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Self-energy correction to the two-photon decay width in hydrogenlike atoms

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We investigate the gauge invariance of the leading logarithmic radiative correction to the two-photon decay width in hydrogenlike atoms. It is shown that an effective treatment of the correction using a Lamb-shift “potential” leads to equivalent results in both the length and velocity gauges provided all relevant correction terms are taken into account. Specifically, the relevant radiative corrections are related to the energies that enter into the propagator denominators, to the Hamiltonian, to the wave functions, and to the energy conservation condition that holds between the two photons; the form of all of these effects is different in the two gauges, but the final result is shown to be gauge invariant, as it should be. Although the actual calculation only involves integrations over nonrelativistic hydrogenic Green functions, the derivation of the leading logarithmic correction can be regarded as slightly more complex than that of other typical logarithmic terms. The dominant radiative correction to the $2S$ two-photon decay width is found to be $-2.020\,536(\alpha/\pi)(Z\alpha)^2 \ln[(Z\alpha)^{-2}]$ in units of the leading nonrelativistic expression. This result is in agreement with a length-gauge calculation [S. G. Karshenboim and V. G. Ivanov, e-print physics/9702027], where the coefficient was given as $-2.025(1)$.

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I. INTRODUCTION

The two-photon decay of the metastable $2S$ level in atomic hydrogen and hydrogenlike systems is a rather intriguing physical phenomenon; it was first investigated by Göppert-Mayer a long time ago [1,2]. The transition involving two quanta limits the lifetime of the metastable $2S$ resonance, at least for low and medium nuclear charge numbers Z . By contrast, the highly suppressed magnetic dipole transition to the $1S$ ground state has a negligible influence on the decay width [3]. In this article, we evaluate the dominant self-energy radiative correction to the two-photon process. We recall here the known leading-order result [4–7]

$$\tau^{-1} \approx \Gamma_0 = 8.229Z^6 \text{ s}^{-1} = 1.310Z^6 \text{ Hz}. \quad (1)$$

For ionized helium ($Z=2$), rather accurate experimental verifications of this result exist [8–10]. Due to its metastability, the $2S$ level in hydrogenlike systems is one of the most accurately defined resonances found in nature. Indeed, it is this very property—the small natural linewidth—which has made possible the high-resolution two-photon spectroscopy of the $1S-2S$ transition [11–16].

The fully relativistic quantum electrodynamic formalism is intricate when applied to bound-state problems [17–20], but it is often possible to gain a rather good understanding of quantum electrodynamic (QED) radiative corrections to a particular process if one uses a simplified, nonrelativistic QED (NRQED) Lagrangian that contains effective operators which then lead to the perturbations that have to be evaluated (see, e.g., [21,22]). Of course, the main difficulty of any bound-state calculation, which is the separation of the two energy scales (scale of binding energy and the energy/mass scale of the free particles), persists in the effective approach. It is necessary to also specify cutoff prescriptions; the artificially introduced scale-separation parameters then cancel at

the end of the calculation [21–24]. An elucidating discussion of the latter point can be found in [25] (Chap. 123) and in [26] (Chap. 11.4, p. 493).

Within nonrelativistic quantum electrodynamics (also referred to as NRQED; see [19,20]), one has the choice between two different forms of the interaction Hamiltonian: the “length” (Yennie) and the “velocity” (Coulomb) gauges. There are certain intriguing issues involved with the gauge invariance in the dynamical nonrelativistic atom-light interaction. Indeed, in order to prove gauge invariance for dynamical processes, it is in many cases necessary to carefully consider the gauge transformation of the atomic wave function in addition to the transformation of the fields. Otherwise, non-gauge-invariant results are obtained off resonance [27–29]. In the current situation of radiative corrections to the two-photon decay width, we will show that it is possible to ignore the transformation of the wave function: the two-photon decay width, including the radiative corrections, is invariant under a “hybrid” gauge transformation [28] which involves only the fields, but ignores the gauge transformation of the wave function. In general, the choice of the gauge and the interpretation of physical operators have to be considered very carefully in time-dependent problems (see [30], p. 268 and Refs. [27–29]).

The gauge invariance of the two-photon decay rate and of the radiative corrections to this effect can be regarded as slightly problematic, partly because the integration over the photon energy is restricted to a finite interval. By contrast, the gauge invariance of the low-energy part of the one-loop self-energy shift, in an effective NRQED treatment, holds only because one may drop terms whose divergence, for large photon frequency, is stronger than logarithmic (this is explained in Sec. 3 of Ref. [23]); in this case gauge invariance would be violated over finite intervals of the virtual photon frequency. It has been one of the main motivations for the current paper to study related questions.

This article is organized as follows: In Sec. II, the leading nonrelativistic contribution to the two-photon decay rate is discussed, together with its relation to the NRQED two-photon self-energy. In Sec. III, the leading logarithmic radiative correction to the two-photon decay rate is formulated; the discussion is based on a perturbation with an effective potential. Explicit expressions are derived in Secs. IV and V for the length and velocity gauges, respectively. Gauge invariance is proved in Sec. VI. Numerical results are presented in Sec. VII. Conclusions are drawn in Sec. VIII. All derivations are presented in some detail, for the sake of transparency.

