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Model Reduction of Linear PDE Systems: A Continuous Time Eigensystem Realization Algorithm

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Abstract—The Eigensystem Realization Algorithm (ERA) is a well known system identification and model reduction algorithm for discrete time systems. Recently, Ma, Ahuja, and Rowley (Theoret. Comput. Fluid Dyn. 25(1) : 233-247, 2011) showed that ERA is theoretically equivalent to the balanced POD algorithm for model reduction of discrete time systems. We propose an ERA for model reduction of continuous time linear partial differential equation systems. The algorithm differs from other existing approaches as it is based on a direct approximation of the Hankel integral operator of the system. We show that the algorithm produces accurate balanced reduced order models for an example PDE system.

I. INTRODUCTION

Model reduction is important for many applications, including design and real time implementation of feedback control laws for complex systems. Balanced truncation is a standard model reduction procedure for linear systems of ordinary differential equations [1], [2]. For large-scale systems, such as those arising from a spatial discretization of a partial differential equation (PDE) system, standard model reduction algorithms are no longer applicable.

Recent research has focused on the development and analysis of balanced truncation model reduction algorithms for large-scale systems. Many researchers are developing new computational linear algebra techniques based on algorithms for approximate solutions of large-scale matrix Lyapunov equations and other approaches; see [3]–[5] and the references therein. Another recent trend has been to use snapshots of simulation data to construct approximate balanced truncated reduced order models. Rowley proposed the balanced POD algorithm for large-scale matrix systems in [6]. The algorithm is related to earlier balanced truncation algorithms proposed by Willcox and Peraire [7] and Lall, Marsden, and Glavaški [8]. The balanced POD algorithm is related to the method of snapshots for standard POD computations [9], however there are now two separate datasets.

Recently, Ma, Ahuja, and Rowley showed [10] that the balanced POD algorithm for finite dimensional *discrete time* linear systems is theoretically identical to the Eigensystem Realization Algorithm (ERA), a system identification method (and also a model reduction algorithm) proposed by Juang and Pappa in [11] (see also [12]). In [10], ERA is compared to discrete time balanced POD for balanced model reduction of a discrete time, spatially discretized, linearized fluid flow PDE system. The authors note that ERA and balanced POD

produce very similar reduced order models, and that ERA requires much less computational cost. On the other hand, the authors discuss that balanced POD produces balancing modes which can be used for model reduction of nonlinear systems, and also balanced POD can be extended to treat unstable systems; ERA is unable to produce the balancing modes without computational cost similar to that of balanced POD, and it is unknown if ERA can directly reduce unstable systems (see [13] for indirect approaches). See [10] for more details and references.

We note that the ERA has been primarily used for identifying matrices in a discrete time model given certain experimental data generated by the actual physical system. In contrast, Ma, Ahuja, and Rowley in [10] used a large scale model to generate specific simulation data and employ the ERA to obtain a low order model.

In this work, we propose an Eigensystem Realization Algorithm for approximate balanced truncation model reduction of *continuous time* linear systems. We consider a general class of PDE systems. Our approach is based on approximating the Hankel operator and exact expressions for the balanced truncation of the continuous time linear system. We do not use a transformation from the discrete time to the continuous time systems.

The ERA has been considered for system identification of continuous time systems [14], [15], e.g., by using a system transformation from discrete time to continuous time. We do not consider the system identification problem in which we have (possibly noisy) experimental data and need to produce a model of the physical system. Instead, we assume we have a linear continuous time (PDE or large-scale) system and can generate specific simulation data.

II. A MODEL PROBLEM

Before we develop the algorithm for a general PDE system, we consider a simple example: a one dimensional convection diffusion equation

$$w_t(t, x) = \mu w_{xx}(t, x) - \kappa w_x(t, x) + b(x)u(t),$$

with Dirichlet boundary conditions

$$w(t, 0) = 0, \quad w(t, 1) = 0,$$

and initial condition

$$w(0, x) = w_0(x).$$

System measurements are taken of the form

$$y(t) = \int_0^1 c(x)w(t, x) dx.$$

We assume the functions $b(x)$ and $c(x)$ are piecewise constant; specifically, $b(x) = b$ when $b_1 < x < b_2$ and $c(x) = c$ when $c_1 < x < c_2$, and both functions are zero otherwise.

