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Hamiltonicity and σ-hypergraphs

Cover Page Footnote
I would like to thank the referees whose comments helped me improve the presentation of this paper.

Abstract

We define and study a special type of hypergraph. A σ-hypergraph $H = H(n, r, q \mid \sigma)$, where σ is a partition of r, is an r-uniform hypergraph having nq vertices partitioned into n classes of q vertices each. If the classes are denoted by V_1, V_2,...,V_n, then a subset K of $V(H)$ of size r is an edge if the partition of r formed by the non-zero cardinalities $\mid K \cap V_i \mid$, $1 \leq i \leq n$, is σ. The non-empty intersections $K \cap V_i$ are called the parts of K, and $s(\sigma)$ denotes the number of parts. We consider various types of cycles in hypergraphs such as Berge cycles and sharp cycles in which only consecutive edges have a nonempty intersection. We show that most σ-hypergraphs contain a Hamiltonian Berge cycle and that, for $n \geq s+1$ and $q \geq r(r-1)$, a σ-hypergraph H always contains a sharp Hamiltonian cycle. We also extend this result to k-intersecting cycles.

1 Introduction

Let $V = \{v_1, v_2, ..., v_n\}$ be a finite set, and let $E = \{E_1, E_2, ..., E_m\}$ be a family of subsets of V. The pair $H = (X, E)$ is called a *hypergraph* with vertex-set $V(H) = V$, and with edge-set $E(H) = E$. When all the subsets are of the same size r, we say that H is an *r-uniform hypergraph*. A *σ-hypergraph* $H = H(n, r, q \mid \sigma)$, where σ is a partition of r, is an r-uniform hypergraph having nq vertices partitioned into n *classes* of q vertices each. If the classes are denoted by V_1, V_2,...,V_n, then a subset K of $V(H)$ of size r is an edge if the partition of r formed by the non-zero cardinalities $\mid K \cap V_i \mid$, $1 \leq i \leq n$, is σ. The non-empty intersections $K \cap V_i$ are called the *parts* of K, and $s = s(\sigma)$ denotes the number of parts. We denote the largest part of σ by $\Delta = \Delta(\sigma)$ and the smallest part by $\delta = \delta(\sigma)$. In order to avoid trivial situations where there are no edges, we shall always assume that $q \geq \Delta$ and $n \geq s$. These hypergraphs were first introduced in [2] and studied further in [3, 4].

We consider Hamiltonian cycles in σ-hypergraphs. In a graph G, a Hamiltonian path is a path which includes every vertex $v \in V(G)$. A Hamiltonian cycle is a closed Hamiltonian path. It is well-known that the problem of determining whether a Hamiltonian cycle exists in a graph is NP-complete. An excellent survey on results related to Hamiltonicity is given in [5].

In hypergraphs, in particular r-uniform hypergraphs, there are several different types of paths and cycles to consider — amongst the first to be defined was the *Berge cycle* [1]. A sequence $C = (v_1, e_1, v_2, e_2, \ldots, v_p, e_p, v_1)$ is a *Berge cycle* if

- $v_1, v_2, \ldots, v_p$ are all distinct vertices
- $e_1, e_2, \ldots, e_p$ are all distinct edges
- $v_k, v_{k+1} \in e_k$ for $k = 1, \ldots, p$ where $v_{p+1} = v_1$

A Berge cycle is Hamiltonian if p is equal to the number of vertices.

Several other types of cycles and Hamiltonian cycles have been described and studied as in [7, 8, 10]. The presentation [6] gives an excellent survey of cycles and paths in hypergraphs. We give the following definitions of cycles and Hamiltonian cycles which are particularly suited to the structure of σ-hypergraphs.

Consider an r-uniform hypergraph H. Let $C = (e_1, \ldots, e_p)$ be a sequence of edges of H. Then C is a *sharp cycle* if $|e_i \cap e_{i+1}| > 0$ for $1 \leq i \leq p$, where addition is modulo p, and $|e_i \cap e_j| = 0$ otherwise.

A sharp cycle C is a *sharp Hamiltonian cycle* if $V(C) = V(H)$.

A sharp Hamiltonian cycle $C = (e_1, e_2, \ldots, e_p)$ is said to be *(t, z)-sharp* if $p = 0 \pmod 2$ and, for some $t, z > 0$, $|e_i \cap e_{i+1}| = t$ when $i = 1 \pmod 2$ and $|e_i \cap e_{i+1}| = z$ when $i = 0 \pmod 2$, for $1 \leq i \leq p$ and addition is modulo p, and $|e_i \cap e_j| = 0$ otherwise. If $t = z$, the cycle is *t-sharp*. These cycles are analogous to t-overlapping cycles as described in [9]. A 1-sharp cycle is often referred in the literature to as a *loose cycle*.

Finally, a *k-intersecting cycle* $C = (e_1, e_2, \ldots, e_p)$ is such that

$$|e_i \cap e_{i+1} \cap \ldots \cap e_{i+k-1}| > 0$$

for $1 \leq i \leq p$, where addition is modulo p, while any other collection of k or more edges has an empty intersection. If $V(C) = V(H)$ then C is a *k-intersecting Hamiltonian cycle*. Thus a sharp Hamiltonian cycle is a 2-intersecting Hamiltonian cycle.

