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Electromagnetic radiation induced by a gravitational wave

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The electromagnetic radiation induced by a gravitational wave incident on a point charge and a static point dipole is calculated using a multipole expansion of the fields. The calculation is easily extended to the case of a relativistic source. The presence of this radiation does not violate the principle of equivalence. Estimates are made of the electromagnetic radiation induced by a plane gravitational wave incident on a plasma. For hot plasmas the radiated power can be considerably enhanced; however, it is still rather small and the frequency of the radiation would probably be too low for this process to be of any astrophysical significance.

INTRODUCTION

Since Einstein first introduced the principle of equivalence, the question of whether or not a charge radiates when falling freely in a gravitational field has generated much discussion. Only recently have explicit calculations been carried out within the framework of Einstein-Maxwell theory which bear upon this problem.¹⁻⁴ These calculations involved the motions of particles near Schwarzschild and Reissner-Nordström black holes. In this paper I will investigate the radiation from a point charge and static point dipole induced by a plane, polarized gravitational wave. The results are interesting examples of how gravity can induce electromagnetic radiation; however, it is doubtful that these processes are of any astrophysical significance.

The mathematical procedure is straightforward. The generalized Maxwell's equations will be solved in a space-time that has been slightly curved by the gravity wave. For a polarized gravitational wave propagating in the \hat{x}_1 direction, the deviation of the metric tensor $g_{\mu\nu}$ from the Minkowskian form $\eta_{\mu\nu}$ is most simply expressed in the special "transverse-traceless" (TT) coordinate system⁵:

$$h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu} = h e^{i(kx_1 - \omega t)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (1)$$

The radiation is assumed to be weak, i.e., $h \ll 1$. In the presence of the gravity wave Maxwell's equations must be expressed in their covariant form,⁶

$$-A_{\alpha}{}^{\beta}{}_{;\beta} + R_{\alpha}{}^{\beta} A_{\beta} = 4\pi J_{\alpha}, \quad (2)$$

where A_{α} is the electromagnetic vector potential, J_{α} is the 4-current density, $R_{\alpha}{}^{\beta}$ is the Ricci tensor, and the semicolon denotes covariant differentiation.

If we assume that the region of space-time of interest has negligible stress-energy, then Einstein's equations tell us that $R_{\alpha}{}^{\beta} = 0$.

COORDINATE CHOICE

Before solving Maxwell's equations one must decide upon an appropriate coordinate system. Although formally the results are independent of the choice, one should pick a frame in which the interpretation of these results is made simple. In general an appropriate frame is one which is close enough to a Lorentz frame that gravitational corrections to the behavior of measuring instruments are negligible compared to the measurement of interest. Only in such a coordinate system will electric and magnetic field vectors have the same physical significance they have in special relativity. From this standpoint the transverse-traceless gauge is acceptable since it deviates from a Lorentz frame by an amount of order $h \ll 1$ at large distances where the electromagnetic field is to be sampled. Corrections due to gravitational effects in measuring instruments will then be of order h and can be neglected.

The specification of the source J^{α} is also coordinate-dependent. The generally covariant expression for the 4-current of a source consisting of point charges is⁷

$$J^{\alpha}(x, t) = g^{-1/2}(x, t) \sum_n q_n \delta(x_n - x) \left(\frac{dx^{\alpha}}{dt} \right)_n, \quad (3)$$

where $g \equiv -\det g_{\alpha\beta}$. The trajectories of noninteracting point particles in the presence of gravity are given by the geodesic equation

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma_{\alpha\beta}^{\mu} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau} = 0.$$

In the TT gauge of Eq. (1) $\Gamma_{00}^{\mu} = 0$, and the solution to this equation for particles initially at rest is $dx^{\alpha}/d\tau = dx^{\alpha}/dt = (1, 0, 0, 0)$. Also, $g^{-1/2} = 1 + \text{terms of order } h^2 \text{ and higher}$. Equation (3), therefore,

reduces to the classical form

$$J^\mu(\vec{x}) = \sum_n q_n \delta(\vec{x}_n - \vec{x}) \delta_{\mu 0} + \text{terms of order } h^2. \quad (4)$$

The above analysis neglected any interaction between the particles. If a dipole consists of two oppositely charged particles separated by a distance d and held apart by a spring, then in general the gravitational wave will excite oscillations of the system. The resulting time-dependent dipole moment will serve as a source for radiation in the usual way. This radiation is well understood classically and will not be considered in this paper. Therefore, we treat the source as consisting of point charges sufficiently free so that Eq. (4) holds.

