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Approximation for single-channel multi-server queues and queuing networks with generally distributed inter-arrival and service times

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APPROXIMATION FOR SINGLE-CHANNEL MULTI-SERVER QUEUES AND QUEUING NETWORKS WITH GENERALLY DISTRIBUTED INTER-AARRIVAL AND SERVICE TIMES

by

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ABSTRACT

This dissertation is divided into two papers. The first paper is related to developing a closed-form approximation for single-channel multiple-server queues with generally distributed inter-arrival and service times, which are often found in numerous settings, e.g., airports and manufacturing systems. Unfortunately, exact models for such systems require distributions for the underlying random variables. Further, data for fitting distributions is sometimes not available, and one only has access to means and variances of the underlying input random variables. Under heavy traffic, excellent approximations already exist for this purpose. In the first paper, a new approximation method for medium traffic is presented. Encouraging numerical evidence for gamma distributed inter-arrival times, often found in many settings, and double-tapering distributions, such as normal, triangular, and gamma, for the service time, is found with the new approximation. In the second paper, a new approximation technique is studied for modeling a two-stage queueing network (QN) in which the first stage contains a multiple-server (G/G/k) queue and the second is composed of multiple single-server queues (G/G/1) in parallel. Airport terminals and other service areas, such as sports stadiums and manufacturing systems, are examples of systems where such two-stage QNs are encountered. The new approximation is rooted in approximating the variance of the service time in a G/G/k queue and leads to encouraging numerical behavior.

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1. INTRODUCTION

Queuing theory has attracted a great deal of research interest in the last 60 years. Applications in manufacturing, airports, supply chain management and many other areas have driven the need for mathematical closed-form models that are both accurate and easy to execute as predictive models.

The history of queuing theory starts in the early 1900s with the transition from manual to automatic telephony (Myska, 1995), when Dutch scientist Agner Krarup Erlang developed a series of measurements to evaluate a telephone circuit: traffic intensity, the B Formula, which determines the probability to lose a call, and the C Formula – also known as the D Formula for "Delay", which determines the probability that a customer has to wait in a queue. He also developed a mathematical proof of the fact that arrival processes in telephone traffic often follow a Poisson distribution. Indeed, he was keenly aware that distributions were significant in determining the performance of a queue. His ultimate goal was to determine how many servers are required in order to satisfy users without wasting resources.

Queuing theory did evolve from the work of Erlang as other numerous contributors built upon his work and created new formulas for additional applications. One of such is the Pollaczek-Khinchine formula that ties the performance measure of a queue (queue length and wait times) to the service time distribution for the M/G/1 queue (Haigh, 2002). Conny Palm made significant contributions to the field of queuing theory in the 1930s and 1940s, with work primarily focused on queuing abandonment, traffic with intensity variations and equilibrium points, where he proved that queues behave in a predictable, stable way in the long-term; this is of particular use in the study of stochastic produces and steady-state probabilities. During World War II, the field of operations research and stochastic processes, which includes queuing theory, were widely used by military planners to aid in the decision process for logistics, scheduling, and training. In 1953, David Kendall brought some standardized notation to the field of queuing theory and introduced what is now known as Kendall's notation commonly used to describe a queue: A/S/c/K/N/D, where "A" is the distribution for the inter-arrival time, "S" is the distribution for the service time, "c" is the number of servers in the queue, "K" is the capacity of the queue, "N" is the size of the population of jobs to be served, and "D" is the queuing disciple (i.e. First-In, First-Out). In 1954, John Little developed a theorem that says: in a stable system, the number of customers in a system is equal to the arrival rate multiplied by the time a customer spends in the system, or $L = \lambda W$ (Little, 1961). In 1958, the book "Queues, Inventory and Maintenance" by Phillip Morse was published and is now considered to be one of the first text books on queuing.

In 1961, John Kingman began the study of generally distributed queues with an approximation for the mean waiting time in heavy traffic (Kingman, 1961). This led to the development of approximations for multi-server queues which have general distributions (G/G/k queue) in medium traffic. Older models such as those of Marchal (1976) and Kraemer and Langenbach-Belz (1976) continue to be useful for G/G/1 queues, but there is a need for additional research on G/G/k queues in the airline, manufacturing and service industries and in queuing networks. Since then, other work, which will be covered in the literature review later in this dissertation, has occurred in the advancement of this model, but for specific cases, and there are still wide gaps in the literature.

Finally, the use of simulations is a common approach to study these systems. While advances in computing power have helped simulations remain relevant and powerful, simulation models have major drawbacks because they require distributions of the input random variables and take a significant amount of computational time.

This dissertation is presented in two papers. The first paper demonstrates a new approximation for G/G/k queues that outperforms past models for gamma distributed inter-arrival times and double-tapering distributions (e.g., normal, triangular) for service times. The second paper presents a new approximation procedure for queueing networks in which the queue in one stage is G/G/k and the set of parallel queues in the second stage are G/G/1.

