V(R_i)| = 9$. Hence $\sum_{0 \leq j \leq 3} c_{2j+1}(R_i - X) = 1 < \frac{2}{3} \cdot 2 + \frac{1}{3} \leq \frac{2}{3}|V(R_i) \cap X| + \frac{1}{3}$. Thus we may assume that $u \in X$.

Let α be the number of components of $R_i - u$ intersecting with X. Since $\alpha \leq 3$, we have $\alpha \leq \frac{2}{3}(\alpha + 1) + \frac{1}{3}$. Furthermore, $\sum_{0 \leq j \leq 3} c_{2j+1}(R_i - X) = c_1(R_i - X) + c_3(R_i - X) \leq \alpha$ and $|V(R_i) \cap X| = |\{u\}| + |(V(R_i) - \{u\}) \cap X| \geq \alpha + 1$. Consequently we get $\sum_{0 \leq j \leq 3} c_{2j+1}(R_i - X) \leq \frac{2}{3}|V(R_i) \cap X| + \frac{1}{3}$. $\square$

Assume for the moment that $V(R_0) \not\subseteq X$. Then $H_n - X$ is connected. Clearly we may assume that $\sum_{0 \leq j \leq 3} c_{2j+1}(H_n - X) = 1$. Then $|X| \geq 2$ because $|V(H_n)| \geq 9$. Hence $\sum_{0 \leq j \leq 3} c_{2j+1}(H_n - X) = 1 < \frac{2}{3} \cdot 2 + \frac{1}{3} \leq \frac{2}{3}|X| + \frac{1}{3}$. Thus we may assume that $V(R_0) \subseteq X$. Then clearly

$$|\mathcal{C}_{2j+1}(H_n - X)| = \sum_{1 \leq i \leq 2n+1} |\mathcal{C}_{2j+1}(R_i - X)|. \tag{5}$$

By Claim 6.1 and (5),

$$\begin{aligned}
\sum_{0 \leq j \leq 3} c_{2j+1}(H_n - X) &= \sum_{0 \leq j \leq 3} \left(\sum_{1 \leq i \leq 2n+1} c_{2j+1}(R_i - X) \right) \\
&\leq \sum_{1 \leq i \leq 2n+1} \left(\frac{2}{3}|V(R_i) \cap X| + \frac{1}{3} \right) \\
&= \frac{2}{3}(|X| - |V(R_0)|) + \frac{1}{3}(2n+1) \\
&= \frac{2}{3}(|X| - n) + \frac{1}{3}(2n+1) \\
&= \frac{2}{3}|X| + \frac{1}{3}.
\end{aligned}$$

Thus we get the desired conclusion. $\square$

From Lemma 6.1, we get the following proposition, which implies that the coefficient of $|X|$ in Theorem 1.2 is best possible in the sense that it cannot be replaced by any number greater than $\frac{2}{3}$.

Proposition 6.1. *There exist infinitely many graphs G having no $\{P_2, P_9\}$-factor such that $\sum_{0 \leq i \leq 3} c_{2i+1}(G - X) \leq \frac{2}{3}|X| + \frac{1}{3}$ for all $X \subseteq V(G)$.*

We now briefly discuss the sharpness of other coefficients. Let $n \geq 8$, and let R_0 be a complete graph of order n. For each i $(1 \leq i \leq n+1)$, let R_i be a graph isomorphic to $K_1 + 2K_2$, and let u_i be the unique cutvertex of R_i. Let H be the graph obtained from $R_0 \cup (\bigcup_{1 \leq i \leq n+1} R_i)$ by joining u_i to all vertices in R_0 for each i $(1 \leq i \leq n+1)$. Then $c_1(H - V(R_0)) + c_3(H - V(R_0)) + \frac{2}{3}c_5(H - V(R_0)) + \frac{1}{3}c_7(H - V(R_0)) = \frac{2}{3}c_5(H - V(R_0)) = \frac{2}{3}|V(R_0)| + \frac{2}{3}$, and $c_1(H - X) + c_3(H - X) + \frac{2}{3}c_5(H - X) + \frac{1}{3}c_7(H - X) \leq \frac{2}{3}|X|$ for all $X \subseteq V(H)$ with $X \neq V(R_0)$, and H has no $\{P_2, P_9\}$-factor. This shows that the coefficient of $c_5(G - X)$ in Theorem 1.2 is best possible in the sense that it cannot be replaced by any number less than $\frac{2}{3}$. Similarly graphs $K_n + (2n+1)K_7$ $(n \geq 1)$ show that the coefficient of $c_7(G - X)$ in Theorem 1.2 is best possible in the sense that it cannot be replaced by any number less than $\frac{1}{3}$.

As for Theorem 1.1, graphs $K_n + (2n+1)(K_1 + 3K_2)$ $(n \geq 1)$ show that the coefficient $\frac{2}{3}$ of $|X|$ is best possible, and graphs $K_n + (2n+1)K_3$ and $K_n + (2n+1)K_5$ $(n \geq 1)$ show that the coefficient $\frac{1}{3}$ of $c_3(G - X)$ and $c_5(G - X)$ are best possible.

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