This problem can be formulated as a continuous time (infinite dimensional) linear system as follows. Let X be the Hilbert space $L^2(0, 1)$ of square integrable functions defined on the interval $(0, 1)$ with standard inner product $(f, g) = \int_0^1 f(x)g(x) dx$ and norm $\|f\| = (f, f)^{1/2}$. Define the convection diffusion operator $A : D(A) \subset X \rightarrow X$ by

$$[Aw](x) = \mu w_{xx}(x) - \kappa w_x(x),$$

Functions in $D(A)$ are twice differentiable and satisfy the above boundary conditions. Define $B : \mathbb{R} \rightarrow X$ and $C : X \rightarrow \mathbb{R}$ by $[Bu](x) = b(x)u$ and $Cw = (w, c)$. In this way, the PDE system can be written as the infinite dimensional system

$$\dot{w}(t) = Aw(t) + Bu(t), \quad w(0) = w_0, \quad y(t) = Cw(t),$$

where the dot denotes a time derivative. We note that many other linear PDE systems can be written in a similar form.

We chose this simple example problem so that we can easily check for convergence of the balanced truncated reduced order model. As we discuss below, the error between the original system and the reduced model is measured using the transfer functions. For this convection diffusion system, the transfer function can be found exactly and is given by the infinite series (compare Proposition 4.1 in [16])

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \sum_{n=1}^{\infty} \frac{bc}{s - \lambda_n} \int_{c_1}^{c_2} f_n(x) dx \int_{b_1}^{b_2} g_n(x) dx, \end{aligned}$$

where

$$\begin{aligned} \lambda_n &= -\mu n^2 \pi^2 - \kappa^2 / 4\mu, \\ f_n(x) &= \sqrt{2} e^{\kappa x / 2\mu} \sin(n\pi x), \\ g_n(x) &= \sqrt{2} e^{-\kappa x / 2\mu} \sin(n\pi x). \end{aligned}$$

III. REPRESENTATIONS OF THE HANKEL OPERATOR AND THE BALANCED REALIZATION

A. Assumptions and Notation

To begin, we consider the general framework of well posed linear systems [17]. Later, we restrict our attention to more specific classes of infinite dimensional systems in order to guarantee that the balanced truncation model reduction problem has a solution.

Let X be a real separable Hilbert space with inner product (\cdot, \cdot) and corresponding norm $\|\cdot\| = (\cdot, \cdot)^{1/2}$. We assume the operator $A : D(A) \subset X \rightarrow X$ generates an exponentially stable C_0 -semigroup e^{At} over X . Let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|A^{-1}x\|$, and let $X_1 = D(A)$ with the graph norm $\|x\|_1 = \|Ax\|$. We have the continuous dense inclusions $X_1 \subset X \subset X_{-1}$

and the semigroup restricts to a semigroup on X_1 and has an extension on X_{-1} . We still denote the restriction and extension by e^{At} .

Assume the operators $B : \mathbb{R}^m \rightarrow X_{-1}$ and $C : X_1 \rightarrow \mathbb{R}^p$ are both bounded. If $B \in \mathcal{L}(\mathbb{R}^m, X)$ and $C \in \mathcal{L}(X, \mathbb{R}^p)$, then B and C are called bounded; otherwise, they are called unbounded.

Furthermore, assume B and C are admissible control and observation operators for e^{At} (see [17]), so that $e^{At}B \in \mathcal{L}(\mathbb{R}^m, X)$ and $Ce^{At} \in \mathcal{L}(X, \mathbb{R}^p)$. The system is well posed if $\|Ce^{At}\|$ and $\|e^{At}B\|$ are both locally square integrable functions over $0 < t < \infty$ [18, Theorem 1].

Define the impulse response $h \in L^1_{\text{loc}}(0, \infty; \mathbb{R}^{p \times m})$ of a well posed system by $h(t) = Ce^{At}B$.

We also require the standard Banach spaces $L^p(0, \infty; \mathbb{R}^d)$, $1 \leq p < \infty$, of all functions x with finite norm

$$\|x\|_{L^p(0, \infty; \mathbb{R}^d)} = \left(\int_0^\infty \|x(t)\|^p dt \right)^{1/p}.$$

For $p = 2$, this is a Hilbert space with inner product

$$(x, y)_{L^2(0, \infty; \mathbb{R}^d)} = \int_0^\infty (x(t), y(t)) dt,$$

where (\cdot, \cdot) and $\|\cdot\|$ denote the standard Euclidean vector inner product and norm on \mathbb{R}^d . We do not use subscripts on inner products and norms whenever the space is understood by the context.