PAPER

I: AN APPROXIMATION FOR MULTI-SERVER QUEUES WITH GAMMA-DISTRIBUTED INTER-ARRIVAL TIMES AND DOUBLE TAPERING SERVICE TIMES IN MEDIUM TRAFFIC

ABSTRACT

Single-channel multiple-server queues with generally distributed inter-arrival and service times (referred to as G/G/k in the literature) are found in numerous settings such as airports and manufacturing systems. Unfortunately, exact models for such systems require distributions for the underlying random variables. Often, data for fitting distributions is not available and one must determine estimates for mean waiting times and queue lengths on the basis of the means and variances of the underlying random variables. Under heavy traffic, excellent approximations already exist for this purpose. The researcher presents a new approximation method for medium traffic, which is based on an appropriate scaling of the coefficient of variation of the service time in the G/G/k queue, as well as on existing single-server approximations for G/G/1 queues from the past work of Kraemer and Langenbach-Belz and that of Marchal. This research finds numerical evidence for gamma distributed inter-arrival times, often found in many settings, and double-tapering distributions, such as normal, triangular, and gamma, for the service time.

1. INTRODUCTION

There is a significant amount of interest among engineers in developing closed-form approximations for queuing systems because one can plug in values into closed-form formulas to obtain estimates for the performance metrics of interest like mean waiting times and queue lengths. In the context of a manufacturing system, such closed-form formulas can be integrated into computerized MRP (Materials Requirements Planning) systems where the manager may be interested in estimating lead times, which in turn can help determine the optimal number of kanbans needed in a production line. Closed-form formulas are also of significant interest to airport managers seeking to optimize an airport queue because they can use such formulas to easily quantify performance measures in the airport system and plug those values into a linear programming model or some other form of optimization model. Lastly, closed-form formulas that are devoid of integrations and work with minimal assumptions have a special appeal in performance evaluation because of their ease of use and white-box nature. Exact and accurate analytical results often require integral calculus, which can be time-consuming and may require specialized software. An alternative is discrete-event simulation, which can also be time-consuming and dependent on specialized software. Also, the approximations that only require two moments, means and variances, of the input random variables are especially beneficial because they can be used with minimal amounts of data, whereas identifying distributions usually requires a significant volume of data. Further, if the approximation error from the formula is within 25% of the actual value, in practice, the estimate often suffices the need of the manager – especially when the estimate is used in combination with factors of safety, for instance, in kanban calculations (Askin and Goldberg, 2002), or in optimization and performance evaluation in an airport setting (Hafizogullari et al., 2003; Manataki and Zografos, 2009). Often, if the approximation is accurate up to the first place after the decimal point in minutes (for waiting times), it meets the demands of managers. As such, there is a need to develop formulas that work for generally distributed inter-arrival times and service times.

For a single-server, single-channel queue with generally distributed inter-arrival times and service times (*GI/G/1* or *G/G/1* queue) there are accurate approximations available (see any standard text on queuing e.g., Medhi, 2003) when traffic intensity is heavy (the server utilization is close to 1). In the case of low-traffic queues, there are some accurate approximations such that in Bloomfield and Cox (1972). In medium traffic, these heavy-traffic and low-traffic approximations are not very accurate, and yet many systems operate under medium traffic. For medium traffic, two important closed-form approximations have been developed by Kraemer and Langenbach-Belz (1976) and Marchal (1976) for the *G/G/1* queue. When multiple, identical servers exist for the single-channel queue, the analysis becomes more involved due to the process by which those approximations are computed; the process will be discussed in detail in section 2.4. The most general case is the *G/G/k* queue in which there are *k* servers and the inter-arrival and the service times have any given distribution. It turns out that the multiple-server setting is commonly experienced in a manufacturing system where there are multiple parallel machines in a flow shop or airport with multiple servers (check-in agents, TSA to check identification document (ID), etc). When the arrival process is Poisson, one has the M/G/k queue for which numerous approximations have been developed, which will be discussed later.

In this paper, the researcher considered the case of a multiple-server queue for inter-arrival times that have (i) the hump shape of gamma distributions and (ii) service times of which the probability density function tapers at both ends ("double tapering," but not necessarily symmetric), such as the normal distribution, the triangular distribution, and the gamma distribution. This case cannot be modeled via the M/G/k queue. For a large number of systems, including in the airport and the manufacturing setting, the conditions described above apply frequently. For instance, in manufacturing systems, the inter-arrival time for a job is often gamma distributed (Benjafaar et al., 2004), while the service time may have the normal distribution in case of automation, which typically leads to low variability, or the gamma distribution in case the machine is failure-prone, which leads to high variability (Das and Sarkar, 1999) . In an airport setting, empirical evidence suggests that inter-arrival times often have the gamma distribution (Khadgi, 2009). For the ID check queue, TSA security line or other service counters, one typically encounters a human server, whose service time is often modeled via the triangular distribution that approximates the beta distribution well (Johnson, 1997).