II. LEADING-ORDER TWO-PHOTON DECAY RATE

The decay width of a bound system may be understood naturally as the imaginary part of the self energy [31]. Indeed, the (negative) imaginary part of the self-energy is just $\Gamma/2$, where Γ is the decay width. We discuss the derivation of the two-photon width based on this concept, within nonrelativistic quantum electrodynamics [32].

The formulation of the two-loop self-energy problem within the context of NRQED has been discussed in [33]. We denote by p^j the Cartesian components of the momentum operator $p = -i\nabla$. The expression for the two-loop self-energy shift reads [33,34]

$$\begin{aligned} \Delta E_{\text{NRQED}} = & - \left(\frac{2\alpha}{3\pi m^2} \right)^2 \int_0^{\epsilon_1} d\omega_1 \omega_1 \int_0^{\epsilon_2} d\omega_2 \omega_2 \left\{ \left\langle p^i \frac{1}{H-E+\omega_1} p^j \frac{1}{H-E+\omega_1+\omega_2} p^i \frac{1}{H-E+\omega_2} p^j \right\rangle \right. \\ & + \frac{1}{2} \left\langle p^i \frac{1}{H-E+\omega_1} p^j \frac{1}{H-E+\omega_1+\omega_2} p^j \frac{1}{H-E+\omega_1} p^i \right\rangle + \frac{1}{2} \left\langle p^i \frac{1}{H-E+\omega_2} p^j \frac{1}{H-E+\omega_1+\omega_2} p^j \frac{1}{H-E+\omega_2} p^i \right\rangle \\ & + \left\langle p^i \frac{1}{H-E+\omega_1} p^i \left(\frac{1}{H-E} \right)' p^j \frac{1}{H-E+\omega_2} p^j \right\rangle - \frac{1}{2} \left\langle p^i \frac{1}{H-E+\omega_1} p^i \right\rangle \left\langle p^i \left(\frac{1}{H-E+\omega_2} \right)^2 p^i \right\rangle - \frac{1}{2} \left\langle p^i \frac{1}{H-E+\omega_2} p^i \right\rangle \\ & \times \left\langle p^j \left(\frac{1}{H-E+\omega_1} \right)^2 p^j \right\rangle - m \left\langle p^i \frac{1}{H-E+\omega_1} \frac{1}{H-E+\omega_2} p^i \right\rangle - \frac{m}{\omega_1+\omega_2} \left\langle p^i \frac{1}{H-E+\omega_2} p^i \right\rangle \\ & \left. - \frac{m}{\omega_1+\omega_2} \left\langle p^i \frac{1}{H-E+\omega_1} p^i \right\rangle \right\}. \end{aligned} \quad (2)$$

All of the matrix elements are evaluated on the reference state $|\varphi\rangle$, for which the nonrelativistic Schrödinger wave function is employed. The expression for the two-photon decay width [Eq. (4) below] now follows in a natural way as the imaginary part generated by the sum of the first three terms in curly brackets in Eq. (2). Specifically, the poles are generated upon ω_2 integration by the propagator

$$\frac{1}{H-E+\omega_1+\omega_2} = \sum_{\varphi'} \frac{|\varphi'\rangle\langle\varphi'|}{E'-E+\omega_1+\omega_2} \quad (3)$$

at $\omega_2 = E - E' - \omega_1$. Alternatively, this condition may be expressed as $E - E' = \omega_1 + \omega_2$ and represents the energy conservation condition for the two-photon decay. The imaginary part generated by the first three terms in curly brackets of the energy shift (2) is thus seen to yield the two-photon decay width [35].

In view of the above discussion and in agreement with Shapiro and Breit (Ref. [4], Eq. (3)), the nonrelativistic expression for the two-photon decay width Γ_0 in the case $|\varphi\rangle = |2S\rangle$ and $|\varphi'\rangle = |1S\rangle$ reads

$$\begin{aligned} \Gamma_0 = & \frac{4}{27} \frac{\alpha^2}{\pi} \int_0^{\omega_{\max}} d\omega_1 \omega_1^3 \omega_2^3 \left| \left\langle \varphi' \left| x^i \frac{1}{H-E+\omega_2} x^i \right| \varphi \right\rangle \right. \\ & \left. + \left\langle \varphi' \left| x^i \frac{1}{H-E+\omega_1} x^i \right| \varphi \right\rangle \right|^2, \end{aligned} \quad (4)$$

where $\omega_2 = \omega_{\max} - \omega_1$ and $\omega_{\max} = E - E'$ is the maximum energy that any of the two photons may have. When comparing this expression to Eq. (2) of [36], it should be noted that the quantity y *ibid.* represents a scaled photon energy. The Einstein summation convention is used throughout this article. Note the identity [27,37]

$$\begin{aligned} & \left\langle \varphi' \left| \frac{p^i}{m} \frac{1}{H-E+\omega_1} \frac{p^i}{m} \right| \varphi \right\rangle + \left\langle \varphi' \left| \frac{p^i}{m} \frac{1}{H-E+\omega_2} \frac{p^i}{m} \right| \varphi \right\rangle \\ & = -\omega_1 \omega_2 m^2 \left\{ \left\langle \varphi' \left| x^i \frac{1}{H-E+\omega_1} x^i \right| \varphi \right\rangle \right. \\ & \left. + \left\langle \varphi' \left| x^i \frac{1}{H-E+\omega_2} x^i \right| \varphi \right\rangle \right\}, \end{aligned} \quad (5)$$

which is valid at exact resonance $\omega_1 + \omega_2 = E - E'$. This identity permits a reformulation of the problem in the velocity-gauge as opposed to the length-gauge form.