In this paper we consider all the above types of Hamiltonian cycles in σ-hypergraphs. We first consider Berge cycles, and then move on to sharp Hamiltonian cycles, and finally to k-intersecting cycles. We give some conditions for the existence and non-existence of the different types of Hamiltonian cycles in σ-hypergraphs, which then lead us to consider conditions on the parameters of $H = H(n, r, q \mid \sigma)$ for which the different types of Hamiltonian cycles always exist.

When constructing sharp Hamiltonian cycles, we will use matchings — the link between matchings and Hamiltonian cycles in r-uniform hypergraphs has been extensively studied [1]. Given an r-uniform hypergraph H, a *matching* is a set of pairwise vertex-disjoint edges $M \subset E(H)$. A *perfect matching* is a matching which covers all vertices of H and we denote the size of the largest matching in an r-uniform hypergraph H by $\nu(H)$.

Matchings in σ-hypergraphs were studied in [4] and, as in that paper we here give more structure to the vertices of the hypergraph $H = H(n, r, q \mid \sigma)$ with $\sigma = (a_1, a_2, \ldots, a_s)$, and $\Delta = a_1 \geq a_2 \geq \ldots \geq a_s = \delta$. The classes making up the vertex set are ordered as $V_1, V_2, \ldots, V_n$ and, within each V_i, the vertices are ordered as $v_{1,i}, v_{2,i}, \ldots, v_{q,i}$. We visualise the vertex set $V(H)$ as a $q \times n$ grid whose first row is $v_{1,1}, v_{1,2}, \ldots, v_{1,n}$. We sometimes refer to the vertices $v_{1,i}, v_{2,i}, \ldots, v_{k,i}$ as the top k vertices of the class V_i, and to $v_{q-k+1,i}, v_{q-k+2,i}, \ldots, v_{q,i}$ as the bottom k vertices of V_i. The vertices $v_{k,i}$ and $v_{k+1,i}$ are said to be consecutive in V_i. The class V_1 is called the first class of vertices, and V_n is the last class; V_i and V_{i+1} are said to be consecutive classes. A set of vertices contained in h consecutive rows and k consecutive classes of $V(H)$ is said to be an $h \times k$ subgrid of $V(H)$. Also, for $\sigma = (a_1, a_2, \ldots, a_s)$, if $a_1 = a_2 = \ldots = a_s = \Delta$, σ is said to be a *rectangular* partition. Furthermore, if σ is rectangular and $\Delta = s(\sigma)$, then σ is a *square* partition.

It is well known that in a graph, the Hamiltonian cycle yields a perfect or near perfect (leaving one vertex unmatched if n is odd) matching. In [4], it was shown that there exist arbitrarily large σ-hypergraphs which do not have a perfect matching, and for which the number of unmatched vertices is quite large. We state a result from this paper:

Lemma 1.1. *Let $H = H(n, r, q \mid \sigma)$, where $\sigma = (a_1, \ldots, a_s)$, $n \geq s$ and $q \geq r$. Suppose $gcd(\sigma) = d \geq 2$, and $q = t \pmod d$ where $1 \leq t \leq d - 1$. Then in a maximum matching of H, there are at least tn vertices left unmatched. Hence $\nu(H) \leq \frac{n(q-t)}{r}$.*

In the sequel, we will show that these σ-hypergraphs, however, still have both a Berge and a sharp Hamiltonian cycle when q and n are large enough.

2 Berge Cycles

Let us first consider this type of cycle, and give necessary and sufficient conditions for the existence of Hamiltonian Berge cycles in σ-hypergraphs.

Theorem 2.1. *Let $H = H(n, r, q \mid \sigma)$ with $\sigma = (a_1, a_2, \ldots, a_s)$, $s \geq 2$ and $\Delta = a_1 \geq a_2 \geq \ldots \geq a_s = \delta \geq 1$. If σ is not rectangular and $q \geq \Delta$ and $n \geq s$, then H has Hamiltonian Berge cycle. If σ is rectangular, then there is a Berge Hamiltonian cycle when $q \geq \Delta + 1$ and $n \geq s + 1$.*

Proof. Let us take a partition σ which is not rectangular — we construct a Berge cycle as follows: for e_1, we take the bottom a_1 vertices in V_1, the bottom a_2 vertices in V_2 and so on up to the bottom a_s vertices in V_s. For e_2, we "shift" the edge one column to the right so that the parts are taken from V_2 to V_{s+1}. We carry on in this fashion, and when we take part a_{s-1} from V_n then we take part a_s from V_1, but "shift" one row up. We carry on in this way, shifting one column to the right each time, and shifting one row up each time — when we reach the top row then we start using the bottom vertices again. In all, we form nq distinct edges in this way. We can now order the vertices by labelling the bottom vertex in V_1 as v_1, the bottom vertex in V_2 as v_2, and so on up to the bottom vertex in V_n as v_n — we then move up one row and label the vertices in this row v_{n+1} up to v_{2n}, from left to right, and we carry on in this fashion until we have labelled all vertices in this way.