The approximation model presented in this paper is based on the single-server approximations of Kramer and Langenbach-Belz (1976) and Marchal (1976) in combination with an aggregation scheme that the researcher developed. The name of the model is *MAGGIE* (*M*ultiple-server *AGG*regation *I*ndex for *E*xpected Values). The underlying idea of the model is to use an aggregation of the servers in order to develop an indexing mechanism that transforms the multiple-server system with k servers into an equivalent, fictitious, single-server system with the same utilization, but subsequently divides the expected value of the length (or waiting time) of the fictitious queue by k to obtain an estimate for the multiple-server queue. In other words, the *k* servers are combined into one fictitious server, which has the same utilization as the original multiple-server queue, but the estimate obtained for this imaginary single-server queue is divided by *k* to obtain the appropriate estimate for the original multiple-server queue. The expected queue length of the fictitious queue is obtained from the selected single-server approximation.

MAGGIE was developed while keeping in mind the gamma distributed inter-arrival times and the double-tapering distributions described above. Most approximations in the literature for G/G/*k* or M/G/*k* systems have relied on scaling factors (Lee & Longton, 1957, Kimura, 1984; Shore, 1988) that serve as coefficients to an existing (possibly exact) formula, where the latter works for a more specific system, less general, than the one for which the approximation is proposed. For instance, approximations for M/G/*k* systems have used the formulas of M/M/*k* systems, while the same approximations for G/G/*k* systems have used the formulas for M/G/*k* systems. In the same spirit, the researcher uses the G/G/1 approximations of Kraemer and Langenbach-Belz (K-L) and Marchal (MAR) in the formulation, along with specific scaling factors dependent on the variability in the inter-arrival and service times, as well as the aggregation approach, alluded to above.

The rest of this report is organized as follows. Section 2.3 provides a detailed literature review for the problem domain. Section 2.4 shows the methodology of the research behind the MAGGIE model. Section 2.5 presents the numerical results of the mathematical model and compares them to the simulation results. Finally, Section 2.6 presents the conclusion of this research as well as directions for future research.

2. LITERATURE REVIEW

A good source for the key papers on closed-form approximations to compute the means of queue lengths (and waiting times) of G/G/k queues is the extensive review of Kimura (1994). Some other excellent sources of relevant material include Whitt (1993) and Medhi (2003), both papers do thorough analysis of single-server a multiple-server queues and the shortcomings of the existing models along with the shortcoming of their own approximations. Most of these approximations rely on the exact and well-known formulas for the mean queue length (and waiting time) for the M/M/k queue, which can be found in any standard undergraduate text on operations research (Hillier & Lieberman, 2001).

There is a body of work on heavy traffic approximations, which originated from the seminal work of Kingman (1962) that was adapted for G/G/k queues by Kimura (1996). There are also numerous other approximations for M/G/k queues: Lee and Longton (1957), Page (1982), and Kimura (1986). Psounis et al. (2005), presented a novel and significantly accurate approximation procedure for the M/G/*k* queue, along with an extensive review of approximations by other researchers for the M/G/k queue. Shore (1988) presented an approximation for the G/G/k queue that showed promise as a simple, yet accurate, model. However, his approximation requires the third and the fourth moment of the inter-arrival times and service times, while in this study the focus is on approximations based on the first and the second moments. Finally, note that “even when exact numerical procedures are available, it is helpful to have simple approximations as concise summaries” (Whitt, 1993).

Two critical papers in the area of generally-distributed inter-arrival and service times are Kraemer and Langenbach-Belz (1976) and Marchal (1976), where two different closed-form approximations for G/G/1 queues have been developed. Kraemer and Langenbach-Belz and Marchal (1985) and used their respective approximations for the G/G/1 queue in combination with the exact formula for the M/M/*k* queue to develop approximations for G/G/k queues. These M/M/*k*-based models used by Kraemer and Langenbach-Belz and Marchal (1985) will be used, in addition to simulation, to benchmark the model formulated in this paper. The empirical work this research shows that these M/M/k-based models work occasionally, but not consistently enough for the specific conditions that are focus in this paper (gamma distributed inter-arrival times and double-tapering distributions for the service times) which are encountered frequently in the real world. As such, there is a need to develop new approximations for the G/G/k queue. This paper seeks to fill this gap in the literature.

Finally, a relevant area to discuss from the literature is the use of distributions. For example, in an airport setting, the inter-arrival times often have the gamma distribution (Suryani et al. 2010) and in the ID check queue or the queue where the bags are checked in, one typically encounters a human server, whose service time is often modeled via the triangular distribution. In manufacturing systems, the inter-arrival time for a job is often gamma distributed while the service time often has the normal distribution (in case of law variability) or the gamma distribution (in case of higher variability). Finally, the research shows that the exponential distribution is a special case of the gamma distribution and thus using the exponential distribution for the inter-arrival time makes the G/G/k queue an M/G/k queue. Nonetheless, the findings in this work are *not* applicable for the exponentially distributed inter-arrival time, but rather for the case where the distribution for the inter-arrival time shows the classical hump seen in the typical gamma distribution. As alluded to above, for the M/G/k queue, a significant body of literature already exists.