B. Balanced Truncation

We provide a brief overview of balanced truncation for infinite dimensional systems. Details of the theory can be found in [19], [20]; see also the review in [21].

Balanced model reduction finds a low order system

$$\dot{a}(t) = A_r a(t) + B_r u(t), \quad y_r(t) = C_r a(t),$$

that is an approximation to the stable system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

in the sense that the input-to-output error is small. The error between the systems is measured by comparing the transfer functions $G(s) = \tilde{C}(sI - A)^{-1}B$ and $G_r(s) = C_r(sI_r - A_r)^{-1}B_r$ of the original and reduced systems.¹ The balanced truncated system is usually found by first balancing the system and then truncating. Specifically, there is another system (A^b, B^b, C^b) , called a balanced realization, which has the same transfer function G as the original system, and the controllability and observability Gramians Z_B^b and Z_C^b of the balanced system are equal and diagonal. The Gramians are the solutions of the Lyapunov equations

$$\begin{aligned} A^b Z_B^b + Z_B^b (A^b)^* + B^b (B^b)^* &= 0, \\ (A^b)^* Z_C^b + Z_C^b A^b + (C^b)^* C^b &= 0, \end{aligned}$$

and the diagonal entries of the Gramians, $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$, are the Hankel singular values of the system. The r state

¹The operator \tilde{C} is an appropriate extension of C [17, Theorem 5.6.5]. In certain situations, we can take C instead of \tilde{C} .

low order system (A_r, B_r, C_r) is found by truncating the balanced realization (A^b, B^b, C^b) .

The transfer function error between the original system and the balanced truncation is bounded by twice the neglected Hankel singular values:

$$\|G - G_r\|_\infty \leq 2 \sum_{k>r} \sigma_k, \quad (1)$$

where the \mathcal{H}_∞ norm is the largest singular value of the function evaluated along the imaginary axis. Therefore, if the Hankel singular values decay rapidly, then the approximation error can be made small for a small value of r .

For the error bound to be meaningful, the sum of the Hankel singular values must be finite. When A generates an exponentially stable C_0 -semigroup, there are three main classes of PDE systems (A, B, C) that are known to yield a finite sum:

- 1) The operators B and C are bounded [22]
- 2) The C_0 -semigroup e^{At} is analytic, $B \in \mathcal{L}(\mathbb{R}^m, X_\beta)$, $C \in \mathcal{L}(X_\alpha, \mathbb{R}^p)$, and $\alpha - \beta < 1$ [22], [23]
- 3) The C_0 -semigroup e^{At} is of class D^p for some $p \geq 1$, $B \in \mathcal{L}(\mathbb{R}^m, X_\beta)$, $C \in \mathcal{L}(X_\alpha, \mathbb{R}^p)$, and $p(\alpha - \beta) < 1$ [18]

The analytic case covers many parabolic PDE systems, and the D^p case covers certain second order PDE systems with weak damping. In both of these cases, the spaces X_α are the interpolation spaces associated with A and therefore the operators B and C are allowed to have some degree of unboundedness. See the above references for more information and examples.

C. The Hankel Operator and the Balanced Realization

For any h in $L^1(0, \infty; \mathbb{R}^{p \times m})$, define the Hankel operator $\mathcal{H} \in \mathcal{L}(L^2(0, \infty; \mathbb{R}^m), L^2(0, \infty; \mathbb{R}^p))$ by

$$[\mathcal{H}u](t) = \int_0^\infty h(t+s)u(s)ds. \quad (2)$$

The Hankel operator is known to be compact [24, Lemma 8.2.4] and therefore there exist singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$ (with repetitions according to multiplicity) and corresponding singular vectors $\{f_k\} \subset L^2(0, \infty; \mathbb{R}^m)$ and $\{g_k\} \subset L^2(0, \infty; \mathbb{R}^p)$ satisfying

$$\mathcal{H}f_k = \sigma_k g_k, \quad \mathcal{H}^*g_k = \sigma_k f_k.$$

The singular vectors are also orthonormal with respect to the L^2 inner product, i.e.,

$$(f_j, f_k)_{L^2(0, \infty; \mathbb{R}^m)} = \int_0^\infty f_j^T(t) f_k(t) dt = \delta_{jk},$$

$$(g_j, g_k)_{L^2(0, \infty; \mathbb{R}^p)} = \int_0^\infty g_j^T(t) g_k(t) dt = \delta_{jk}.$$

