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John A. Burns

John R. Singler
Missouri University of Science and Technology, singlerj@mst.edu

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Feedback Control of Low Dimensional Models of Transition to Turbulence

John A. Burns John Singler

Center for Optimal Design and Control
Interdisciplinary Center for Applied Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061-0531

Abstract— **The problem of controlling or delaying transition to turbulence in shear flows has been the subject of numerous papers over the past twenty years. This period has seen the development of several low dimensional models for parallel shear flows in an attempt to explain the failure of classical linear hydrodynamic stability theory to correctly predict transition. In recent years, ideas from robust control theory have been employed to attack this problem. In this paper we use these models to develop a scenario for transition that employs both classical bifurcation theory and robust control theory. In addition, we present numerical results to illustrate the ideas and to show how feedback can be used to delay transition. We close with a specific conjecture and discuss some previous results along this line.**

I. Introduction and Problem Formulation

During the past decade we have seen enormous advances in the development of new approaches to the problem of transition to turbulence. Although there is no single mathematical framework that describes transition to turbulence for all possible flows, new approaches to (non-classical) linear hydrodynamic stability theory have provided improvements in the fundamental understanding of this process. This new linear theory replaces eigenvalue analysis with pseudo-spectrum and uses ideas from robust control theory to deal with system sensitivity and uncertainty. In fact, one of the most important potential applications of these new approaches is to the problem of designing feedback flow controllers.

In the late 1980's and early 1990's Henningson, Reddy, Schmid, Trefethen and co-workers began to develop a new approach to hydrodynamic stability that is based on a linear theory, but differs from classical linear hydrodynamic stability in that pseudo-spectrum plays the central role in their work. The observation that linearization about a nontrivial laminar flow leads to a non-normal problem is the key to this theory. The references [2], [17], [18], [19], [20] and [27] provide the foundations for this work and the recent book by Schmid and Henningson [24] provides an excellent and modern treatment of this area. Much of this work focuses on the idea that small initial conditions can produce large transient growth due to the non-normality of the linear part of the equations and eventually the nonlinear terms become important. The exact role (other than the mixing property) that the nonlinearity plays in producing transition has not been clarified. Motivated by flow control problems, Bamieh, Dahleh, Farrell and Ioannou (see [4], [5], [6], [13], [14]) and others focused on the linear response to small random forcing at the boundary as a mechanism for transition. This effort is important because it also suggests that boundary control has the potential to significantly delay or eliminate transition in a wide variety of shear flows. Almost all of this work focuses on linear input-output theory and again the nonlinearity is not fully investigated.

One reason the nonlinearity is relegated to a minor role in the mostly linear theory is that the nonlinear term F is conservative, i.e. $\langle F(z), z\rangle = 0$ where $\langle\cdot,\cdot\rangle$ is an energy inner product on an appropriate state space. Thus, the nonlinear term conserves energy and it is argued that the response to the non-normal linearized system dominates in determining the onset of transition. In this short note we discuss this issue and use some low dimensional model problems to illustrate how the nonlinear term can greatly impact system sensitivity, transition and control design. During the past ten years several low dimensional models have been proposed to illustrate the ideas and to test the scenarios that come from this linear analysis (see [3], [7], [16], [21], [28], [29], [30]). Paper [3] by Baggett and Trefethen provides an excellent comparison of these models. We focus on low dimensional models that are known to exhibit robustness problems. These models provide some insight into the role that the nonlinear term plays in the mechanism that leads to transition. Also, we illustrate that feedback may be used to control a fully developed flow.