Contributions of this paper: The focus in this paper is on (i) G/G/k queues, rather than M/G/k queues, (ii) medium traffic, where heavy traffic approximations do not perform well, (iii) gamma-distributed inter-arrival times and (iv) service times with one of the following distributions: triangular, normal or gamma. For a large number of systems, including airport and manufacturing settings, the conditions described above frequently apply.

3. METHODOLOGY AND MATHEMATICAL MODEL

In this section, the researcher will describe some basic queuing notation, the existing conventional models (and their attributes), and afterwards, the researcher will introduce MAGGIE, the queuing model that is the main contribution of this research.

3.1. BASIC QUEUING THEORY NOTATION

To ensure consistent communication through this paper, let us begin with some notation:

k: Number of servers in the queue

λ: Mean rate of arrival $= \frac{1}{\mathrm{E(inter-arrival\ time)}}$

μ: Mean service rate $= \frac{1}{\mathrm{E(service\ time)}}$

ρ: Utilization of the servers $= \frac{\lambda}{k\mu}$

$L_q^{G/G/k}$: Mean number of entities in the multi-server queue

$L^{G/G/k}$: Mean number of customers in the multi-server system

$W_q^{G/G/k}$: Mean wait time in the multi-server queue

$W^{G/G/k}$: Mean wait time in the multi-server system

σ_a^2: Variance of the inter-arrival process

σ_s^2: Variance of the service process

$\mathrm{C_a^2} = \frac{\sigma_a^2}{\left(\frac{1}{\lambda}\right)^2}$: Squared coefficient of variation for the inter-arrival time

$C_s^2 = \frac{\sigma_s^2}{\left(\frac{1}{\mu}\right)^2}$: Squared coefficient of variation for the service time

From Little's rule and queuing basics, see the following formulas:

$$L = \lambda W$$

$$L_q = \lambda W_q$$

$$W = W_q + \frac{1}{\mu}$$

3.2. MARCHAL APPROXIMATION

The classical approximation developed by Marchal (1976) for generally distributed inter-arrival times and service times, for a single queue (G/G/1) is shown below:

$$L_q^{G/G/1} \cong \frac{\rho^2 \,(1 + C_s^2)(C_a^2 + \rho^2 C_s^2)}{2\,(1 - \rho)(1 + \rho^2 C_s^2)} \qquad (1)$$

He then used the existing M/M/k approximation developed by Lee and Haughton (1959):

$$P_0 = \sum_{m=0}^{k-1} \frac{k\rho^m}{m!} \qquad (2)$$

$$L_q^{M/M/k} = \frac{P_0\left(\frac{\lambda}{\mu}\right)^k \rho}{k!\,(1-\rho)^2} \qquad (3)$$

Marchal (1985) developed a scaling factor that exploits his own G/G/1 approximation (Marchal, 1976), which, when used with M/M/k approximation, serves as a formula for the G/G/k queue: The scaling factor, SF, is defined as follows:

$$SF = \frac{(1+C_S^2)\left(C_S^2+(\rho^2 C_S^2)\right)}{2(\rho^2 C_S^2)} \qquad (4)$$

Combining Equations 3 and 4 results in the following formula for the G/G/k queue:

$$L_q^{G/G/k} = SF.L_q^{M/M/k} = \frac{P_0\left(\frac{\lambda}{\mu}\right)^k \rho}{k!\,(1-\rho)^2}\,\frac{(1+C_S^2)\left(C_S^2+(\rho^2 C_S^2)\right)}{2(\rho^2 C_S^2)} \qquad (5)$$

3.3. KRAEMER AND LANGENBACK-BELZ (K-L)

In the same year that Marchal developed his approximation, Kraemer and Langenbach-Belz (1976) developed a different approximation for the G/G/1 queue:

$$L_q^{G/G/1} \cong \frac{\rho^2\,(C_a^2 + C_s^2)\,g}{2(1-\rho)} \tag{6}$$

where $g = exp\left[\frac{-2\,(1-\rho)\,(1 + C_s^2)2}{3\,\rho\,(C_a^2 + C_s^2)}\right]$ when $C_a^2 \leq 1$; (7)

and $g = exp\left[\frac{-1\,(1-\rho)\,(C_a^2 - 1)}{3\,\rho\,(C_a^2 + C_s^2)}\right]$ when $C_a^2 > 1$. (8)

Similarly, they developed a multi-server approximation that uses the M/M/c approximation and their single-server approximation. In this case, the scaling factor K-L developed was:

$$SF = \frac{g(c_a^2 + C_s^2)}{2} \tag{9}$$

where g is as defined in Equations 7 and 8.

Finally, the K-L approximation for a multi-server system for a general distribution service and inter-arrival time (G/G/k) is derived by combining Equations 3 and 9

$$L_q = \frac{P_0\left(\frac{\lambda}{\mu}\right)^k \rho}{k!\,(1-\rho)^2}\,\frac{g(c_a^2 + C_s^2)}{2} \tag{10}$$

where g is as defined in Equations 7 and 8.