III. RADIATIVE CORRECTIONS

We consider a hydrogenlike atom and employ natural units with $\hbar = \epsilon_0 = c = 1$. In order to analyze the radiative correction to the two-photon decay width, one could write down all Feynman diagrams which contribute to the process and start evaluating them. However, a much more economical understanding into the problem can be gained by considering an approach inspired by effective field theory or nonrelativistic quantum electrodynamics [19,20], in which the leading effect due to radiative photons is described by an effective Lamb-shift potential [38,39]

$$\delta V_{\text{Lamb}} = \frac{4}{3} \alpha (Z\alpha) \ln[(Z\alpha)^{-2}] \frac{\delta^{(3)}(r)}{m^2}. \quad (6)$$

In this work we will consider a ‘‘standard normalized perturbative local potential’’ [40]

$$\delta V = \frac{\pi(Z\alpha)}{m^2} \delta^{(3)}(r). \quad (7)$$

which is related to δV_{Lamb} by a simple prefactor:

$$\delta V_{\text{Lamb}} = \frac{4}{3} \frac{\alpha}{\pi} \ln[(Z\alpha)^{-2}] \delta V. \quad (8)$$

The corrections to the Hamiltonian, to the energy, and to the wave function, incurred by the perturbative potential (7), read as follows:

$$E \rightarrow E + \delta E, \quad (9a)$$

$$\delta E = \langle \varphi | \delta V | \varphi \rangle, \quad (9b)$$

$$H \rightarrow H + \delta V, \quad (9c)$$

$$|\varphi\rangle \rightarrow |\varphi\rangle + |\delta\varphi\rangle, \quad (9d)$$

$$|\delta\varphi\rangle = \left(\frac{1}{E - H} \right)' \delta V |\varphi\rangle \quad (9e)$$

The standard potential (7) leads to a ‘‘normalized’’ energy shift with unit prefactors:

$$\delta E(nS) = \frac{(Z\alpha)^4 m}{n^3}. \quad (10)$$

IV. LENGTH GAUGE

According to Eq. (4), the two-photon decay rate Γ_0 of the metastable $2S$ state is given by

$$\frac{\Gamma_0}{A} = \int_0^{\omega_{\max}} d\omega_1 \omega_1^3 \omega_2^3 \zeta^2, \quad (11)$$

where we use the definition

$$A = \frac{4}{27} \frac{\alpha^2}{\pi}, \quad (12)$$

as well as $\omega_2 \equiv E_{2S} - E_{1S} - \omega_1$ and $\omega_{\max} \equiv E_{2S} - E_{1S}$. The quantity ζ is given by

$$\zeta = \zeta_1 + \zeta_2, \quad (13a)$$

$$\zeta_1 = \left\langle 1S \left| x^i \frac{1}{H - E_{2S} + \omega_1} x^i \right| 2S \right\rangle, \quad (13b)$$

$$\zeta_2 = \left\langle 1S \left| x^i \frac{1}{H - E_{1S} - \omega_1} x^i \right| 2S \right\rangle. \quad (13c)$$

The perturbation (9) leads to the following replacements, which include the first-order corrections to the various quantities that are relevant to the $2S$ decay width:

$$E_{1S} \rightarrow E_{1S} + \delta E_{1S}, \quad \delta E_{1S} = \langle 1S | \delta V | 1S \rangle,$$

$$E_{2S} \rightarrow E_{2S} + \delta E_{2S}, \quad \delta E_{2S} = \langle 2S | \delta V | 2S \rangle,$$

$$|1S\rangle \rightarrow |1S\rangle + \left(\frac{1}{E_{1S} - H} \right)' \delta V |1S\rangle,$$

$$|2S\rangle \rightarrow |2S\rangle + \left(\frac{1}{E_{2S} - H} \right)' \delta V |2S\rangle,$$

$$\omega_2 \rightarrow \omega_2 + \delta\omega_2, \quad \delta\omega_2 = \delta E_{2S} - \delta E_{1S}. \quad (14)$$

The latter correction ensures that a perturbed energy conservation condition is fulfilled:

$$\omega_1 + \omega_2 + \delta\omega_2 = E_{2S} - E_{1S} + (\delta E_{2S} - \delta E_{1S}), \quad (15)$$

i.e., that the two-photon frequencies add up to the perturbed transition frequency.