Then the cycle $v_1, e_1, v_2, e_2, \ldots, v_{nq}, e_{nq}, v_1$ is a Hamiltonian Berge cycle.

If σ is rectangular and $q = \Delta$ and $n = s$, then there is only one edge and hence no Berge Hamiltonian cycle, otherwise the Berge Hamiltonian cycle can be constructed as above.

□

Note: The conditions are easily seen to be necessary, that is if H has a Berge Hamiltonian cycle then necessarily $q \geq \Delta$ and $n \geq s$ otherwise H has no edges, while when σ is rectangular, q must be at least $\Delta + 1$ and $n \geq s + 1$, otherwise there are not enough edges.

Figure 1 gives an example of a Hamiltonian Berge cycle for $H = H(n, r, q \mid \sigma)$ with $\sigma = (2, 1)$, $s = 2$ and $q = n = 3$. The cycle has 9 edges. The shaded vertices form the edge in each case, and the vertices are numbered in cyclic order.

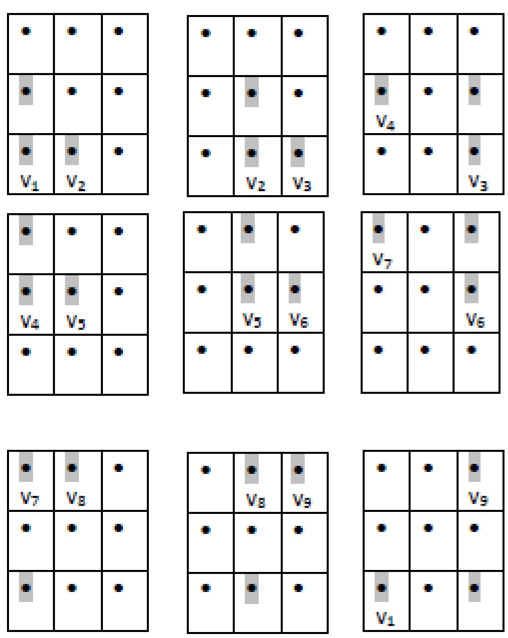

Figure 1: Berge Hamiltonian Cycle - shaded vertices represent the linking edges

3 Sharp Hamiltonian Cycles

We have given necessary and sufficient conditions for a σ-hypergraph to contain a Berge Hamiltonian cycle. Hence we now turn our attention to sharp Hamiltonian cycles which prove to be more challenging. Although we shall be studying, in a later section, k-intersecting Hamiltonian cycles, we want to treat separately sharp cycles first, which are k-intersecting for $k = 2$, because they illustrate very clearly the main techniques used in this paper and also because stronger results are possible with sharp cycles, when, in some cases, we prove that the Hamiltonian cycles obtained are either t-sharp or (t, z)-sharp.

We first present some basic observations about sharp Hamiltonian cycles in r-uniform hypergraphs.

Lemma 3.1. *Let H be a r-uniform hypergraph and let C be a sharp Hamiltonian cycle in H. Then*

1. $\frac{2|V(H)|}{r} \geq |E(C)| \geq \frac{|V(H)|}{r-1}$
2. $\nu(H) \geq \nu(C) = \left\lfloor \frac{|E(C)|}{2} \right\rfloor$.
3. *If* $2\nu(H) + 1 < \frac{nq}{r-1}$, *there is no sharp Hamiltonian cycle in H.*

Proof.

1. Consider $C = (e_1, e_2, \ldots, e_p)$. Clearly, each edge of C intersects the next edge, so each edge contributes at most $r-1$ vertices to $V(C) = V(H)$, and hence $|V(H)| \leq p(r-1)$ which imples $p = |E(C)| \geq \frac{|V(H)|}{r-1}$.

Now consider the degrees of the vertices in C. No vertex can have degree greater than 2, so using the well known fact that

$$r|E(C)| = \sum deg_C(v) \leq 2|V(H)|$$

we get the result $|E(C)| \leq \frac{2|V(H)|}{r}$.

2. By the definition of a sharp cycle, the subset of $E(C)$ $\{e_{2j+1} : 0 \leq j \leq \left\lfloor \frac{|E(C)|}{2} \right\rfloor\}$ is a maximal matching in C, as the edges are distinct, and clearly any other edge in C will intersect one of these edges. Hence

$$\nu(H) \geq \nu(C) = \left\lfloor \frac{|E(C)|}{2} \right\rfloor.$$

3. If we apply the result in part 1 to σ-hypergraphs, where $V(H) = nq$, we get

$$\frac{2nq}{r} \geq |E(C)| \geq \frac{nq}{r-1}.$$

Clearly, by the assumption $2\nu(H) + 1 < \frac{nq}{r-1}$,

$$|E(C)| \geq \frac{nq}{r-1} > 2\nu(H) + 1 \geq 2\nu(C) + 1.$$

But $|E(C)|$ is an integer, hence

$$\begin{aligned} |E(C)| &\geq 2\nu(H) + 2 \geq 2\nu(C) + 2 \\ &= 2\left\lfloor \frac{|E(C)|}{2} \right\rfloor + 2 \geq 2\left(\frac{|E(C)| - 1}{2}\right) + 2 \\ &= |E(C)| + 1, \end{aligned}$$

a contradiction. □

3.1 Examples of σ-hypergraphs with no Hamiltonian cycle.

Let us consider examples of σ-hypergraphs in which there is no sharp Hamiltonian cycle.