A. The Motivating Flow Control Problem

Consider the incompressible Navier-Stokes equations defined on a channel $\Omega = R \times (0,1) \times R$ by

$$\frac{\partial}{\partial t}\overrightarrow{u}(t) + (\overrightarrow{u}(t)\cdot\nabla)\overrightarrow{u}(t) = -\nabla p(t) + \frac{1}{\mathrm{Re}}\Delta\overrightarrow{u}(t), \quad (1)$$

$$div\,\overrightarrow{u}(t) = 0, \quad (2)$$

where $\overrightarrow{u}(t) = [u(t,x,y,z), v(t,x,y,z), w(t,x,y,z)]^T$ and $(x,y,z) \in \Omega$. Let $\overrightarrow{U}(x,y,z) = [U(y),0,0]^T$ be a laminar flow with stream-wise (x-direction) velocity $u(y)$ varying only in the cross-stream direction (y-direction) and define $\widetilde{u}$

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by

$$\overrightarrow{u} = \overrightarrow{U} + \widetilde{u}.$$

The fluctuation equations for $\widetilde{u}$ are given by

$$\frac{\partial}{\partial t}\widetilde{u} + (\widetilde{u}\cdot\nabla)\widetilde{u} = -\nabla\widetilde{p} + \frac{1}{\mathrm{Re}}\Delta\widetilde{u} - (\overrightarrow{U}\cdot\nabla)\widetilde{u} - (\widetilde{u}\cdot\nabla)\overrightarrow{U}$$

and the linearized equations become

$$\frac{\partial}{\partial t}\widetilde{u} = -\nabla\widetilde{p} + \frac{1}{\mathrm{Re}}\Delta\widetilde{u} - (\overrightarrow{U}\cdot\nabla)\widetilde{u} - (\widetilde{u}\cdot\nabla)\overrightarrow{U}.$$

Representing the wall normal velocity v and wall normal vorticity ω in terms of Fourier modes in the streamwise x-direction and spanwise z-direction, the linearized equations may be written in operator form

$$\frac{d}{dt}\begin{bmatrix} \widehat{\omega} \\ \widehat{v} \end{bmatrix} = A(\mathrm{Re})\begin{bmatrix} \widehat{\omega} \\ \widehat{v} \end{bmatrix},$$

where

$$A(\mathrm{Re}) = \begin{bmatrix} L_{sq} & L_c \\ 0 & L_{os} \end{bmatrix},$$

with L_{sq}, L_{os} and L_c are the Squire, Orr-Sommerfeld and coupling operators, respectively (see [22] and [24]). The important point is that the operator $A(\mathrm{Re})$ is highly non-normal and has the form $A(\mathrm{Re}) = [\frac{1}{\mathrm{Re}}A_0 + \mathcal{R}]$ where A_0 is a negative definite self-adjoint differential operator and $\mathcal{R}$ is a bounded linear operator defined on an appropriate Hilbert (state) space Z.

If one applies a control on a subset Γ_c of the boundary $\Gamma = \partial\Omega$ of the channel Ω and includes the nonlinear term, then the controlled fluctuation equation has the form

$$\dot{z}(t) = [A_0(\mathrm{Re}) + \mathcal{R}]z(t) + F(z(t)) + Bu(t) + G\varepsilon,$$

where B is a unbounded linear operator and $F(\cdot)$ is a conservative non-linear function in the sense that

$$\langle F(z), z\rangle = 0 \tag{3}$$

for all $z \in Z$ (see [10] for details). Here, the operator G will also be unbounded if the "small" external constant disturbance ε is located on the boundary $\Gamma = \partial\Omega$. In order to develop practical and convergent numerical algorithms for computing feedback control laws, one should consider the non-normality of the linear operator $A(\mathrm{Re})$. Moreover, it is important to understand the role that the nonlinear term plays in the stability and robustness of the resulting closed-loop system. For example, it is known (see [1]) that such systems can be infinitely sensitive to small perturbations at the boundary. We shall focus on a specific low dimensional model of the type commonly found in the literature cited above to illustrate this sensitivity and to demonstrate how feedback control can be employed to stabilize a fully developed chaotic flow.