When h is the impulse response of the system (A, B, C) , i.e., $h(t) = Ce^{At}B$, the Hankel singular values and singular vectors can be used to form the balanced realization of the system. If $h \in L^2 \cap L^1$, then the Hankel singular vectors are known to be continuous for $0 \leq t \leq \infty$ and differentiable

with derivatives in L^1 [19], [20]. The balanced realization is given as follows:

Theorem 1 ([19], [20]): If $h \in L^2 \cap L^1(0, \infty; \mathbb{R}^{p \times m})$ and the Hankel singular values are distinct, then a balanced realization (A^b, B^b, C^b) is given by

$$A_{ij}^b = \left(\frac{\sigma_j}{\sigma_i}\right)^{1/2} \int_0^\infty g_i^T(t) \dot{g}_j(t) dt,$$

$$= \left(\frac{\sigma_i}{\sigma_j}\right)^{1/2} \int_0^\infty \dot{f}_i^T(t) f_j(t) dt, \quad (3)$$

$$B^b = [\sigma_1^{1/2} f_1(0), \sigma_2^{1/2} f_2(0), \dots]^T, \quad (4)$$

$$C^b = [\sigma_1^{1/2} g_1(0), \sigma_2^{1/2} g_2(0), \dots], \quad (5)$$

and A^b can also be expressed as

$$A_{ij}^b = \frac{(\sigma_i \sigma_j)^{\frac{1}{2}}}{\sigma_i^2 - \sigma_j^2} (\sigma_j f_i^T(0) f_j(0) - \sigma_i g_i^T(0) g_j(0)), \quad i \neq j, \quad (6)$$

$$A_{ii}^b = -\frac{1}{2} f_i^T(0) \dot{f}_i(0) = -\frac{1}{2} g_i^T(0) \dot{g}_i(0). \quad (7)$$

The representation (3)-(5) was proved by Curtain and Glover in [19]. A similar formula to the alternate representation of A^b given in (6)-(7) was proved by Glover, Curtain, and Partington in [20, proof of Lemma 4.3] for the output normal realization; their argument is easily modified for the standard balanced realization to give the above result.

This result has recently been extended by Guiver and Opmeer [25] to the case where h is only in L^1 . (They also prove that if the Hankel operator is nuclear, then it must take the form of the integral operator (2) with $h \in L^1$.)

IV. CONTINUOUS TIME EIGENSYSTEM REALIZATION ALGORITHM

We present a continuous time Eigensystem Realization Algorithm (ERA) that approximates the singular values and singular vectors of the Hankel operator and uses these quantities to approximate the balanced truncated realization of Theorem 1. As far as the author is aware, the results in Theorem 1 have not been previously used to approximate reduced order models of PDE systems.

Since the continuous time algorithm is similar in nature to the discrete time version, we first provide a brief overview of this algorithm. For a stable discrete time system $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k)$, the ERA for model reduction roughly proceeds as follows [11], [12]:

- 1) Approximate the impulse response (i.e., the Markov parameters $Y_k = CA^{k-1}B$) of the system.
- 2) Form the generalized Hankel matrix H consisting of the Markov parameters.
- 3) Compute the singular value decomposition of H , and use the singular values to determine the order of the reduced model.
- 4) Form the reduced order model using the singular values, singular vectors, and Markov parameters.

For the precise formulas, see the above references.

Next, we give a natural version of ERA for model reduction of continuous time systems and provide implementation details.

A. Model Reduction ERA: Continuous Time Systems

For a stable continuous time system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, we propose the following model reduction Eigensystem Realization Algorithm (ERA):

- 1) Approximate the impulse response $h(t) = Ce^{At}B$ of the system.
- 2) Approximate the Hankel singular values $\{\sigma_k\}$ and singular vectors $\{f_k, g_k\}$ of the Hankel operator.
- 3) Use the singular values to determine the order r of the reduced model.
- 4) Form the reduced order model (A_r, B_r, C_r) using the approximate Hankel singular values and singular vectors using the formulas from Theorem 1.

Theorem 1 gives two different representations of the matrix A_r . Either representation may be used in the final step of the algorithm.

The algorithm hinges on approximating the impulse response and approximating the Hankel singular values and singular vectors. We discuss these two tasks in more detail below.