For the first example we use Lemma 3.1 — consider $H = H(n, r, q \mid \sigma)$ with $\sigma = (\Delta, \Delta, \ldots, \Delta)$ and $s(\sigma) = \Delta \geq 2$, that is σ is a square partition. Let $n = q = 2\Delta - 1$.

In this case, it is easy to see that $\nu(H) = 1$, while $\frac{nq}{r-1} = \frac{(2\Delta-1)^2}{\Delta^2-1}$. Hence, if $\frac{(2\Delta-1)^2}{\Delta^2-1} > 3$, that is for $\Delta \geq 3$, there is no sharp Hamiltonian cycle in H.

As a second example, consider $H = H(n, r, q \mid \sigma)$ with $\sigma = (\Delta, 1, \ldots, 1)$ where $s(\sigma) = \Delta = \frac{r+1}{2} \geq 4$ and $q = n = \frac{r+1}{2}$. Consider an edge E_1 with the first part of size Δ taken from V_1, and the other parts taken from V_2 to V_n respectively. It is clear that any other edge intersects this edge — we need at least four edges for a sharp Hamiltonian cycle, but this is impossible since all these edges intersect E_1 and hence the cycle is not sharp.

In view of the above examples, our goal is to deal with the following problem which now arises naturally:

Problem 3.1. *Let $H = H(n, r, q \mid \sigma)$, where $s(\sigma) \geq 2$. Does there exists $q(\sigma)$ and $n(\sigma)$ such that for all $q \geq q(\sigma)$ and $n \geq n(\sigma)$, H has a sharp Hamiltonian cycle?*

We will supply an affirmative solution to this problem. So we begin with some results which will then allow us to find a solution to this problem.

Lemma 3.2. *Let $H = H(n, r, q \mid \sigma)$ with $\sigma = (a_1, a_2, \ldots, a_s)$, $s \geq 2$ and $\Delta = a_1 \geq a_2 \geq \ldots \geq a_s = \delta \geq 1$. Let $1 \leq p < s$, and let*

$$t = \sum_{i=1}^{i=p} a_i \text{ and } z = \sum_{i=p+1}^{i=s} a_i = r - t.$$

If $r|q$ and $n \geq s + 1$, then H has a (t, z)-sharp Hamiltonian cycle. Moreover if there exists p such that

$$t = \sum_{i=1}^{i=p} a_i = \sum_{i=p+1}^{i=s} a_i,$$

then the cycle is t-sharp.

Proof. Let us consider the first r vertices in $V_1, \ldots, V_n$ as an $r \times n$ grid of vertices. We will construct two perfect matchings M and M^*, whose edges will then be used to form a sharp Hamiltonian cycle C_1 with $2n$ edges. Observe that we need $s = s(\sigma) \geq 2$, otherwise if $s = 1$, that is $\sigma = (r)$, then for $n \geq 2$ H is not connected, while for $n = 1$, H is the complete r-uniform hypergraph on q vertices, which is trivially Hamiltonian for $q \geq r + 1$.

For the first matching M, let each column of r vertices be partitioned into s consecutive parts of sizes $a_1, a_2, \ldots, a_s$ (from top to bottom). The part a_i in V_j will be referred to as the i^{th} part in V_j. The edge E_1 is formed by taking the top a_1 vertices from V_1, the second part of size a_2 from V_2 and so on, "in diagonal fashion". This is repeated for E_2 by "shifting one class to the right", taking the top a_1 vertices from V_2, the second part from V_3 etc. In general, the edge E_j, $1 \leq j \leq n$, takes the first part from V_j, the second part from V_{j+1} and in general the k^{th} part from V_{j+k-1}, for $1 \leq k \leq s$, with addition modulo n. This gives a perfect matching M with n edges.

For the second matching M^*, let $1 \leq p < s$, and let

$$t = \sum_{i=1}^{i=p} a_i \text{ and } z = \sum_{i=p+1}^{i=s} a_i = r - t$$

Then we take edge E_1^* such that parts a_1 to a_p are taken as in edge E_1, while part a_{p+1} to a_s are taken as in edge E_2 - this is possible since $n \geq s+1$ and hence E_1^* is different from all E_i in M. In general, E_i^* has parts a_1 to a_p as in edge E_i, and parts a_{p+1} to a_s as in edge E_{i+1}, where addition is modulo n. It is clear that these edges form another perfect matching, and that they are distinct from the edges taken in M.