The first-order self-energy correction $\delta\Gamma$ to the two-photon decay rate may be expressed as

$$\frac{\delta\Gamma}{B} = 2 \int_0^{\omega_{\max}} d\omega_1 \omega_1^3 \omega_2^3 \zeta \delta\zeta + 3 \delta\omega_2 \int_0^{\omega_{\max}} d\omega_1 \omega_1^3 \omega_2^2 \zeta^2, \quad (16)$$

where the correction $\delta\zeta$ is the sum of six terms,

$$\delta\zeta = \sum_{j=1}^6 \delta\zeta_j, \quad (17)$$

to be defined as follows, and the second term on the right-hand side of Eq. (16) is due to perturbed energy conservation condition. The quantity B may be inferred from Eqs. (4), (7), and (8) as

$$B = \frac{16}{81} \frac{\alpha^3}{\pi^2} \ln[(Z\alpha)^{-2}]. \quad (18)$$

The terms $\delta\zeta_1$ and $\delta\zeta_2$ are related to energy perturbations to the matrix elements,

$$\delta\zeta_1 = \left\langle 1S \left| x^i \left(\frac{1}{H - E_{2S} + \omega_1} \right)^2 x^i \right| 2S \right\rangle \langle 2S | \delta V | 2S \rangle, \quad \Gamma_0 = \Gamma'_0 \quad (19a)$$

$$\delta\zeta_2 = \langle 1S | \delta V | 1S \rangle \left\langle 1S \left| x^i \left(\frac{1}{H - E_{1S} - \omega_1} \right)^2 x^i \right| 2S \right\rangle, \quad \xi = -\omega_1 \omega_2 \zeta. \quad (19b)$$

whereas the terms $\delta\zeta_{3,4,5,6}$ are perturbations to the initial and final-state wave functions:

$$\delta\zeta_3 = \left\langle 1S \left| x^i \frac{1}{H - E_{2S} + \omega_1} x^i \left(\frac{1}{E_{2S} - H} \right)' \delta V \right| 2S \right\rangle, \quad (19c)$$

$$\delta\zeta_4 = \left\langle 1S \left| x^i \frac{1}{H - E_{1S} - \omega_1} x^i \left(\frac{1}{E_{2S} - H} \right)' \delta V \right| 2S \right\rangle, \quad (19d)$$

$$\delta\zeta_5 = \left\langle 1S \left| \delta V \left(\frac{1}{E_{1S} - H} \right)' x^i \frac{1}{H - E_{2S} + \omega_1} x^i \right| 2S \right\rangle, \quad (19e)$$

$$\delta\zeta_6 = \left\langle 1S \left| \delta V \left(\frac{1}{E_{1S} - H} \right)' x^i \frac{1}{H - E_{1S} - \omega_1} x^i \right| 2S \right\rangle. \quad (19f)$$

V. VELOCITY GAUGE

We now discuss the evaluation of radiative corrections in the velocity gauge, where the interaction Hamiltonian is given by

$$H'_{\text{int}} = -e \frac{p \cdot A}{m} + e^2 \frac{A^2}{2m^2}. \quad (20)$$

According to Eqs. (4) and (5), the leading-order decay rate in the velocity gauge is

$$\frac{\Gamma'_0}{A} = \int_0^{\omega_{\max}} d\omega_1 \omega_1 \omega_2 \xi^2, \quad (21)$$

where A is defined in Eq. (12), $\omega_2 \equiv E_{2S} - E_{1S} - \omega_1$ and $\omega_{\max} \equiv E_{2S} - E_{1S}$. The quantity ξ is the sum of two terms:

$$\xi = \xi_1 + \xi_2, \quad (22a)$$

where

$$\xi_1 = \left\langle 1S \left| \frac{p^i}{m} \frac{1}{H - E_{2S} + \omega_1} \frac{p^i}{m} \right| 2S \right\rangle, \quad (22b)$$

$$\xi_2 = \left\langle 1S \left| \frac{p^i}{m} \frac{1}{H - E_{1S} - \omega_1} \frac{p^i}{m} \right| 2S \right\rangle.$$

Gauge invariance of the leading-order decay rate [see Eqs. (11) and (21)]

immediately follows from Eq. (5); this equation may be rewritten in a compact form as

$$\xi = -\omega_1 \omega_2 \zeta. \quad (24)$$

Equation (24) may be proven easily by repeated application of the commutator relation(s)

$$\frac{p^i}{m} = i[H - E_{2S} + \omega_1, x^i] = i[H - E_{1S} - \omega_1, x^i]. \quad (25)$$

Now the first-order correction to the two-photon decay rate, in the velocity gauge, is

$$\frac{\delta\Gamma'}{B} = 2 \int_0^{\omega_{\max}} d\omega_1 \omega_1 \omega_2 \xi \delta\xi + \delta\omega_2 \int_0^{\omega_{\max}} d\omega_1 \omega_1 \xi^2, \quad (26)$$

where the prime denotes the velocity-gauge form of the correction and B is defined in Eq. (18). We desire to show that $\delta\Gamma = \delta\Gamma'$.