B. Approximating the Impulse Response

To begin, assume B and C are bounded operators. Then B must take the form

$$Bu = \sum_{j=1}^m u_j b_j,$$

where each b_j is in X . Then $e^{At}B$ is given by

$$e^{At}Bu = \sum_{j=1}^m u_j w_j(t), \quad w_j(t) = e^{At}b_j.$$

Therefore, each w_j is the solution of the differential equation

$$\dot{w}_j(t) = Aw_j(t), \quad w_j(0) = b_j. \quad (8)$$

Finally, compute $Cw_j(t)$ to approximate the components of the impulse response. Note that the time history of $w_j(t)$ does not need to be stored when using a time stepping method to approximate the solution of the differential equations.

If the operators B and C are unbounded, then one must likely utilize the nature of the specific type of problem being considered to make such a procedure rigorous. However, this procedure does extend directly to the analytic semigroup case discussed in Section III-B due to the smoothing property of the semigroup.

C. Approximating the Hankel Singular Values and Singular Vectors

Recall that the Hankel operator can be expressed as

$$[\mathcal{H}f](t) = \int_0^\infty h(t+s)f(s)ds.$$

The Hilbert adjoint operator is given by

$$[\mathcal{H}^*g](t) = \int_0^\infty h^T(t+s)g(s)ds.$$

We note here that if the Hankel operator is nuclear (i.e., the sum of the Hankel singular values is finite), then $h(t)$ is continuous for all $t > 0$ [20, Corollary 2.1].² However, if B and/or C are unbounded, then $h(t)$ can have a singularity at $t = 0$.

We use existing techniques from the field of integral equations (see, e.g., [26]–[28]) to approximate the singular values and singular vectors of the Hankel operator. Specifically, in this work we approximate the integral with quadrature to obtain a matrix singular value decomposition problem. Many other existing numerical methods for integral equations can certainly be used.

For now we also assume h is continuous at $t = 0$. If h has a singularity at $t = 0$, then it may be beneficial to use existing special numerical methods for singular integral equations to obtain the approximations. We leave this for future work.

Let $\{\alpha_k, \tau_k\}_{k=1}^N$ be the nodes and weights of a single quadrature rule. The weights must all be positive. We take $\tau_1 = 0$ in order to directly approximate the Hankel singular vectors at $t = 0$ (which is needed to approximate the balanced truncated model). We take τ_N large enough so that $h(\tau_N)$ is nearly zero so that we essentially truncate the integral over $(0, \infty)$ to a finite interval.

Apply the quadrature rule to $\mathcal{H}f = \sigma g$ and $\mathcal{H}^*g = \sigma f$ and evaluate at the quadrature nodes to obtain the approximate equations

$$\begin{aligned} \sum_{\ell=1}^N \alpha_\ell h(\tau_k + \tau_\ell) f(\tau_\ell) &\approx \sigma g(\tau_k), \\ \sum_{\ell=1}^N \alpha_\ell h^T(\tau_k + \tau_\ell) g(\tau_\ell) &\approx \sigma f(\tau_k). \end{aligned}$$

Replace the above approximate equations by equalities and multiply the resulting equations by $\alpha_k^{1/2}$. Let $U_\ell \in \mathbb{R}^m$ and $V_\ell \in \mathbb{R}^p$ be the vectors

$$U_\ell = \alpha_\ell^{1/2} f(\tau_\ell), \quad V_\ell = \alpha_\ell^{1/2} g(\tau_\ell). \quad (9)$$

For each k and ℓ , let $\Gamma_{k\ell}$ be the $p \times m$ matrix

$$\Gamma_{k\ell} = \alpha_k^{1/2} \alpha_\ell^{1/2} h(\tau_k + \tau_\ell). \quad (10)$$

We have

$$\sum_{\ell=1}^N \Gamma_{k\ell} U_\ell = \sigma V_k, \quad \sum_{\ell=1}^N \Gamma_{k\ell}^T V_\ell = \sigma U_k.$$

Let Γ be the $pN \times mN$ block matrix with $k\ell$ block $\Gamma_{k\ell}$, i.e.,

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1N} \\ \Gamma_{21} & \Gamma_{22} & \cdots & \Gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{N1} & \Gamma_{N2} & \cdots & \Gamma_{NN} \end{bmatrix}. \quad (11)$$

Define the stacked vectors $u \in \mathbb{R}^{mN}$ and $v \in \mathbb{R}^{pN}$ by

$$u = [U_1^T, U_2^T, \dots, U_N^T]^T, \quad v := [V_1^T, V_2^T, \dots, V_N^T]^T.$$

²Technically, $h(t)$ is equal almost everywhere to a function continuous for $t > 0$.