3.4. MAGGIE (MULTIPLE-SERVER AGGREGATION INDEX FOR EXPECTED VALUES)

MAGGIE is a new approximation technique that was developed in this research for estimating values of the key performance metrics of a queue, namely, the mean length of the queue (L_q) and the mean waiting time in queue (W_q). The underlying principle of MAGGIE is to aggregate a multi-server queue into a single server queue and then use existing approximations for G/G/1 queues to estimate the performance metrics. Now, here is a detailed description of the MAGGIE in detail.

The standard coefficient of variation for the service time in a G/G/1 queue is calculated as follows:

$$C_S^2 = \frac{\sigma_S}{\left(\frac{1}{\mu}\right)^2} \tag{11}$$

An aggregation procedure will be adopted to compute an adjusted squared coefficient of variation for a multi-server queue in order for it to be treated as a single-server queue. The aggregation procedure will modify/adjust the coefficient of variation of the service time of each server in the multi-server queue and can be done in one of three ways, depending on the variability in the inter-arrival time:

a. Aggregating the means *when the variability in the inter-arrival time is high*:

$$C_{S\_adj}^2 = \frac{\sigma_S}{\left(\frac{1}{k\,\mu}\right)^2} \tag{12}$$

b. Aggregating the variance *when the variability in the inter-arrival time is medium*:

$$C_{S\_adj}^2 = \frac{k\,\sigma_S}{\left(\frac{1}{\mu}\right)^2} \tag{13}$$

c. Scaling the coefficient *when the variability in the inter-arrival time is low*:

$$C_{S\_adj}^2 = \frac{\sigma_S}{(k\left(\frac{1}{\mu}\right))^2} \tag{14}$$

The main steps underlying MAGGIE are described next:

Step 1: The hypothetical aggregated server will be a single server whose parameters (mean and variance of the service time) will be computed using one of the three approaches discussed via Equations 12 through 14.

Step 2: The expected length of this aggregated queue will be computed using either MAR or the K-L approach for the G/G/1 queue. See Figure 3.1. below for a visual representation of the aggregation principle.

Step 3: The expected value of the queue length (or waiting time) of this hypothetical G/G/1 queue will be divided by *k* in order to obtain the same value for the original G/G/k system.

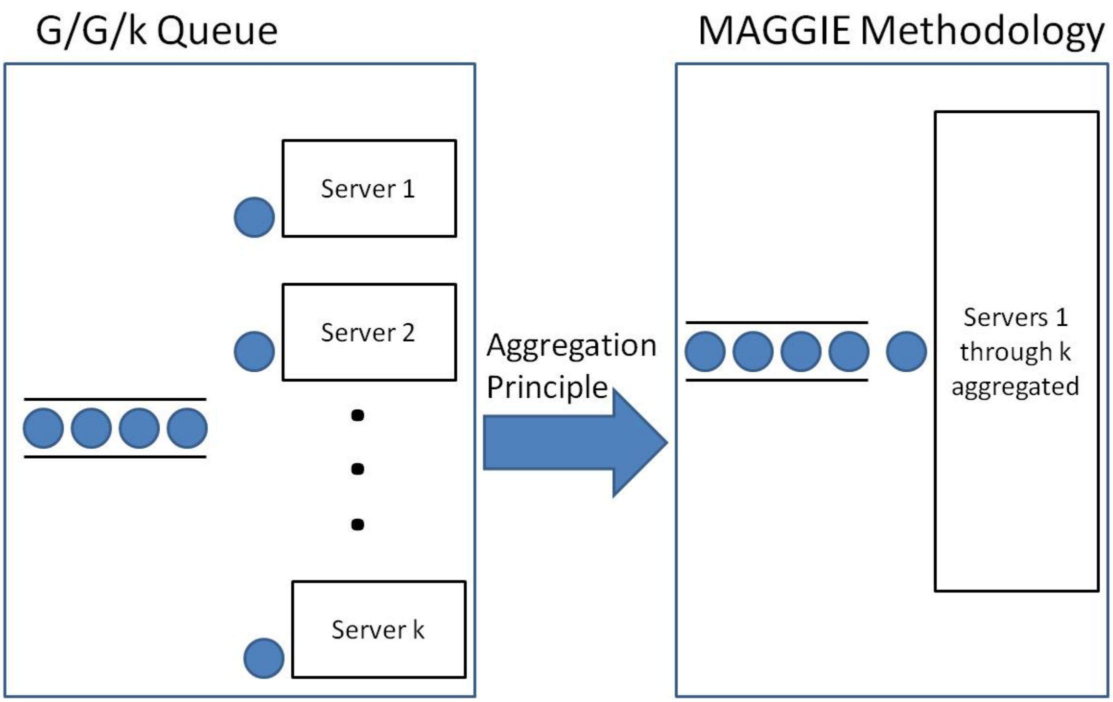

Figure 3.1. Representation of the Aggregation Principle

In what follows, details of the MAGGIE model are provided. The areas below were derived empirically after obtaining results from numerous experiments. Different combinations of single-server approximation and different approaches for computing the adjusted squared coefficient of variation of the service time were tried. The following rules provide the best results for the gamma distributed inter-arrival times and the double-tapering service times. The research found that there are 14 different cases, where each case represents an area in the quadrant where the x-axis is C_a^2 and the y-axis is C_s^2.