The correction $\delta\xi$ finds a natural representation as the sum of eight terms:

$$\delta\xi = \sum_{j=1}^8 \delta\xi_j. \quad (27)$$

In analogy to Eqs. (19a) and (19b), $\delta\xi_1$ and $\delta\xi_2$ are energy perturbations:

$$\delta\xi_1 = \left\langle 1S \left| \frac{p^i}{m} \left(\frac{1}{H - E_{2S} + \omega_1} \right)^2 \frac{p^i}{m} \right| 2S \right\rangle \langle 2S | \delta V | 2S \rangle, \quad (28a)$$

$$\delta\xi_2 = \langle 1S | \delta V | 1S \rangle \left\langle 1S \left| \frac{p^i}{m} \left(\frac{1}{H - E_{1S} - \omega_1} \right)^2 \frac{p^i}{m} \right| 2S \right\rangle. \quad (28b)$$

The terms $\delta\xi_{3,4,5,6}$ are perturbations to the initial and final-state wave functions:

$$\delta\xi_3 = \left\langle 1S \left| \frac{p^i}{m} \frac{1}{H - E_{2S} + \omega_1} \frac{p^i}{m} \left(\frac{1}{E_{2S} - H} \right)' \delta V \right| 2S \right\rangle, \quad (28c)$$

$$\delta\xi_4 = \left\langle 1S \left| \frac{p^i}{m} \frac{1}{H - E_{1S} - \omega_1} \frac{p^i}{m} \left(\frac{1}{E_{2S} - H} \right)' \delta V \right| 2S \right\rangle, \quad (28d)$$

$$\delta\xi_5 = \left\langle 1S \left| \delta V \left(\frac{1}{E_{1S} - H} \right)' \frac{p^i}{m} \frac{1}{H - E_{2S} + \omega_1} \frac{p^i}{m} \right| 2S \right\rangle, \quad (28e)$$

$$\delta\xi_6 = \left\langle 1S \left| \delta V \left(\frac{1}{E_{1S} - H} \right)' \frac{p^i}{m} \frac{1}{H - E_{1S} - \omega_1} \frac{p^i}{m} \right| 2S \right\rangle. \quad (28f)$$

Finally, $\delta\xi_{7,8}$ are due to the seagull term:

$$\delta\xi_7 = -\frac{3}{m} \left\langle 1S \left| \left(\frac{1}{E_{2S} - H} \right)' \delta V \right| 2S \right\rangle, \quad (28g)$$

$$\delta\xi_8 = -\frac{3}{m} \left\langle 1S \left| \delta V \left(\frac{1}{E_{1S} - H} \right)' \right| 2S \right\rangle. \quad (28h)$$

VI. PROOF OF GAUGE INVARIANCE

Here, we merely present the results of the analysis carried out in detail in the Appendix. Indeed, using Eqs. (A1a)–(A1f), as well as Eqs. (A2a) and (A2b), we obtain the compact relation

$$\delta\xi = -\omega_1\omega_2\delta\zeta - \delta\omega_2\omega_1\zeta. \quad (29)$$

In view of this relation, we can rewrite Eqs. (16) and (26) using Eq. (29):

$$\begin{aligned} \frac{\delta\Gamma'}{B} &= 2 \int_0^{\omega_{\max}} d\omega_1 \omega_1 \omega_2 \xi \delta\xi + \delta\omega_2 \int_0^{\omega_{\max}} d\omega_1 \omega_1 \xi^2 \\ &= 2 \int_0^{\omega_{\max}} d\omega_1 \omega_1 \omega_2 (-\omega_1 \omega_2 \zeta) [-\omega_1 \omega_2 \delta\zeta - \delta\omega_2 \omega_1 \zeta] \\ &\quad + \delta\omega_2 \int_0^{\omega_{\max}} d\omega_1 \omega_1^3 \omega_2^2 \zeta^2 \\ &= 2 \int_0^{\omega_{\max}} d\omega_1 \omega_1^3 \omega_2^3 \zeta \delta\zeta + (2+1) \delta\omega_2 \int_0^{\omega_{\max}} d\omega_1 \omega_1^3 \omega_2^2 \zeta^2 \\ &= \frac{\delta\Gamma}{B}. \end{aligned} \quad (30)$$

This proves the gauge invariance $\delta\Gamma = \delta\Gamma'$ of the logarithmic radiative corrections to the two-photon decay rate of the metastable $2S$ state in hydrogenlike systems. The gauge invariance of the leading-order decay rate ($\Gamma_0 = \Gamma'_0$) has been indicated in Eq. (23).