The above equations give $\Gamma u = \sigma v$ and $\Gamma^T v = \sigma u$.

Let $\{\sigma_k, u_k, v_k\}$ be the singular values and orthonormal singular vectors of Γ . Then $\{\sigma_k\}$ approximate the Hankel singular values. The values of the k th (approximate) Hankel singular vectors f_k and g_k at the quadrature nodes can be recovered from (9) above:

$$f_k(\tau_\ell) = \alpha_\ell^{-1/2} U_\ell^k, \quad g_k(\tau_\ell) = \alpha_\ell^{-1/2} V_\ell^k,$$

where $U_\ell^k \in \mathbb{R}^m$ is the ℓ th vector block of U^k , $V_\ell^k \in \mathbb{R}^p$ is the ℓ th vector block of V^k , and the k th singular vectors are partitioned as above:

$$u_k = [(U_1^k)^T, (U_2^k)^T, \dots, (U_N^k)^T]^T, \\ v_k = [(V_1^k)^T, (V_2^k)^T, \dots, (V_N^k)^T]^T.$$

V. NUMERICAL RESULTS

For our numerical experiments with the convection diffusion model problem outlined above, we chose $\mu = 0.1$ and $\kappa = 1$. The constants for the piecewise constant functions $b(x)$ and $c(x)$ were chosen to be $b = 4$, $b_1 = 0$, $b_2 = 0.5$, $c = 2$, $c_1 = 0.5$, and $c_2 = 1$.

To approximate the solutions of the partial differential equation (8), we used standard piecewise linear finite elements for the spatial discretization with constant mesh spacing h and the trapezoid rule for the time discretization with constant time step $\Delta t = h$. The trapezoid quadrature rule was used to approximate the Hankel integral operator, and second order finite differences was used to approximate the derivatives of the Hankel singular vectors.

The approximate Hankel singular values are shown in Figure 1 for various numbers of equally spaced finite element nodes. Here, the PDE (8) was integrated in time until the magnitude of the approximate impulse response was less than 10^{-4} . The largest approximate Hankel singular values are converging as the mesh is refined, but the smaller singular values are leveling off and not converging to zero. (Recall that the exact Hankel singular values must decay to zero.) Of course, the approximate balanced reduced model is constructed using the largest approximate singular values, therefore the behavior of the smaller approximate singular values may not cause problems with the construction of the reduced model.

To look at this phenomenon more closely, the approximate Hankel singular values are shown again in Figure 2; here, we fixed the number of finite element nodes (143) and integrated the PDE (8) until the magnitude of the approximate impulse response was less than various tolerances ($10^{-3}, \dots, 10^{-6}$). Now it can be seen that the approximate Hankel singular values are converging to zero as the tolerance is decreased.

As discussed earlier, we have essentially truncated the infinite time interval to approximate the integral operator. This strategy is not generally advisable. However, since the kernel of the integral operator is not known analytically and is approximated by integrating in time, truncating the infinite interval appears to be a natural strategy. However, other strategies are possible and they may give better results.

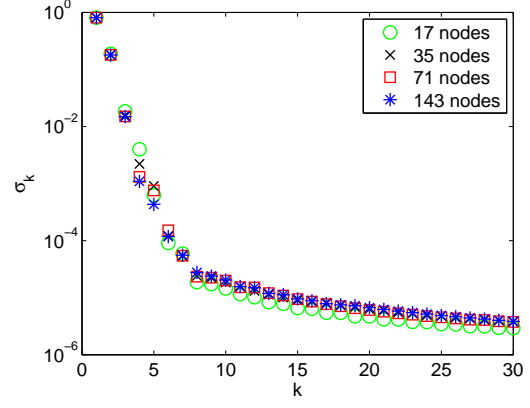


Fig. 1. Approximate Hankel singular values σ_k computed using various numbers of equally spaced finite element nodes.

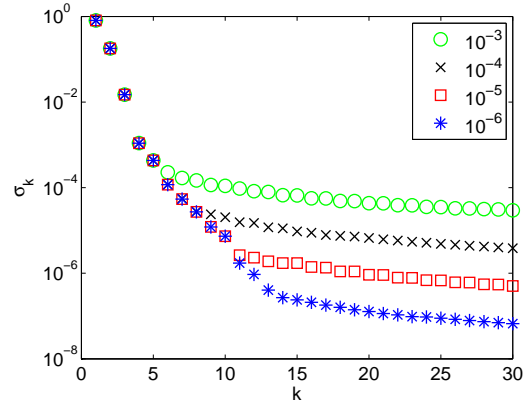


Fig. 2. Approximate Hankel singular values σ_k computed using 143 equally spaced finite element nodes and various tolerances.