For the case, $C_s^2 > 0.25$, i.e., $\sigma_s^2 > 0.25/\mu^2$, it is not possible to find appropriate values for the parameters of the normal or the triangular distribution. For instance, if $\mu = \frac{1}{10}$, then one must have that $\sigma_s^2 > 25$. But in order to have such a high value of variance, the smallest value in the distribution will be forced to be negative, but the inter-arrival time cannot be negative. Hence, the research model is designed for the range $0 < C_s^2 \leq 0.25$ and $0 < C_a^2 \leq 1$. Note that for other distributions, values of $C_s^2 > 0.25$ cannot be ruled out, but those distributions are beyond the scope of this study.

Area 1: $C_a^2 < 0.30$:

- $k < 5$
- $C_s^2 \leq 0.15$
- $C_{s\_adj}^2$ computed using Equation 14
- Marchal's approximation for the hypothetical aggregated server via

Equation 1 in which $C_{s\_adj}^2$ is used instead of C_s^2 and ρ is computed as $\frac{\lambda}{k\mu}$

Area 2: $0.3 \leq C_a^2 < 0.60$:

- $k < 5$
- $C_s^2 \leq 0.15$

- $C^2_{s\_adj}$ computed using Equation 13
- Marchal's approximation for the hypothetical aggregated server via Equation 1 in which $C^2_{s\_adj}$ is used instead of C^2_s and ρ is computed as $\frac{\lambda}{k\mu}$

Area 3: $0.60 \leq C^2_a < 0.75$:

- $k < 5$
- $C^2_s \leq 0.15$
- $C^2_{s\_adj}$ computed using Equation 13
- K-L approximation for the hypothetical aggregated server via Equation 6, using the g-value in Equation 8, in which $C^2_{s\_adj}$ is used instead of C^2_s and ρ is computed as $\frac{\lambda}{k\mu}$

Area 4: $0.75 \leq C^2_a \leq 1.00$:

- $k < 5$
- $C^2_s \leq 0.15$
- $C^2_{s\_adj}$ Computed using Equation 12
- K-L approximation for the hypothetical aggregated server via Equation 6, using the g-value in Equation 8, in which $C^2_{s\_adj}$ is used instead of C^2_s and ρ is computed as $\frac{\lambda}{k\mu}$

Area 5: $C^2_a < 0.30$

- $k < 5$

- $0.15 < C_s^2 \leq 0.25$
- $C_{s\_adj}^2$ Computed using Equation 14
- K-L approximation for the hypothetical aggregated server via

Equation 6, using the g-value in Equation 8, in which $C_{s\_adj}^2$ is used instead of C_s^2 and ρ is computed as $\frac{\lambda}{k\mu}$

Area 6: $0.30 \leq C_a^2 < 0.60$:

- $k < 5$
- $0.15 < C_s^2 \leq 0.25$
- $C_{s\_adj}^2$ Computed using Equation 13
- Marchal's approximation for the hypothetical aggregated server via

Equation 1 in which $C_{s\_adj}^2$ is used instead of C_s^2 and ρ is computed as $\frac{\lambda}{k\mu}$

Area 7: $0.60 \leq C_a^2 < 0.75$:

- $k < 5$
- $0.15 < C_s^2 \leq 0.25$
- $C_{s\_adj}^2$ Computed using Equation 13
- K-L approximation for the hypothetical aggregated server via

Equation 6, using the g-value in Equation 8, in which $C_{s\_adj}^2$ is used instead of C_s^2 and ρ is computed as $\frac{\lambda}{k\mu}$

Area 8: $0.75 \leq C_a^2 \leq 1.00$:

- $k < 5$
- $0.15 < C_s^2 \leq 0.25$
- $C_{s\_adj}^2$ computed using Equation 13
- K-L approximation for the hypothetical aggregated server via

Equation 6, using the g-value in Equation 8, in which $C_{s\_adj}^2$ is used instead of C_s^2 and ρ is computed as $\frac{\lambda}{k\mu}$. This produces the first aggregated performance measure ($L_{q,K-L}^{G/G/k}$)

- $C_{s\_adj}^2$ Computed using Equation 12
- Marchal's approximation for the hypothetical aggregated server via

Equation 1 in which $C_{s\_adj}^2$ is used instead of C_s^2 and ρ is computed as $\frac{\lambda}{k\mu}$. This produces the second aggregated performance measure ($L_{q,MAR}^{G/G/k}$)

- Finally, compute the arithmetic mean of the two performance

measures as follows: $L_q^{G/G/k} = \frac{L_{q,K-L}^{G/G/k} + L_{q,MAR}^{G/G/k}}{2}$

Area 9: $C_a^2 < 0.40$:

- $5 \leq k < 10$
- $C_s^2 \leq 0.15$
- $C_{s\_adj}^2$ computed using Equation 14
- Marchal's approximation for the hypothetical aggregated server via