SOLVING MAXWELL'S EQUATIONS

If we expand the covariant derivatives in terms of the metric tensor in Eq. (1), Maxwell's equations become

$$\square A_\alpha - (\eta_{\beta\beta}/\eta_{\gamma\gamma}) h_{\beta\gamma} A_{\alpha,\beta\gamma} - \eta_{\beta\beta} (\Gamma_{\alpha\beta,\beta}^\delta A_\delta + 2\Gamma_{\alpha\beta}^\delta A_{\delta,\beta} + \Gamma_{\beta\beta}^\delta A_{\alpha,\delta}) = -4\pi J_\alpha, \quad (5)$$

where terms of order h^2 and higher have been neglected (note $\Gamma_{\alpha\beta}^\gamma \propto h$), $\Gamma_{\alpha\beta}^\gamma$ are the usual Riemann-Christoffel symbols, and the comma denotes ordinary differentiation. Since $h \ll 1$ we expect A_α to be close to the flat-space-time solution A_α^0 where

$$\square A_\alpha^0 = -4\pi J_\alpha. \quad (6)$$

For both of the sources treated in this paper A_α^0 is of the form $A_\alpha^0 = (\phi, 0, 0, 0)$. Consider a perturbation ϵA_α to this solution. Substituting $A_\alpha = A_\alpha^0 + \epsilon A_\alpha$ into Eq. (5), using Eq. (6), and dropping terms of order ϵh , we obtain

$$\square A_\alpha = -4\pi J_\alpha^i, \quad (7)$$

where ϵ has been set equal to 1 for convenience, the connection coefficients $\Gamma_{\alpha\beta}^\gamma$ are expressed in terms of h , and we have introduced the 4-vector J_α^i ,

$$J_0^i = -\frac{h}{4\pi} e^{i(kx_1 - \omega t)} \left(\frac{\partial^2 \phi}{\partial x_2^2} - \frac{\partial^2 \phi}{\partial x_3^2} \right),$$

$$J_1^i = 0,$$

$$J_2^i = \frac{i\omega h}{4\pi} e^{i(kx_1 - \omega t)} \left(\frac{\partial \phi}{\partial x_2} \right), \quad (8)$$

$$J_3^i = -\frac{i\omega h}{4\pi} e^{i(kx_1 - \omega t)} \left(\frac{\partial \phi}{\partial x_3} \right).$$

The fields from A_α^0 fall off as r^{-2} or faster; consequently, any radiation will be associated with the perturbation A_α . In effect, the gravitational wave induces a 4-current density J_α^i in a flat space-time background, and it is this current which acts as a source for electromagnetic radiation. One can check that J_α^i satisfies the important requirement that it be conserved, i.e., $J^{\alpha,\alpha} = 0$.

Although J_α^i can be expressed analytically, it is not a simple source and the only practical approach to the problem is to expand the radiation field in multipoles. The power radiated by a source $\vec{J}(\vec{x}, t) = e^{-i\omega t} \vec{J}(\vec{x})$ can be expressed by the sum⁷

$$P = (8\pi\omega^2)^{-1} \sum_{lm} [|A_M(l, m)|^2 + |A_E(l, m)|^2]. \quad (9)$$

A_M and A_E are the coefficients of the magnetic and electric multipole fields and are given by the expressions

$$A_E(l, m) = 4\pi i k \int j_l(kr) \vec{X}_{lm}^* \cdot (\vec{\nabla} \times \vec{J}) d^3x, \quad (10)$$

$$A_M(l, m) = -4\pi \int j_l(kr) \vec{X}_{lm}^* \cdot (\vec{\nabla} \times \vec{\nabla} \times \vec{J}) d^3x,$$

where j_l are spherical Bessel functions and \vec{X}_{lm} are vector spherical harmonics.

RADIATION FROM A POINT CHARGE

The scalar potential for a point charge q is $\phi = -q/r$ (the minus sign is a result of using the covariant form of the vector potential); therefore, from Eq. (8) the induced current density becomes

$$\vec{J}^i = \left(0, \frac{ix_2}{4\pi r^3} \omega q h e^{i(kx_1 - \omega t)}, \frac{-ix_3}{4\pi r^3} \omega q h e^{i(kx_1 - \omega t)} \right). \quad (11)$$