B. Low Dimensional Models of Parallel Shear Flows

We consider a 2D and 3D system that is typical of those found in the papers [3], [7], [16], [21], [28], [29] and [30]. However, we focus on the role that small constant disturbances play in transition and illustrate how feedback can delay or eliminate transition in these cases. Both systems have the form

$$\dot{z}(t) = A(R)z(t) + \|z(t)\|\, Sz(t) + Bu(t) + G\varepsilon, \tag{4}$$

where $A(R) = [\frac{1}{R}A_0 + \mathcal{R}]$, $A_0 < 0$ is diagonal and $S = -S^*$ is skew-adjoint. In particular, the 2 dimensional system is defined by

$$A(R) = \begin{bmatrix} -\alpha/R & 1 \\ 0 & -\beta/R \end{bmatrix}, S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

and

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{6}$$

The 3 dimensional system is defined by

$$A(R) = \begin{bmatrix} -\alpha/R & 1 & 0 \\ 0 & -\beta/R & 1 \\ 0 & 0 & -\gamma/R \end{bmatrix}, \tag{7}$$

$$S = \begin{bmatrix} 0 & -b_1 & -b_2 \\ b_1 & 0 & b_3 \\ b_2 & -b_3 & 0 \end{bmatrix} \tag{8}$$

and

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, G = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{9}$$

where all constants are positive. Both models have the property that the linear operator $A(R)$ is stable for all $R > 0$ and the 2 dimensional nonlinear model is also dissipative. In particular, the nonlinear 2 dimensional system defined by (5)-(6) has a compact global attractor. The non-linear 3 dimensional system defined by (7)-(9) is more complex, but exhibits features very similar to those one finds in Plane Couette flows.

II. Model Problems

As noted above, the problem with classical linear analysis is that it fails to predict the correct critical Reynolds number that yields transition. For plane Couette flows the linearized equations are always stable and theoretically one should not see transition if the initial flow state is sufficiently close to the Plane Couette flow. However, if one views a "small" constant disturbance as a perturbation of the **conservative nonlinear term**, then standard bifurcation theory under uncertainty yields a transition scenario which matches many flow cases. Understanding this mechanism is crucial to the development of feedback control laws. The following simple models are sufficient to illustrate the basic ideas and to demonstrate how feedback might be useful in the delaying of transition.

A. The 2 Dimensional Model

In this case we set $\alpha = 1.2$ and $\beta = 1.4$. We call the eigenvector $z_{TS} = [\ 1 \quad 0\ ]^T$ corresponding to the smallest eigenvalue $-\alpha/R$ the TS state because of the similarity to the Tollmien-Schlichting waves in plane Poiseuille flows. We refer to the vector $z_{OB} = [\ 1 \quad 1\ ]^T$ as the oblique state. Observe that $A(R)$ is stable for all $R > 0$. In addition, one can show that this 2 dimensional system has a compact global attractor (see Figures 1 and 2). If $\varepsilon = 0$, then the zero $z_0 = 0$ equilibrium is locally asymptotically stable for all R. However, the radius $\delta(R)$ of the largest ball about z_0 that lies in the domain of attraction converges to 0 and is approximately given by $\delta(R) = O(R^{-2})$. Figure 3 shows how and why the oblique initial state transitions before the TS initial state as observed in [24]. When one adds a small "uncertainty" such as an $\varepsilon = .0001$ perturbation to the nonlinear term, there is a subcritical bifurcation near $R = 6$ as illustrated in Figure 4. In this case all initial states near $z_0 = 0$ transition. In Figure 5 one sees the "tunnelling effect" observed in many flows (see [24]). Finally, Figure 6 shows that if one applies a LQR feedback control to this system, then the closed-loop system looks much like the $R = 4$ open-loop system. Here feedback delays the transition. The LQR control was computed with weighting matrices $Q = I_2$ and $r = 25$.

We turn now to the 3D system to illustrate the same transition scenarios and to investigate the application of feedback to a fully developed chaotic flow.