VII. NUMERICAL RESULTS

Leading order. We recall that, according to Eq. (4), the well-known leading-order nonrelativistic effect Γ_0 is of the order of $\alpha^2(Z\alpha)^6$. The result for the two-photon decay width of the metastable $2S$ state is

$$\Gamma_0 = 0.001\,318\,222\alpha^2(Z\alpha)^6 m. \quad (31)$$

This translates into

$$\Gamma_0 = 8.229\,351\,997Z^6 \text{ s}^{-1} \quad (32a)$$

$$= 1.309\,742\,049Z^6 \text{ Hz}. \quad (32b)$$

Radiative correction. In view of Eqs. (7) and (8), the leading

logarithmic radiative correction $\delta\Gamma$ is of the order of

$$\delta\Gamma \sim \alpha^3(Z\alpha)^8 \ln[(Z\alpha)^{-2}]m, \quad (33)$$

i.e., of relative order $\alpha(Z\alpha)^2 \ln[(Z\alpha)^{-2}]$ with respect to Γ_0 . In the length gauge, the relevant expression for $\delta\Gamma$ can be found in Eq. (16). [For clarity, we would like to indicate that the correction $\delta\omega_2$ occurring in the expression (16) is defined in Eq. (14), the quantity ζ can be found in Eq. (13), and the terms $\delta\zeta_i$ ($i=1, \dots, 6$) are defined in Eq. (19).] In the velocity gauge, the relevant expression for $\delta\Gamma'$ can be found in Eq. (26), with the $\delta\xi_i$ ($i=1, \dots, 8$) being defined in Eq. (28).

According to Eqs. (16) and (26), both $\delta\Gamma$ as well as $\delta\Gamma'$ find a natural representation as the sum of two terms, the first of which summarizes the perturbations to the matrix elements and the second is a consequence of the perturbed energy conservation condition for the transition. Gauge invariance $\delta\Gamma = \delta\Gamma'$ has been shown in Sec. VI, yet it is instructive to observe that there are indeed considerable cancellations among the two contributions to $\delta\Gamma$ and $\delta\Gamma'$. Specifically, we have from the first and second terms on the right-hand sides Eqs. (16) and (26), respectively,

$$\frac{\delta\Gamma}{\Gamma_0} = (29.542 - 31.562) \frac{\alpha}{\pi} (Z\alpha)^2 \ln[(Z\alpha)^{-2}], \quad (34)$$

$$\frac{\delta\Gamma'}{\Gamma'_0} = (8.500 - 10.521) \frac{\alpha}{\pi} (Z\alpha)^2 \ln[(Z\alpha)^{-2}]. \quad (35)$$

(The cancellations appear to be typical for radiative corrections to decay rates; this has recently been observed in connection with radiative corrections to the *one*-photon decay of P states [41].)

The final result for the leading logarithmic correction to the decay width of the metastable $2S$ state is

$$\frac{\delta\Gamma}{\Gamma_0} = \frac{\delta\Gamma'}{\Gamma'_0} = -2.020\,536 \frac{\alpha}{\pi} (Z\alpha)^2 \ln[(Z\alpha)^{-2}]. \quad (36)$$

The calculation of $\delta\zeta$ and $\delta\xi$ involves expressions analogous to those encountered in [40]. In [36], the coefficient has been given as $-2.025(1)$, which is in agreement with the current calculation. [There is a misprint in the overall sign of the correction as given in the abstract of [36]; one should follow the sign indicated in Eq. (8) *ibid.*]

The result (36), converted to hertz and/or inverse seconds, reads

$$\delta\Gamma = -3.273 \times 10^{-7} Z^8 \ln[137^2 Z^{-2}] \text{ Hz} \quad (37a)$$

$$= -2.057 \times 10^{-6} Z^8 \ln[137^2 Z^{-2}] \text{ s}^{-1}. \quad (37b)$$

For low Z , the highly suppressed $M1$ one-photon decay $2S \rightarrow 1S$ is numerically smaller than the radiative correction (37) to the two-photon decay (see Refs. [42–46]) because it lacks the large logarithm:

$$\Gamma_{M1} = 2.496 \times 10^{-6} Z^{10} \text{ s}^{-1}. \quad (38)$$

All results indicated in this article for Γ and $\delta\Gamma$ relate to the metastable $2S$ state; however, the approach may easily be

generalized to the two-photon decay of other states.

VIII. CONCLUSIONS

In the current investigation, the derivation of the leading radiative correction to the two-photon decay width of the metastable $2S$ state in hydrogenlike atoms has been based on the effective “radiative potential” (6) discussed in Sec. III. It has been shown that the gauge invariance of the corrections holds due to the interplay of corrections to the transition matrix elements on the one hand and corrections due to perturbed energy conservation conditions on the other hand [first and second terms on the right-hand sides of Eqs. (16) and (26), respectively]. The corrections to the transition matrix elements are again divided into corrections to the wave function (these were referred to as the d terms in [36]) and to the energies that enter into the propagator denominators, which were termed f in the length-gauge calculation [36]. The length- and velocity-gauge forms of the correction are discussed in Secs. IV and V. The gauge invariance of the radiative correction holds (even) on the level of the effective treatment as implied by the radiative potential (6), as shown in Sec. VI. All derivations are presented in some detail, for clarity and transparency. The numerical evaluation in Sec. VII follows immediately.

There are two more results of the current paper, probably of rather minor importance, which should only briefly be mentioned: first of all, the relativistic result (without radiative corrections) for the decay rate at $Z=1$ has previously been indicated as 8.229 s^{-1} [47–50], whereas in [4], the (non-relativistic) result has been indicated as $8.226 \pm 0.001 \text{ s}^{-1}$. The current investigation [Eq. (32)] confirms that the discrepancy has been due to a certain overestimation of the numerical accuracy in the early nonrelativistic calculation [4], not due to a conceivable large relativistic shift. Second, the discussion in Sec. II clarifies that the concept of a decay width as an imaginary part of a self-energy [31] generalizes to the two-loop self-energy shift, in which case the imaginary part gives rise to the two-photon decay width.