Now a sharp cycle C_1 is formed by taking the edges in $M \cup M^*$ in this order:

$$E_1, E_1^*, E_2, E_2^*, \ldots, E_i, E_i^*, E_{i+1}, E_{i+1}^*, \ldots, E_n, E_n^*.$$

In general, $|E_i \cap E_i^*| = a_1 + \ldots + a_p = t$ and $|E_i^* \cap E_{i+1}| = a_{p+1} + \ldots + a_s = z = r - t$, and at the end, E_n^* has parts a_{p+1} to a_s to coincide with the same parts in E_1 to close the cycle. It is clear that the cycle is (t,z)-sharp since the only intersections between edges are the ones described. If there exists p such that

$$t = \sum_{i=1}^{i=p} a_i = \sum_{i=p+1}^{i=s} a_i = z = r - t,$$

then the cycle is t-sharp.

Now if $q \geq r$, then for the next r vertices in V_1 to V_n we can create another sharp cycle C_2 in the same way. To link C_1 to C_2, for the last edge of M^* in C_1, we change the parts a_{p+1} to a_s to coincide with the equivalent parts in the first edge of M in C_2, thus linking the two cycles.

Hence if $q = xr$, we proceed this way to obtain a sharp Hamiltonian cycle $C = C_1 \cup C_2 \cup \ldots \cup C_x$, where we change the parts a_{p+1} to a_s in the last edge of C_x to coincide with the corresponding parts in the first edge of C_1, thus forming a (t,z)-sharp Hamiltonian cycle in H.

Now if there exists p such that

$$t = \sum_{i=1}^{i=p} a_i = \sum_{i=p+1}^{i=s} a_i,$$

it is clear that the cycle is t-sharp.

□

Lemma 3.3. *Let $H = H(n, r, q \mid \sigma)$ with $\sigma = (a_1, a_2, \ldots, a_s)$, $s \geq 2$ and $\Delta = a_1 \geq a_2 \geq \ldots \geq a_s = \delta \geq 1$. Let $1 \leq p < s$, and let*

$$t = \sum_{i=1}^{i=p} a_i \text{ and } z = \sum_{i=p+1}^{i=s} a_i = r - t.$$

If $r+1 | q$ and $n \geq s+1$, then H has an $(t-1, z)$-sharp Hamiltonian cycle.

Proof. Let us first consider the first $r+1$ vertices in $V_1, \ldots, V_n$ as an $(r+1) \times n$ grid of vertices. As in the previous Lemma, we will construct two matchings M and M^*, whose edges can be used to form a sharp cycle C_1 with $2n$ edges.

The first matching M is constructed in exactly the same way as M was constructed in Lemma 3.2 to cover the top $r \times n$ grid of vertices, having n edges and leaving out the vertices in the $(r+1)^{th}$ row.

For the second matching M^*, again we take $1 \leq p < s$, and let

$$t = \sum_{i=1}^{i=p} a_i \text{ and } z = \sum_{i=p+1}^{i=s} a_i = r - t.$$

The edges are then formed as follows: for edge E_1^*, part a_1 is taken as in E_1, but replacing the last vertex in this part with the $(r+1)^{th}$ vertex in the same class. Parts a_2 to a_p are taken as per edge E_1, while parts a_{p+1} to a_s are taken as per edge E_2. Therefore, in general, E_i^* has part a_1 taken from V_i to include the top $a_1 - 1$ vertices, and the last vertex in the class, parts a_2 to a_p as per edge E_i , and parts a_{p+1} to a_s as in edge E_{i+1}.

Now the sharp cycle is $C_1 = (E_1, E_1^*, E_2, E_2^*, \ldots, E_n, E_n^*)$ so that $|E_i \cap E_i^*| = t-1$, and $|E_i^* \cap E_{i+1}| = z = r - t$. The last edge E_n^* intersects E_1 in parts a_{p+1} to a_s.

Now if $q \geq r+1$ and $(r+1)|q$, then for the next $r+1$ vertices in V_1 to V_n we can create another cycle C_2 in the same way. To link C_1 to C_2, for E_n^*, we change the parts a_{p+1} to a_s to coincide with the equivalent parts in the first edge of M in C_2, thus linking the two cycles.

Hence if $q = x(r+1)$, we proceed this way to obtain a sharp Hamiltonian cycle $C = C_1 \cup C_2 \cup \ldots \cup C_x$, where we change the parts a_{p+1} to a_s in the last edge of C_x to coincide with the corresponding parts in the first edge of C_1, thus forming a $(t-1, z)$-sharp Hamiltonian cycle in H.