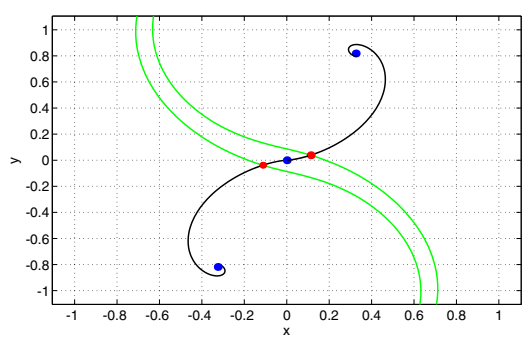

Fig. 1. Phase portrait without disturbance ($\alpha = 1.2$, $\beta = 1.4$, $R = 4$, $\varepsilon = 0$). The green lines are the stable manifold and the black lines are the unstable manifolds for the hyperbolic critical points. The union of the five equilibrium and unstable manifolds is the global attractor. The basin of attraction for the zero equilibrium lies between the stable manifolds.

B. The 3 Dimensional Model

Here we have a more complex system and, for various values of the parameter $R > 1$, this system exhibits periodic, quasi-periodic and chaotic attractors. For all the runs presented below, we set $\alpha = .5$, $\beta = .75$, $\gamma = 1.0$, $b_1 = 1$, $b_2 = .5$ and $b_3 = .25$. We denote the eigenvector $z_{TS} = [\ 1 \quad 0 \quad 0\ ]^T$ corresponding to the smallest eigenvalue $-\alpha/R$, the TS state. The vector $z_{OB} = [\ 1 \quad 1 \quad 1\ ]^T$ is called the oblique state. If $9.5 < R < 23$, then there is a

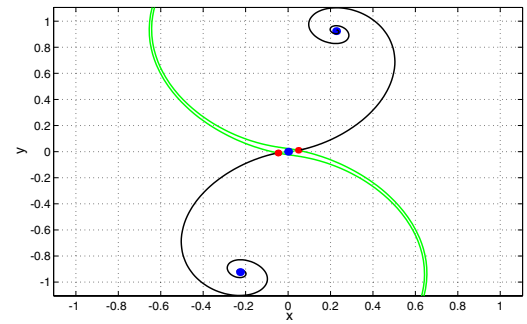

Fig. 2. Phase portrait without disturbance ($\alpha = 1.2$, $\beta = 1.4$, $R = 6$, $\varepsilon = 0$). The green lines are the stable manifold and the black lines are the unstable manifolds for the hyperbolic critical points. The union of the five equilibrium and unstable manifolds is the global attractor. The basin of attraction for the zero equilibrium lies between the stable manifolds.

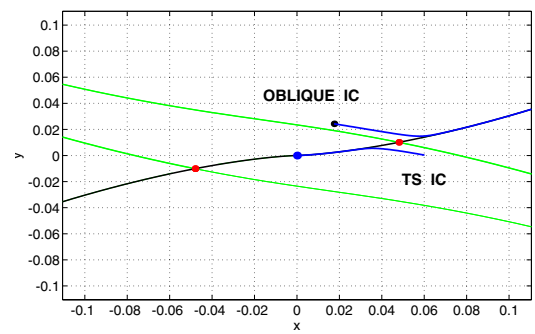

Fig. 3. Phase portrait without disturbance ($\alpha = 1.2$, $\beta = 1.4$, $R = 6$, $\varepsilon = 0$). Here we zoom in near the zero equilibrium. The green lines are the stable manifold and the black lines are the unstable manifolds for the hyperbolic critical points. The oblique initial condition transitions to the stable equilibrium while the TS initial condition remains stable. The union of the five equilibrium and unstable manifolds is the global attractor. The basin of attraction for the zero equilibrium lies between the stable manifolds.

chaotic (local) attractor and all solutions with initial states $\bar{z}$ satisfying $\|\bar{z}\| < 1$ will either approach this attractor or the zero equilibrium. All the results presented below are based on $R = 10$ and initial states $\bar{z}$ satisfying $\|\bar{z}\| = 10^{-4}$. In Figure 7 one sees that the oblique initial state z_{OB} transitions to the chaotic attractor with a transition time of approximately 50 seconds. However, Figure 8 shows that the TS initial state z_{TS} returns to the zero state. As for the 2D model, if one sets $\varepsilon = 10^{-6}$, then the TS initial state z_{TS} also transitions to the chaotic attractor. As illustrated in Figure 9 the transition time increases to approximately 100 seconds.