The leading-order nonrelativistic contribution to the two-photon decay width is of the order of $\alpha^2(Z\alpha)^6 mc^2$ (see Sec. II). The self-energy radiative correction to the two-photon decay is of the order of $\alpha^3(Z\alpha)^8 \ln[(Z\alpha)^{-2}] mc^2$, as discussed in Sec. VII, with explicit results indicated in Eqs. (36) and (37). It would be interesting to evaluate also the constant term of relative order $\alpha(Z\alpha)^2$. This term supplements the logarithm evaluated here which is of relative order $\alpha(Z\alpha)^2 \ln[(Z\alpha)^{-2}]$. According to our experience, in bound-state calculations, the nonlogarithmic, constant term has an opposite sign as compared to the leading logarithm, and its magnitude is 2–3 times larger than the coefficient of the logarithm. This is true for radiative corrections [41] as well as Lamb-shift effects [51,52].

One should note a rather general interest in various intriguing details related to the two-photon decay process, which are not restricted to the search for conceivable parity admixtures to the $2S$ state (see, e.g., [53,54]). Although accurate measurements of integrated decay rates are difficult [55], there is some hope that in low- Z and middle- Z ionic

systems, experiments will eventually profit from the possibilities offered by electron-beam ion traps, especially when combined with conceivable x-ray lasers that could be used in order to excite the trapped ions into the metastable states.

Finally, we recall that accurate measurements of the two-photon decay width test the $2S$ state for parity-violating $2P$ admixtures and can therefore be used as a test for a conceivable electron or nuclear (electric) dipole moment or for interactions via “anapole” or “pseudocharge” currents [56–58]. One particularly interesting investigation on hydrogenlike Ar^{17+} , with an elucidating discussion of the issues related to parity admixtures, has been given in [59].

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APPENDIX: RELATIONS AMONG MATRIX ELEMENTS

In this appendix, we present in detail the relations needed for the proof of the identity (29). For $\delta\xi_1$ as defined in Eq. (28a), we have

$$\begin{aligned} & \left\langle 1S \left| \frac{p^i}{m} \left(\frac{1}{H - E_{2S} + \omega_1} \right)^2 \frac{p^i}{m} \right| 2S \right\rangle \\ &= -\omega_1 \omega_2 \left\langle 1S \left| x^i \left(\frac{1}{H - E_{2S} + \omega_1} \right)^2 x^i \right| 2S \right\rangle + (\omega_2 - \omega_1) \\ & \quad \times \left\langle 1S \left| x^i \frac{1}{H - E_{1S} + \omega_1} x^i \right| 2S \right\rangle + \langle 1S | x^i x^i | 2S \rangle. \end{aligned} \quad (\text{A1a})$$

We notice the term $\delta\xi_1$ emerge on the right-hand side [see Eq. (19a)]. The corresponding relation for $\delta\xi_2$ reads

$$\begin{aligned} & \left\langle 1S \left| \frac{p^i}{m} \left(\frac{1}{H - E_{1S} - \omega_1} \right)^2 \frac{p^i}{m} \right| 2S \right\rangle \\ &= -\omega_1 \omega_2 \left\langle 1S \left| x^i \left(\frac{1}{H - E_{1S} - \omega_1} \right)^2 x^i \right| 2S \right\rangle + \omega_1 - \omega_2 \\ & \quad \times \left\langle 1S \left| x^i \frac{1}{H - E_{1S} - \omega_1} x^i \right| 2S \right\rangle + \langle 1S | x^i x^i | 2S \rangle. \end{aligned} \quad (\text{A1b})$$

For $\delta\xi_3$, the following relation is useful:

$$\begin{aligned}
 & \left\langle 1S \left| \frac{p^i}{mH - E_{2S} + \omega_1} \frac{p^i}{m} \left(\frac{1}{E_{2S} - H} \right)' \delta V \right| 2S \right\rangle \\
 &= -\omega_1 \omega_2 \left\langle 1S \left| x^i \frac{1}{H - E_{2S} + \omega_1} x^i \left(\frac{1}{E_{2S} - H} \right)' \delta V \right| 2S \right\rangle \\
 & - \omega_2 \left\langle 1S \left| x^i \frac{1}{H - E_{2S} + \omega_1} x^i \right| 2S \right\rangle \langle 2S | \delta V | 2S \rangle \\
 & + \left\langle 1S \left| x^i (H - E_{2S} + \omega_2) x^i \left(\frac{1}{E_{2S} - H} \right)' \delta V \right| 2S \right\rangle \\
 & - \langle 1S | x^i x^i | 2S \rangle \langle 2S | \delta V | 2S \rangle. \tag{A1c}
 \end{aligned}$$