If one substitutes Eq. (11) into Eq. (10) and using the expansion

$$e^{ikx_1} = \sum_{l'=0}^{\infty} [4\pi(2l'+1)]^{1/2} i^{l'} Y_{l',0} j_{l'}(kr),$$

the integrals can be evaluated and one obtains a rather lengthy expression for $A_E(l, m)$:

$$\begin{aligned}
A_E = & i\omega^2 q \hbar (\pi^3/2)^{1/2} (-1)^m \left[\frac{2l+1}{l(l+1)} \right]^{1/2} \\
& \times \sum_{l'} i^{l'} (2l'+1) \left\{ 2m\sqrt{3} \begin{pmatrix} l' & l & 2 \\ 0 & 0 & 0 \end{pmatrix} \left[\begin{pmatrix} l' & l & 2 \\ 0 & -m & 2 \end{pmatrix} - \begin{pmatrix} l' & l & 2 \\ 0 & -m & -2 \end{pmatrix} \right] \left[(l'+l+2)(l'+l)\Gamma\left(\frac{l'-l+3}{2}\right)\Gamma\left(\frac{l-l'+3}{2}\right) \right]^{-1} \right. \\
& + i \begin{pmatrix} l' & l & 1 \\ 0 & 0 & 0 \end{pmatrix} \left[[(l+m)(l-m+1)]^{1/2} \begin{pmatrix} l' & l & 1 \\ 0 & -m+1 & 1 \end{pmatrix} + [(l-m)(l+m+1)]^{1/2} \begin{pmatrix} l' & l & 1 \\ 0 & -m-1 & -1 \end{pmatrix} \right] \\
& \times \left[(l'+l+1)\Gamma\left(\frac{l'-l+2}{2}\right)\Gamma\left(\frac{l-l'+2}{2}\right) \right]^{-1} \\
& - \sqrt{3} \begin{pmatrix} l' & l & 2 \\ 0 & 0 & 0 \end{pmatrix} \left[[(l+m)(l-m+1)]^{1/2} \begin{pmatrix} l' & l & 2 \\ 0 & -m+1 & 1 \end{pmatrix} + [(l-m)(l+m+1)]^{1/2} \begin{pmatrix} l' & l & 2 \\ 0 & -m-1 & -1 \end{pmatrix} \right] \\
& \left. \times \left[(l'+l+2)(l'+l)\Gamma\left(\frac{l'-l+3}{2}\right)\Gamma\left(\frac{l-l'+3}{2}\right) \right]^{-1} \right\},
\end{aligned}$$

where

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

is the Wigner 3- j coefficient and $\Gamma(n)$ is the gamma function. The magnetic multipole coefficients $A_M(l, m)$ all vanish. From the above expression one can see that the only nonvanishing $A_E(l, m)$ are $l \geq 2$ and $m = \pm 2$. This is not too surprising considering the spin-2 nature of the graviton. Figure 1 is a plot of the multipole coefficients and the power radiated in each multipole,

$$P_l = (8\pi\omega^2)^{-1} \sum_m \left[|A_M(l, m)|^2 + |A_E(l, m)|^2 \right]$$

For large l , $P_l \propto 1/l$ so the sum $\sum P_l$ diverges. Therefore, an infinite plane wave incident on a point charge induces an infinite amount of electromagnetic radiation. The divergence is avoided if either the charge is screened at some distance d or the gravity wave is in the form of a wave packet of dimension d . In each case the sum will be effectively terminated at $l \approx kd$. Physically this is because the large angular momenta associated with high multipoles arise from radiation incident with large angular momenta, i.e., large impact parameters. An approximate expression for the radiation emitted with the sum terminated at multipole number l is

$$P = [0.35 + \frac{1}{2} \ln(l/10)] \omega^2 q^2 \hbar^2 \text{ for } l \geq 10. \quad (12)$$

The cross section σ for the production of electromagnetic radiation is obtained by dividing Eq. (12) by the flux of incident gravitational radiation,⁵ $F = (1/32\pi)\omega^2 \hbar^2$:

$$\sigma = P/F \approx [35 + 4\pi \ln(l/10)] q^2. \quad (13)$$

For a particle of mass m and charge q , σ is ap-

proximately a factor of q^2/m^2 less than the cross section for scattering electromagnetic radiation (Thomson scattering) and approximately q^2/m^2 times greater than the cross section for scattering gravitational radiation.⁸ If the particle is an electron, $q^2/m^2 \approx 5 \times 10^{42}$ and $\sigma \approx 10^{-66}$ cm².