In order to test the feedback control, we computed a LQR controller and used a "capturing" algorithm that turns on the control only if $t > 150$ and the trajectory "wanders" into the domain of attraction for the closed-loop system. A version of this method was suggested Yorke and co-workers in the papers [25] and [26].

Remark It is interesting to note that even this "simple" 3D model problem is more complex than it might first seem. For example, it not obvious that this system (for the given parameters) is dissipative. Although there is numerical

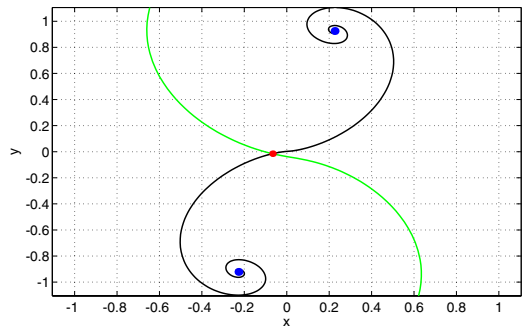

Fig. 4. Phase portrait with disturbance ($\alpha = 1.2$, $\beta = 1.4$, $R = 6$, $\varepsilon = 0.0001$). The disturbance of size $\varepsilon = 0.0001$ produces a subcritical bifurcation and there are only three critical points. The green lines are the stable manifold and the black lines are the unstable manifolds for the single hyperbolic critical point. The union of the three equilibrium and the unstable manifolds is the global attractor.

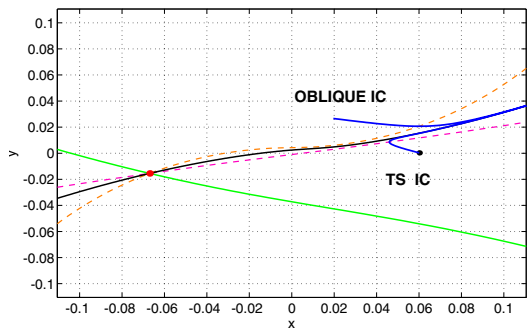

Fig. 5. Phase portrait with disturbance ($\alpha = 1.2$, $\beta = 1.4$, $R = 6$, $\varepsilon = 0.0001$). Here we zoom in near the zero equilibrium. The dashed lines are the nullclines and the disturbance of size $\varepsilon = 0.0001$ produces a subcritical bifurcation. There are only three critical points. The green lines are the stable manifold and the black lines are the unstable manifolds for the single hyperbolic critical point. The union of the three equilibrium and the unstable manifolds is the global attractor. However, all initial states above the stable manifold transition to the distant stable equilibrium.

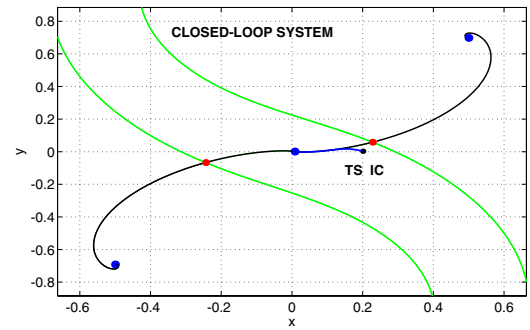

Fig. 6. Phase portrait with disturbance ($\alpha = 1.2$, $\beta = 1.4$, $R = 6$, $\varepsilon = 0.0001$) with a LQR feedback controller. The disturbance of size $\varepsilon = 0.0001$ no longer produces a subcritical bifurcation and again there are five critical points. The green lines are the stable manifold and the black lines are the unstable manifolds for the two hyperbolic critical points. The union of the three equilibrium and the unstable manifolds is the global attractor. The basin of attraction for the zero equilibrium lies between the stable manifolds and is much greater than the open loop system with no disturbance.