Equation 1 in which $C_{s\_adj}^2$ is used instead of C_s^2 and ρ is computed as $\frac{\lambda}{k\mu}$

Area 10: $0.40 \leq C_a^2 < 0.75$:

- $5 \leq k < 10$
- $C_s^2 \leq 0.15$
- $C_{s\_adj}^2$ computed using Equation 13
- Marchal's approximation for the hypothetical aggregated server via

Equation 1 in which $C_{s\_adj}^2$ is used instead of C_s^2 and ρ is computed as $\frac{\lambda}{k\mu}$

Area 11: $0.75 \leq C_a^2 < 1.00$:

- $5 \leq k < 10$
- $C_s^2 \leq 0.15$
- $C_{s\_adj}^2$ is not used; rather, the conventional C_s^2 is used
- The conventional K-L model for the multi-server queues is to be

used for Equation 10

Area 12: $C_a^2 < 0.40$:

- $5 \leq k < 10$
- $0.15 < C_s^2 \leq 0.25$
- $C_{s\_adj}^2$ computed using Equation 14
- K-L approximation for the hypothetical aggregated server via

Equation 6, using the g-value in Equation 8, in which $C_{s\_adj}^2$ is used instead of C_s^2 and ρ is computed as $\frac{\lambda}{k\mu}$

Area 13: $0.40 \leq C_a^2 < 0.75$:

- $5 \leq k < 10$
- $0.15 < C_s^2 \leq 0.25$
- $C_{s\_adj}^2$ computed using Equation 13
- Marchal's approximation for the hypothetical aggregated server via

Equation 1 in which $C_{s\_adj}^2$ is used instead of C_s^2 and ρ is computed as $\frac{\lambda}{k\mu}$

Area 14: $0.75 \leq C_a^2 < 1.00$:

- $5 \leq k < 10$
- $C_s^2 \leq 0.15$
- $C_{s\_adj}^2$ is not used; rather, the conventional C_s^2 is used
- The conventional K-L model for the multi-server queues is to be

used for Equation 10

In the case that $C_s^2 > 0.25$, i.e., $\sigma_s^2 > 0.25/\mu^2$, it is not possible to find appropriate values for the parameters of the normal or the triangular distribution. For instance, if $\mu = \frac{1}{10}$, then one must have that $\sigma_s^2 > 25$. But in order to have such a high value of variance, the smallest value in the distribution will be forced to be negative, but the inter-arrival time cannot be negative. For this reason, the model is designed for the range $0 < C_s^2 \leq 0.25$ and $0 < C_a^2 \leq 1$. Note that for other distributions, values of $C_s^2 > 0.25$ cannot be ruled out, but those distributions are beyond the scope of this study.

The logic for the queuing model presented in this paper were summarized in the previous section and the section below presents the summary of those results. Figures 3.2 and 3.3 represent the results of the model.