We note that the approximate Hankel singular values computed using balanced POD appear to converge more quickly for this example (not shown). Furthermore, the balanced POD approximate Hankel singular values do converge to zero (or at least to machine precision), and also they are not sensitive to the truncation of the infinite integration interval.

Next, we examine the \mathcal{H}_∞ norm between the exact transfer function (computed using 160 terms in the series) and the transfer function of the reduced model. Table I gives the approximate value of the transfer function error $\|G - G_r^N\|_\infty$ with $r = 4$ and G_r^N computed by three different methods: the continuous time ERA using (6)-(7) for A_r , balanced POD using a similar quadrature approach as ERA, and a standard balanced truncation algorithm applied to matrix approximations of (A, B, C) . The standard matrix approximation algorithm is not applicable for large scale discretized systems, but we include it here for comparison purposes. The ERA gives accurate approximations to the exact transfer function and gives similar approximation errors as the balanced POD approximate transfer function. For larger N , ERA gives slightly better results. The approximation errors are also close to the matrix approximation approach.

We note that the ERA using the integral formula (3) for

TABLE I

APPROXIMATE TRANSFER FUNCTION ERRORS $\|G - G_r^N\|_\infty$ WITH $r = 4$ AND VARIOUS VALUES OF N , THE NUMBER OF FINITE ELEMENT NODES. METHODS: “cERA” IS THE ALGORITHM PROPOSED HERE; “BPOD” IS BALANCED POD; “M” IS A MATRIX APPROXIMATION APPROACH.

method	$N = 17$	$N = 35$	$N = 71$	$N = 143$
cERA	1.5×10^{-2}	4.2×10^{-3}	9.1×10^{-4}	3.1×10^{-4}
BPOD	4.6×10^{-3}	2.7×10^{-3}	2.0×10^{-3}	9.3×10^{-4}
M	4.1×10^{-3}	8.9×10^{-4}	3.3×10^{-4}	2.1×10^{-4}

A_r was less accurate than using (6)-(7) for A_r . This is not surprising since accuracy can be lost as the derivatives of the Hankel singular vectors are approximated by finite differences and the integral is also approximating by quadrature.

Increasing the value of r gives a change in the behavior of the errors of the methods; see Table II. ERA again give accurate approximations to the exact transfer function. However, the approximations are not quite as accurate as those produced by balanced POD algorithm or matrix approximations of the operators.

TABLE II

APPROXIMATE TRANSFER FUNCTION ERRORS $\|G - G_r^N\|_\infty$ WITH $r = 5$ AND VARIOUS VALUES OF N , THE NUMBER OF FINITE ELEMENT NODES.

method	$N = 17$	$N = 35$	$N = 71$	$N = 143$
cERA	1.5×10^{-2}	5.8×10^{-3}	2.3×10^{-3}	8.9×10^{-4}
BPOD	4.1×10^{-3}	1.0×10^{-3}	5.2×10^{-4}	3.2×10^{-4}
M	4.2×10^{-3}	9.1×10^{-4}	2.0×10^{-4}	7.8×10^{-5}

VI. DISCUSSION AND FUTURE WORK

We proposed an Eigensystem Realization Algorithm (ERA) for model reduction of continuous time linear partial differential equation systems. Instead of using discrete time ERA on the time discretized system, we used a quadrature approach to directly approximate the Hankel integral operator of the continuous time system. We showed that this method gives accurate balanced reduced order models for a simple example problem, however balanced POD was more accurate as the order of the reduced model was increased.

It is possible that using alternate methods from the field of integral equations to approximate the Hankel integral operator will yield better accuracy. Also, special methods may be required when the impulse response of the system has a singularity at $t = 0$ (which can occur for unbounded input and/or output operators). Furthermore, additional comparison remains to be performed between the continuous time ERA proposed here, discrete time ERA using a transformation to continuous time, and balanced POD. Convergence theory is also of interest. Moreover, the algorithm needs to be thoroughly tested on more complex multidimensional PDE systems. We leave these topics for future work.

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