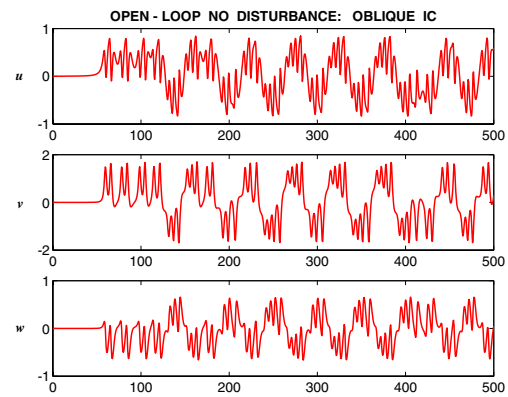

Fig. 7. Solutions of the 3D system ($\alpha = .5$, $\beta = .75$, $\gamma = 1.0$, $R = 10$, $\varepsilon = 0.0$) with oblique initial data of norm $\|z_0\| = 10^{-4}$. This initial data transitions to a chaotic attractor.

evidence that suggests there is an chaotic attractor inside the ellipsoid

$$ELIP(\rho) = \left\{(u, v, w) : u^2 + v^2/4 + w^2 < \rho\right\}$$

for for $\rho = 1.02$, we do not have a rigorous proof to justify this statement. On the other hand, if $\delta > 0$ and the non-linear term is modified by adding a dissipative term of the form

$$F_\delta(z) = F(z) - \delta z^p$$

where

$$z^p \triangleq \begin{bmatrix} u^p & v^p & w^p \end{bmatrix}^T$$

and p is an odd integer greater than 1, then it is easy to prove that the perturbed dynamical system is dissipative and there exist a global chaotic attractor.

To illustrate the impact of feedback for the case here, we wait until the flow is fully chaotic ($t > 150$ for both initial states) and then only turn on the feedback control law when $\|z(t)\| < 1$. The weighting matrices for the LQR problem were $Q = I_3$ and $r = 1$. Figures 10, 11 and 12 show the closed-loop responses, a comparison of open-loop and closed-loop energies and the control history, respectively. Here, the feedback control with capturing stabilizes the fully developed flow.

Note it is the perturbation of the conservative nonlinear term that provides the transition mechanism. Even a small perturbation to the condition

$$\langle F(z), z\rangle = 0$$

can produce changes in the bifurcation diagrams such as shown in Figures 13 and 14. In particular, if $\varepsilon \neq 0$ then the perturbed nonlinear term becomes

$$F_\varepsilon(z) = F(z) + G\varepsilon$$

so that

$$\langle F_\varepsilon(z), z\rangle = \langle F(z), z\rangle + \langle G\varepsilon, z\rangle = \langle G\varepsilon, z\rangle = \varepsilon(u + v + w)$$

is no longer conservative and not definite.

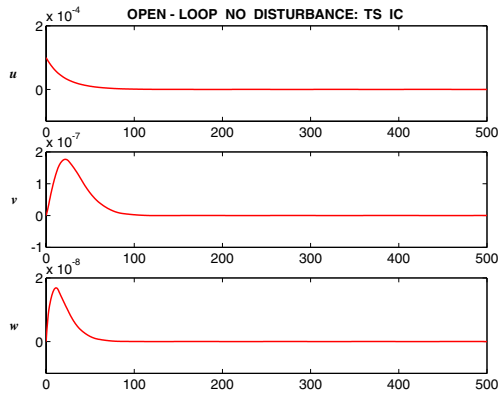

Fig. 8. Solutions of the 3D system ($\alpha = .5$, $\beta = .75$, $\gamma = 1.0$, $R = 10$, $\varepsilon = 0.0$) and the TS initial data. If the TS initial data has norm $\|z_0\| = 10^{-4}$, then there is no transition.