Again, if there exists p such that

$$t - 1 = \left(\sum_{i=1}^{i=p} a_i\right) - 1 = \sum_{i=p+1}^{i=s} a_i,$$

then the cycle is $(t-1)$-sharp. □

We shall use the following classical theorem by Frobenius which states:

Theorem 3.4. *Let a_1, a_2 be positive integers with $gcd(a_1, a_2) = 1$. Then for $n \geq (a_1 - 1)(a_2 - 1)$, there are nonnegative integers x and y such that $xa_1 + ya_2 = n$.*

Using Lemmas 3.2 and 3.3 combined with Theorem 3.4, we can now present an affirmative solution to Problem 3.1, which we restate as a Theorem:

Theorem 3.5. *Let $H = H(n, r, q \mid \sigma)$, where $s(\sigma) \geq 2$. If $q \geq r(r-1)$ and $n \geq s+1$, then H has a sharp Hamiltonian cycle.*

Proof. By Theorem 3.4, we know that if $q \geq r(r-1)$, there exist nonnegative integers x and y such that $xr + y(r+1) = q$, since r and $r+1$ are always coprime. Divide the $q \times n$ grid into x consecutive grids of size $r \times n$, followed by y consecutive grids of size $(r+1) \times n$. If we consider the $xr \times n$ grid first, we know that by Lemma 3.2, there is a sharp cycle C_1 covering these vertices, and by Lemma 3.3, there is a sharp cycle C_2 covering the $y(r+1) \times n$ grid. If $x = 0$ or $y = 0$, then C_1, respectively C_2 give the required sharp Hamiltonian cycle. So we may assume that both x and y are greater than 0. We now need to look at linking C_1 to C_2 and viceversa. Firstly, instead of linking the last edge in C_1 with the first one, we link it to the first edge in C_2, by changing the parts a_{k+1} to a_s for this last edge and choosing them to coincide with the parts a_{k+1} to a_s in the first edge of C_2. The last edge of C_2, must be linked to the first edge in C_1. So we change the parts a_{k+1} to a_s of the last edge in C_2 and let them coincide with the same parts in the first edge in C_1. Thus $C_1 \cup C_2$ form a sharp Hamiltonian cycle in H. □

4 k-intersecting Hamiltonian cycles

We now turn to k-intersecting cycles and generalise the results obtained in the previous section to k-intersecting Hamiltonian cycles in σ-hypergraphs. Recall that a *k-intersecting cycle* $C = (e_1, e_2, \ldots, e_p)$ is such that

$$|e_i \cap e_{i+1} \cap \ldots \cap e_{i+k-1}| > 0$$

for $1 \le i \le p$, where addition is modulo p, while any other collection of k or more edges has an empty intersection. If $V(C) = V(H)$ then C is a *k-intersecting Hamiltonian cycle.*

Lemma 4.1. *Let $H = H(n, r, q \mid \sigma)$ with $\sigma = (a_1, a_2, \ldots, a_s)$, $s \ge 2$ and $\Delta = a_1 \ge a_2 \ge \ldots \ge a_s = \delta \ge 1$. Let $2 \le k \le s$. If $r|q$ and $n \ge s+1$, then H has a k-intersecting Hamiltonian cycle.*

Proof. Let us consider the first r vertices in $V_1, \ldots, V_n$ as an $r \times n$ grid of vertices. We construct k perfect matchings $M_1, \ldots, M_k$ which we then use to construct a k-intersecting Hamiltonian cycle. Recall that $2 \le k \le s$, and $k = 2$ is equivalent to a sharp Hamiltonian cycle.

The first matching M_1 is equivalent to the matching M in Lemma 3.2, with edges labelled as $E_{1,1}, E_{1,2}, \ldots, E_{1,n}$.

In the matching M_2, we take edges $E_{2,1}$ to $E_{2,n}$ so that edge $E_{2,i}$ has parts a_1 to a_{k-1} as in edge $E_{1,i}$, while parts a_k to a_s are as in edge $E_{1,i+1}$.

In the matching M_3, we take edges $E_{3,1}$ to $E_{3,n}$ so that edge $E_{3,i}$ has parts a_1 to a_{k-2} as in edge $E_{1,i}$, while parts a_{k-1} to a_s are as in edge $E_{1,i+1}$.

In general, in the matching M_j, we take edges $E_{j,1}$ to $E_{j,n}$ so that edge $E_{j,i}$ has parts a_1 to a_{k-j+1} as in edge $E_{1,i}$, while parts a_{k-j+2} to a_s are as in edge $E_{1,i+1}$, for $2 \le j \le k$. Since $n \ge s+1$, this is possible for all values of k between 2 and s.

Now we form a k-intersecting cycle C_1 by taking the edges in the following order

$$E_{1,1}, E_{2,1}, \ldots, E_{k,1}, E_{1,2}, \ldots, E_{k,2}, \ldots, E_{1,i}, E_{2,i}, \ldots, E_{k,i}, E_{1,i+1}, \ldots, E_{k,i+1}, \ldots, E_{1,n}, \ldots, E_{k,n}.$$

We now look at the intersections:

$E_{1,1}, E_{2,1}, \ldots, E_{k,1}$ intersect in part a_1 in V_1.

$E_{2,1}, E_{3,1}, \ldots, E_{1,2}$ intersect in parts a_k to a_s in V_{k+1} to V_{s+1}.

$E_{3,1}, E_{4,1}, \ldots, E_{2,2}$ intersect in part a_{k-1} in V_k.

In general, for the remaining edges, $E_{j,i}, E_{j+1,i}, \ldots, E_{j-1,i+1}$ intersect in part a_{k-j+2} for $3 \le j \le k$. The last $k-1$ edges intersect $E_{1,1}$ in parts a_k to a_s in V_k to V_s, making this cycle a k-intersecting cycle. Any other sets of k or more edges will have an empty intersection since they will always include two edges from one matching, which have an empty intersection by definition.