The results presented above are readily generalized to the case where the source is a highly relativistic, $\gamma \gg 1$, point charge. The metric tensor of the gravity wave in the rest frame of a particle moving with velocity $\vec{v} = -v\hat{x}_1$ takes exactly the same form as in Eq. (1), except that the frequency is Doppler-shifted, $\omega' = 2\gamma\omega$. Therefore, the power radiated in the rest frame of the source is

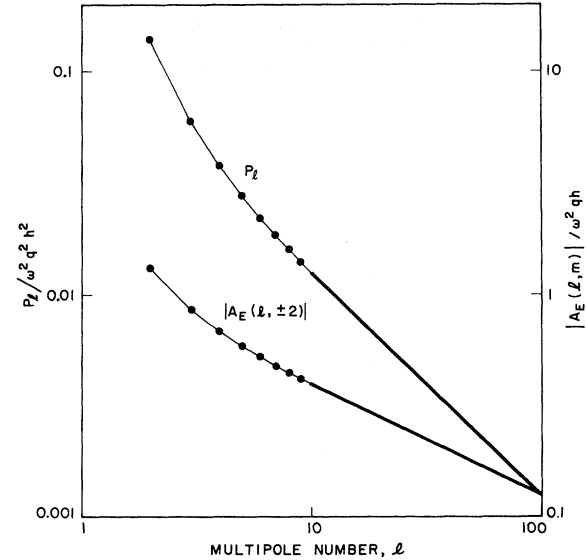


FIG. 1. Power P_l and coefficients $A_E(l, m)$ of the multipole components of the induced radiation field of a point charge.

$$P \approx [1.4 + \frac{1}{2} \ln(l/10)] \omega^2 \gamma^2 q^2 \hbar^2. \quad (14)$$

We recall from relativistic kinematics that in the original frame of the observer most of this radiation will be emitted within an angle $\theta = 1/\gamma$ of the forward direction $-\hat{x}_1$ and with a frequency between $\gamma\omega'$ and $2\gamma\omega'$. From the transformation properties of the fields the intensity of the radiation is increased by γ^2 for radiation in the forward direction. The total power emitted in the observer's frame is then of the same order as that in the particle's rest frame. Hence, by Eq. (14) the cross section σ for a relativistic particle is up by γ^2 from that for the nonrelativistic case. In the forward direction the intensity is increased by a factor of γ^4 and at a frequency of about γ^2 times the frequency of the gravity wave.

RADIATION FROM A POINT DIPOLE

Now consider a static point dipole which has the usual flat-space-time scalar potential $\phi = -(\vec{p} \cdot \vec{x})/r^3$. To simplify the calculation I have assumed $\vec{p} = p\hat{x}_1$. $A_E(l, m)$ and $A_M(l, m)$ are calculated using the same procedure as above. In this case, however, the dipole field falls off rapidly enough that for large l , $P_l \propto 1/l^3$ and the infinite sum converges. The first term is again quadrupolar,

$$P_{l=2} \approx 5.6 \times 10^{-4} p^2 \omega^4 \hbar^2. \quad (15)$$

For a dipole formed by two charges $+q$ and $-q$ separated by a distance d , this power is roughly a factor $(kd)^2$ less than that of a single charge q . It is interesting to compare this case with the radiation implied by changes in the "proper" length d of a finite dipole. Both the expression for proper length⁹ and the equation of geodesic deviation tell us that particles lying on the x_1 axis are not accelerated with respect to one another and from this point of view should not radiate, contrary to the above calculation. The point is that radiation is a global, not local, phenomenon, and cannot be determined by investigating the relative accelerations of charged particles in a local inertial frame.

Recall that in this analysis all charges are assumed to be noninteracting. Suppose we attach the point charges to the ends of a stiff spring (i.e., a spring with a resonant frequency larger than the frequency of the gravitational wave) of length d . It is a well-known result¹⁰ that in a TT frame the spring responds to a gravitational plane wave with an oscillation of amplitude $\sim hd$ if the spring is transverse to the direction of propagation. Thus, the dipole moment has a time-dependent component $dp/dt \sim \omega h q d$. From classical electrodynamics the power radiated from an oscillating dipole is $\frac{1}{3} |\delta p|^2 \omega^4 \sim p^2 \omega^4 \hbar^2$. This is of the same order given

in Eq. (15). Consequently, this type of radiation must not be neglected for tightly bound dipoles lying perpendicular to the propagation vector of the gravity wave.