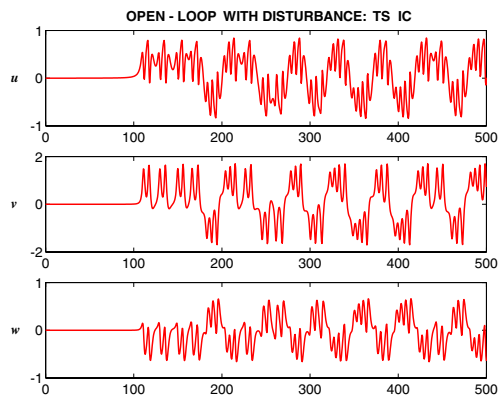

Fig. 9. Solutions of the 3D system with constant disturbance ($\alpha = .5$, $\beta = .75$, $\gamma = 1.0$, $R = 10$, $\varepsilon = 10^{-6}$), The initial state is given by the TS initial data with norm $\|z_0\| = 10^{-4}$. There is a subcritical transition to a chaotic attractor.

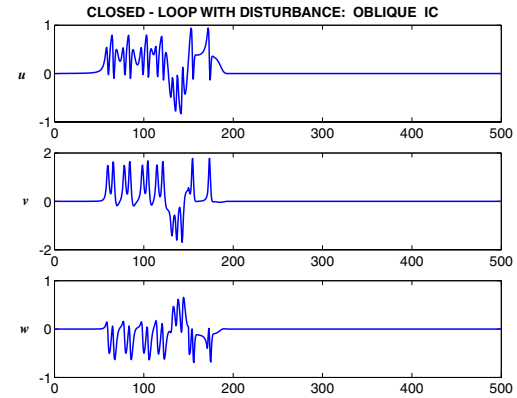

Fig. 10. Solutions of the closed-loop 3D system with constant disturbance ($\alpha = .5$, $\beta = .75$, $\gamma = 1.0$, $R = 10$, $\varepsilon = 10^{-6}$). The initial state is given by the oblique initial data with norm $\|z_0\| = 10^{-4}$. The capturing feedback control law is turned on at $t = 150$ and the fully developed flow is stabilized by $t = 190$ seconds.

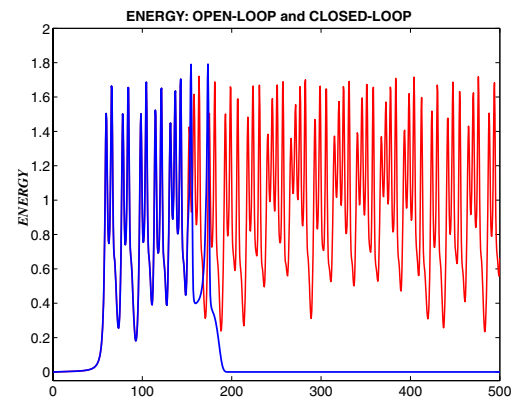

Fig. 11. Open-loop and closed-loop energy for the 3D system with constant disturbance ($\alpha = .5$, $\beta = .75$, $\gamma = 1.0$, $R = 10$, $\varepsilon = 10^{-6}$). The initial state is given by the oblique initial data with norm $\|z_0\| = 10^{-4}$. The red line is open-loop energy and the blue line is closed-loop energy.

III. Conclusions

The two models considered here have mathematical structures and features common to many shear flow control problems. The examples above clearly show that it might be possible to develop a rigorous theoretical framework to explain some transition scenarios as a subcritical "bifurcation under uncertainly". The linear part of such non-normal systems is extremely important in understanding sensitivity and control design.