Now if $q \ge r$ and $r|q$, then for the next r vertices in V_1 to V_n we can create another cycle C_2 in the same way. To link C_1 and C_2, we must consider the last $k-1$ edges taken in C_1, that is edge $E_{2,n}$ to $E_{k,n}$. For edge $E_{2,n}$,we change parts a_k to a_s and choose them to coincide with the same parts in the first edge in C_2, and in general, for edge $E_{j,n}$ we change parts a_{k-j+2} to a_s and choose them to coincide with the same parts in the first edge in C_2.

Hence if $q = pr$, we have a k-intersecting Hamiltonian cycle $C = C_1 \cup C_2 \cup \ldots \cup C_p$, with the last $k-1$ edges of C_p intersecting the first edge of C_1 by changing the respective parts in these edges in a similar way as described for C_1 intersecting C_2. □

Figure 2 shows the first two edges in the four perfect matchings required for a 4-intersecting Hamiltonian cycle when $\sigma = (a_1, a_2, \ldots, a_6)$, $q = r$ and $n = 7$. The boxes represent the parts a_1 to a_6 ordered from top to bottom — $E_{j,1}$ is shaded in light grey while $E_{j,2}$ is shaded in dark grey, for $1 \le j \le 4$. Each matching has seven distinct edges.

Lemma 4.2. *Let $H = H(n, r, q \mid \sigma)$ with $\sigma = (a_1, a_2, \ldots, a_s)$, $s \ge 2$ and $\Delta = a_1 \ge a_2 \ge \ldots \ge a_s = \delta \ge 1$. Let $2 \le k \le s$. If $(r+1)|q$ and $n \ge s+1$, then H has a k-intersecting Hamiltonian cycle.*

Proof. Let us consider the first $r+1$ vertices in $V_1, \ldots, V_n$ as an $(r+1) \times n$ grid of vertices. We construct k matchings $M_1, \ldots, M_k$ which we then use to construct a k-intersecting Hamiltonian cycle, using a method similar to that used in the previous lemma.

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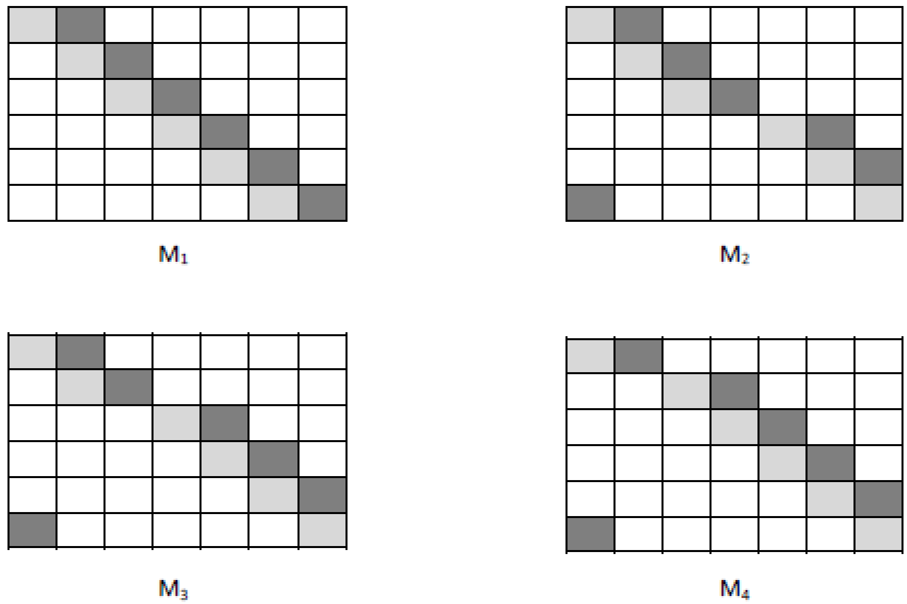

Figure 2: $\sigma = (a_1, a_2, \ldots, a_6)$ - matchings M_1 to M_4

The first matching M_1 is for the top r vertices in V_1 to V_n, and is equivalent to the matching M in Lemma 4.1, with edges labelled as $E_{1,1}, E_{1,2}, \ldots, E_{1,n}$.

In the matching M_2 for the $(r+1) \times n$ grid , we take edges $E_{2,1}$ to $E_{2,n}$ so that edge $E_{2,i}$ has part a_1 as in $E_{1,i}$, but replacing the last vertex in this part with the $(r+1)^{th}$ vertex in the same class, parts a_2 to a_{k-1} as in edge $E_{1,i}$, while parts a_k to a_s are as in edge $E_{1,i+1}$.

In the matching M_3, we take edges $E_{3,1}$ to $E_{3,n}$ so that edge $E_{3,i}$ has part a_1 as in edge $E_{2,i}$, parts a_2 to a_{k-2} as in edge $E_{1,i}$, while parts a_{k-1} to a_s are as in edge $E_{1,i+1}$.