For $\delta\xi_4$, we have

$$\begin{aligned}
 & \left\langle 1S \left| \frac{p^i}{mH - E_{1S} - \omega_1} \frac{p^i}{m} \left(\frac{1}{E_{2S} - H} \right)' \delta V \right| 2S \right\rangle \\
 &= -\omega_1 \omega_2 \left\langle 1S \left| x^i \frac{1}{H - E_{1S} - \omega_1} x^i \left(\frac{1}{E_{2S} - H} \right)' \delta V \right| 2S \right\rangle \\
 & - \omega_1 \left\langle 1S \left| x^i \frac{1}{H - E_{1S} - \omega_1} x^i \right| 2S \right\rangle \langle 2S | \delta V | 2S \rangle \\
 & + \left\langle 1S \left| x^i (H - E_{1S} - \omega_2) x^i \left(\frac{1}{E_{2S} - H} \right)' \delta V \right| 2S \right\rangle \\
 & - \langle 1S | x^i x^i | 2S \rangle \langle 2S | \delta V | 2S \rangle. \tag{A1d}
 \end{aligned}$$

The term $\delta\xi_5$ may be reformulated according to

$$\begin{aligned}
 & \left\langle 1S \left| \delta V \left(\frac{1}{E_{1S} - H} \right)' \frac{p^i}{mH - E_{2S} + \omega_1} \frac{p^i}{m} \right| 2S \right\rangle \\
 &= -\omega_1 \omega_2 \left\langle 1S \left| \delta V \left(\frac{1}{E_{1S} - H} \right)' x^i \frac{1}{H - E_{2S} + \omega_1} x^i \right| 2S \right\rangle \\
 & + \omega_1 \langle 1S | \delta V | 1S \rangle \left\langle 1S \left| x^i \frac{1}{H - E_{1S} - \omega_1} x^i \right| 2S \right\rangle \\
 & + \left\langle 1S \left| \delta V \left(\frac{1}{E_{1S} - H} \right)' x^i (H - E_{1S} - \omega_1) x^i \right| 2S \right\rangle
 \end{aligned}$$

$$- \langle 1S | \delta V | 1S \rangle \langle 1S | x^i x^i | 2S \rangle. \tag{A1e}$$

Finally, we have, for $\delta\xi_6$,

$$\begin{aligned}
 & \left\langle 1S \left| \delta V \left(\frac{1}{E_{1S} - H} \right)' \frac{p^i}{mH - E_{1S} - \omega_1} \frac{p^i}{m} \right| 2S \right\rangle \\
 &= -\omega_1 \omega_2 \left\langle 1S \left| \delta V \left(\frac{1}{E_{1S} - H} \right)' x^i \frac{1}{H - E_{1S} - \omega_1} x^i \right| 2S \right\rangle \\
 & + \omega_2 \langle 1S | \delta V | 1S \rangle \left\langle 1S \left| x^i \frac{1}{H - E_{1S} - \omega_1} x^i \right| 2S \right\rangle \\
 & + \left\langle 1S \left| \delta V \left(\frac{1}{E_{1S} - H} \right)' x^i (H - E_{2S} + \omega_1) x^i \right| 2S \right\rangle \\
 & - \langle 1S | \delta V | 1S \rangle \langle 1S | x^i x^i | 2S \rangle. \tag{A1f}
 \end{aligned}$$

However, the relations (A1a)–(A1f) are not yet sufficient in order to proceed with the proof of gauge invariance. We also need

$$\begin{aligned}
 & \left\langle 1S \left| x^i [(H - E_{1S}) + (H - E_{2S})] x^i \left(\frac{1}{E_{2S} - H} \right)' \delta V \right| 2S \right\rangle \\
 &= \frac{3}{m} \left\langle 1S \left| \left(\frac{1}{E_{2S} - H} \right)' \delta V \right| 2S \right\rangle + \langle 1S | x^i x^i | 2S \rangle \langle 2S | \delta V | 2S \rangle, \tag{A2a}
 \end{aligned}$$

$$\begin{aligned}
 & \left\langle 1S \left| \delta V \left(\frac{1}{E_{1S} - H} \right)' x^i [(H - E_{1S}) + (H - E_{2S})] x^i \right| 2S \right\rangle \\
 &= \frac{3}{m} \left\langle 1S \left| \delta V \left(\frac{1}{E_{1S} - H} \right)' \right| 2S \right\rangle + \langle 1S | \delta V | 1S \rangle \langle 1S | x^i x^i | 2S \rangle. \tag{A2b}
 \end{aligned}$$

We notice that (the negative of) the seagull terms (28g) and (28h) emerge.

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result by a factor of 2π . This result may alternatively be obtained by dividing Γ —i.e., the imaginary part of the energy—by \hbar , not h . The general paradigm is that in order to evaluate an energy in units of Hz, one should use the relation $E = h\nu$, whereas for a conversion of an imaginary part of an energy to the inverse lifetime, one should use $\Gamma = \hbar \tau^{-1}$. As calculated in Refs. [4,6], the width of the metastable $2S$ state in atomic hydrogenlike systems is $8.229Z^6 s^{-1}$ (inverse seconds). At $Z=1$, this is equivalent to the “famous” value of 1.3 Hz which is nowadays most frequently quoted in the literature.

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