Clearly these model problems do not provide anything closely resembling a theoretical foundation for infinite dimensional flows. However, the examples do provide insight in to such problems. The papers [11], [12] and [22] provide more realistic applications of similar control ideas to turbulent boundary layers. The book [15] by Gad el Hak is a valuable source of flow control applications. Also, the papers [8], [9] provide examples where infinite dimensional theory can be applied to such systems. Moreover, in view of recent rigorous resolvent estimates for plane Couette flows (see [23] it is reasonable to conjecture that a similar analysis of the nonlinearity might be successful for this infinite dimensional system.

Finally, it is interesting to note that providing rigorous proofs that these "simple" models are dissipative is **not simple**. In fact, the existence of a chaotic attractor is clearly dependent on the choice of the parameters in the mixing matrix S above.

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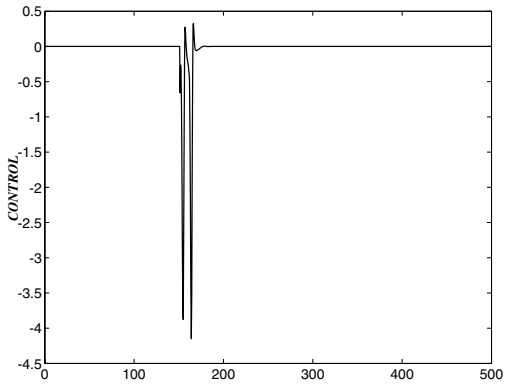

Fig. 12. Optimal LQR control for the 3D system with constant disturbance ($\alpha = .5$, $\beta = .75$, $\gamma = 1.0$, $R = 10$, $\varepsilon = 10^{-6}$). The initial state is given by the oblique initial data with norm $\|z_0\| = 10^{-4}$.

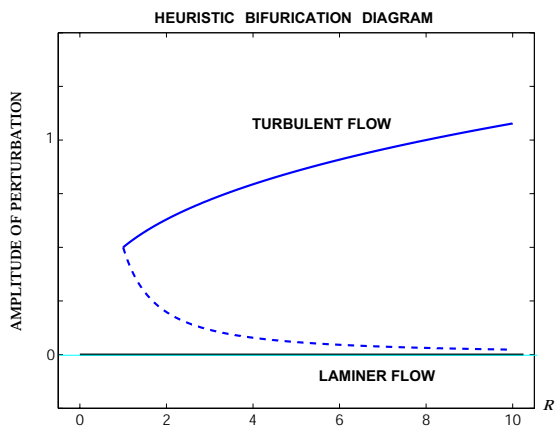

Fig. 13. Heuristic bifurcation diagram for low dimensional models. The laminar flow is stable for all $R > 0$ but the stability radius decays to 0 as $R \longrightarrow +\infty$. An initial state must be above the dashed blue line to transition.

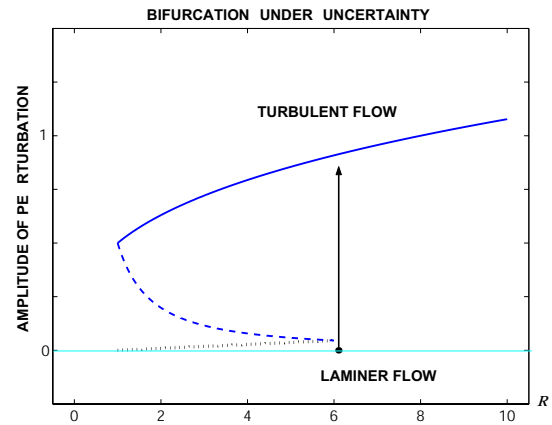

Fig. 14. A bifurcation under uncertainty. The small constant disturbance produces a non-conservative nonlinear term which leads to a subcritical bifurcation. The laminar flow state is no longer an equilibrium for $R > R_{crit}$ and transition occurs.

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