In general, in the matching M_j, we take edges $E_{j,1}$ to $E_{j,n}$ so that edge $E_{j,i}$ has parts a_1 as in edge $E_{2,i}$, parts a_2 to a_{k-j+1} as in edge $E_{1,i}$, while parts a_{k-j+2} to a_s are as in edge $E_{1,i+1}$, for $3 \leq j \leq k$.

Now we form a k-intersecting cycle C_1 by taking the edges in the following order:
$E_{1,1}, E_{2,1}, \ldots, E_{k,1}, E_{1,2}, \ldots, E_{k,2}, \ldots, E_{1,i}, E_{2,i}, \ldots, E_{k,i}, E_{1,i+1}, \ldots, E_{k,i+1}, \ldots, E_{1,n}, \ldots, E_{k,n}$.

We now look at the intersections:
$E_{1,1}, E_{2,1}, \ldots, E_{k,1}$ intersect in $a_1 - 1$ vertices in part a_1 in V_1.
$E_{2,1}, E_{3,1}, \ldots, E_{1,2}$ intersect in parts a_k to a_s in V_{k+1} to V_{s+1}.
$E_{3,1}, E_{4,1}, \ldots, E_{2,2}$ intersect in part a_{k-1} in V_k.

In general, $E_{j,i}, E_{j+1,i}, \ldots, E_{j-1,i+1}$ intersect in part a_{k-j+2} for $3 \leq j \leq k$. The last $k-1$ edges intersect $E_{1,1}$ in parts a_k to a_s in V_k to V_s, making this cycle a k-intersecting cycle. Any other sets of k or more edges will have an empty intersection since they will always include two edges from one matching, which have an empty intersection by definition.

Now if $q \geq r+1$ and $(r+1)|q$, then for the next $r+1$ vertices in V_1 to V_n we can create another cycle C_2 in the same way. To link C_1 and C_2, we must consider the last $k-1$ edges taken in C_1, that is edge $E_{2,n}$ to $E_{k,n}$. For edge $E_{2,n}$,we change parts a_k to a_s and choose them to coincide with the same parts in the first edge in C_2, and in general, for edge $E_{j,n}$ we change parts a_{k-j+2} to a_s and choose them to coincide with the same parts in he first edge in C_2.

Hence if $q = p(r+1)$, we have a k-intersecting Hamiltonian cycle $C = C_1 \cup C_2 \cup \ldots \cup C_p$, with the last $k-1$ edges of C_p intersecting the first edge of C_1 by changing the respective parts in these edges in a similar way as described for C_1 intersecting C_2. □

We can now prove a generalised form of Theorem 3.5:

Theorem 4.3. *Let $H = H(n, r, q \mid \sigma)$, where $s(\sigma) \geq 2$. For $2 \leq k \leq s$, if $q \geq r(r-1)$ and $n \geq s+1$, then H has a k-intersecting Hamiltonian cycle.*

Proof. By Theorem 3.4, we know that if $q \geq r(r-1)$, there exist nonnegative integers x and y such that $xr + y(r+1) = q$, since r and $r+1$ are always coprime. So let us divide the $q \times n$ grid into x consecutive grids of size $r \times n$, followed by y consecutive grids of size $(r+1) \times n$. If we consider the $xr \times n$ grid first, we know that by Lemma 4.1, there is a k-intersecting cycle C_1 covering these vertices, and by Lemma 4.2, there is a k-intersecting cycle C_2 covering the $y(r+1) \times n$ grid. If $x = 0$ or $y = 0$, then C_1, respectively C_2 give the required k-intersecting Hamiltonian cycle. So we may assume that both x and y are greater than 0. We now need to look at linking C_1 to C_2 and vice versa. Firstly, instead of linking the last $k-1$ edges in C_1 with the first one, we link them to the first edge in C_2, as follows: for edge $E_{2,n}$,we change parts a_k to a_s and take them to coincide with the same parts in the first edge in C_2, and in general, for edge $E_{j,n}$ we change parts a_{k-j+2} to a_s and take them toncoincide with the same parts in the first edge in C_2.

The last $k-1$ edges of C_2, must be linked to the first edge in C_1. So we change the respective parts in these edges and take them to coincide with the same parts in the first edge in C_1, in a similar way to how we linked the last $k-1$ edges in C_1 to the first edge in C_2. Thus $C_1 \cup C_2$ form a k-intersecting Hamiltonian cycle in H. □

5 Conclusion

The paper [2] defined σ-hypergraphs and started their study in order to investigate what are known as mixed colourings or Voloshin colouring [11] of hypergraphs. In the colourings in [2], no edge was allowed to have all vertices having the same colour, or all vertices having different colours. This study was continued in [3]. These papers demonstrated the versatility of σ-hypergraphs in obtaining interesting results on mixed colourings. In [4], the study of σ-hypergraphs was extended to two other classical areas of graph and hypergraph theory: matchings and independence. In this paper we continue in this vein, showing that σ-hypergraphs can also give elegant results on Hamiltonicity.

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