ASTROPHYSICAL SIGNIFICANCE

The incredibly small cross sections together with the over-all electrical neutrality of matter indicate that the processes described in this paper are of little astrophysical significance. Still, let us estimate the radiation produced by a plane wave incident on a plasma of large dimension. For our purposes a simple model of a plasma is a distribution of randomly oriented dipoles of magnitude $p = qn^{-1/3}$, where n is the number density of charged particles. This model is the easiest way to incorporate the constraint of zero net charge of the plasma. Because of random orientation the radiated power per unit volume will be roughly n times the power emitted by a single dipole,

$$\frac{dP}{dV} \approx n P_{\text{dipole}} \approx n p^2 \omega^4 \hbar^2 = q^2 \omega^4 \hbar^2 n^{1/3}.$$

Within a tenuous plasma, electromagnetic radiation with a frequency less than the plasma frequency, $\omega_p = 4\pi n q^2 m^{-1}$ (where m is the electron mass), is exponentially damped with a penetration depth of $\delta = 1/\omega_p$.¹¹ Consequently, we expect only the outer layer δ of the plasma to emit the gravitationally induced radiation. Therefore, the induced flux is on the order of

$$F_{\text{induced}} \sim \frac{dP}{dV} \delta \sim \omega^4 q \hbar^2 n^{-1/6} m^{1/2}.$$

The ratio of this to the incident flux F_{GW} of gravity waves is

$$R \equiv F_{\text{induced}}/F_{\text{GW}} \sim q \omega^2 \hbar^{-1/6} m^{1/2}.$$

If we choose the typical values of $m = \text{electron mass}$, $q = \text{electron charge}$, $\omega = 10^5 \text{ sec}^{-1}$, and $n = 10^{12}/\text{cm}^3$, then $R \sim 10^{-96}$. If the plasma is very hot the electrons may be relativistic and consequently emit a rather broad band spectrum of radiation while the heavier positive charges will emit at the primary frequency. Then the two components of the plasma act independently and one would expect coherent radiation to occur for regions of dimension $\lambda^2 \delta$ on the outer layer of the plasma (where λ denotes the radiation wavelength). The power emitted by such a region is, from Eq. (10),

$$P \sim (n \delta / \omega^2)^2 \omega^2 q^2 \hbar^2,$$

which results in a flux of roughly $F \sim mn \hbar^2$. The ratio of this to the flux of gravity waves is $R \sim mn / \omega^2$. For the values of m , n , and ω chosen above $R \sim 10^{-26}$. This ratio is still too small to

be interesting; furthermore, the expected frequency of gravity waves is so low that the electromagnetic radiation produced would have trouble propagating through interstellar space and certainly would never penetrate the ionosphere of the earth. Highly relativistic particles, $\gamma \gg 1$, as demonstrated, would raise the frequency of the induced radiation by a factor $\sim \gamma^2$ and the radiated flux by $\sim \gamma^4$; however, it is difficult to imagine a source which would produce a sufficient density of such particles to give this process astrophysical significance.

DISCUSSION

Though apparently of little astrophysical significance, the processes treated in this paper are interesting examples of the interaction of gravitational and electromagnetic waves. Strictly speaking, a gravity wave cannot interact with a point particle. A point charge, however, has a non-vanishing quadrupole moment due to the distribution of field energy about the charge. It is therefore able to absorb energy from the gravitational wave. The subsequent reradiation of this energy as electromagnetic waves was the subject treated in this paper. Since the energy quadrupole moment of a point charge diverges we might expect an infinite amount of energy to be absorbed from the gravity wave. This agrees with the divergent cross section for electromagnetic emission calculated above.

Momentum as well as energy is transferred in this process. Although it would take a detailed calculation to determine the momentum carried away by the particle, it is quite possible that it is comparable to that of the radiation. If this is the

case the particle will experience a radiation reaction force on the order of $\hbar^2 q^2 \omega^2$. By attaching an accelerometer to the charge one could in principle detect this force. The magnitude of the response depends only on the force $\sim \hbar^2 q^2 \omega^2$, which is defined at a point, and the response time is a property of the accelerometer alone. Consequently, this measurement might be construed as a violation of the equivalence principle as it is usually stated, i.e., "no local measurement can detect the curvature of space-time." This statement is rather vague in that no specification of what constitutes a local measurement is given. Possibly one could prescribe conditions of locality which the above measurement fails to meet. A much more satisfactory resolution is to restate the equivalence principle in the only way general relativists ever use it, namely, "all the fundamental laws of physics reduce to their special-relativistic form at the origin of a local inertial frame, i.e., a frame such that $\Gamma_{\alpha\beta}^{\mu} = 0$ at the spatial origin." The phenomena dealt with in this paper are not laws of physics but rather global solutions to the laws of physics and depend on their structure in a large region of space-time. The presence of a radiation reaction force in no way affects the local structure of physical laws and does not constitute a violation of the equivalence principle.

ACKNOWLEDGMENTS

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