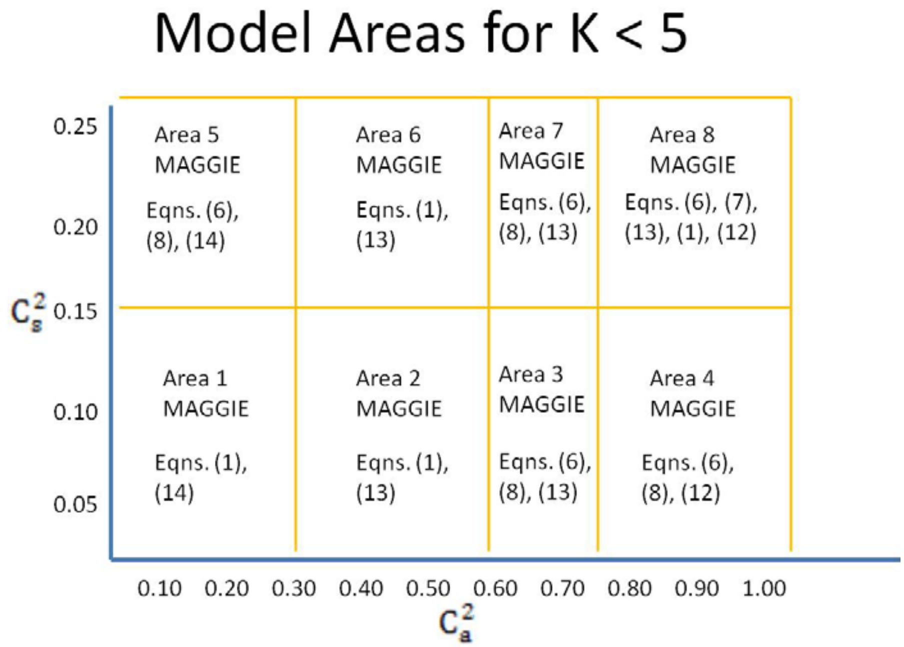

Figure 3.2. Model for k < 5 Servers

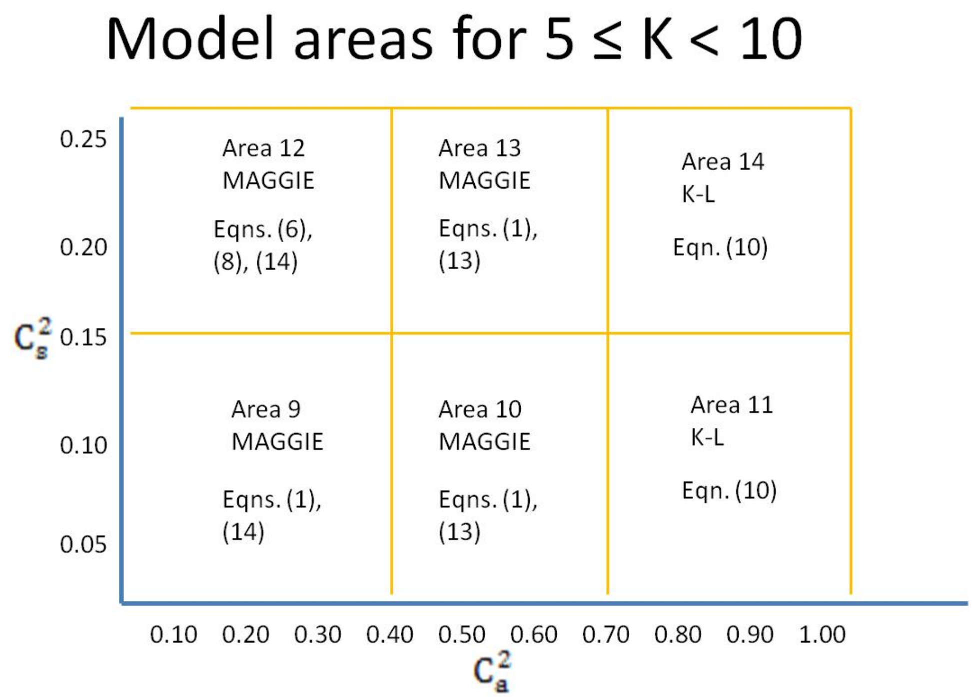

Figure 3.3. Model for 5 ≤ k < 10 Servers

The numerical results of the model are presented in the next section.

3.5. NUMERICAL RESULTS

The results of the computational study are provided in this section. Thousands of scenarios were tested for each of the “areas” discussed above. The results from MAGGIE are shown via Tables 3.1 through 3.28, with two tables per area; the top table shows the inputs used in simulation and model trials and the bottom table shows the results of the models with its respective percent-error. The waiting time in the queue as estimated by a given model is denoted by W_q, and the error percentage in MAGGIE, K-L and MAR models using simulation as a benchmark was calculated as follows:

$$Error\ \% = \frac{|W_q - W_q^{Simulation}|}{W_q^{Simulation}}\ X\ 100$$

Area 1

Table 3.1. Input data for Area 1 in which the inter-arrival time has the gamma distribution, $\lambda = \frac{1}{5};\ \mu = 0.15.$

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
1	2	0.10	Triangular	0.11	0.212229
2	2	0.15	Triangular	0.11	0.320377
3	2	0.20	Triangular	0.11	0.436271
4	2	0.25	Triangular	0.11	0.563239
5	2	0.10	Normal	0.10	0.19811
6	2	0.15	Normal	0.10	0.31108
7	2	0.20	Normal	0.10	0.423196
8	2	0.25	Normal	0.10	0.552683
9	2	0.10	Normal	0.15	0.276078
10	2	0.15	Normal	0.15	0.389045
11	2	0.20	Normal	0.15	0.517875
12	2	0.25	Normal	0.15	0.646325

Table 3.2. Results of MAGGIE, MAR, and K-L, and their errors from simulation

	MAGGIE	MAGGIE	MAR	MAR	K-L	K-L
Case	W_q	Error	W_q	Error	W_q	Error
1	0.2121	0.06%	0.4153	95.68%	0.1492	29.70%
2	0.2979	7.02%	0.5562	73.61%	0.2678	16.41%
3	0.3836	12.07%	0.6971	59.79%	0.4079	6.50%
4	0.4694	16.66%	0.8380	48.78%	0.5623	0.17%
5	0.2092	5.60%	0.4057	104.79%	0.1383	30.19%
6	0.2948	5.23%	0.5461	75.55%	0.2544	18.22%
7	0.3804	10.11%	0.6865	62.22%	0.3929	7.16%
8	0.4660	15.68%	0.8269	49.62%	0.5462	1.17%
9	0.2312	16.26%	0.4792	73.57%	0.2264	17.99%
10	0.3179	18.29%	0.6229	60.11%	0.3585	7.85%
11	0.4046	21.87%	0.7667	48.05%	0.5074	2.02%
12	0.4913	23.99%	0.9104	40.86%	0.6675	3.28%

In the first area, one can see the difference in performance between MAGGIE and MAR, where MAR does not perform well in the selected interval. When comparing MAGGIE and K-L models, one can see that they perform comparatively well. MAGGIE performs better on the lower bounds and K-L performs better on the upper bounds of the defined region. But after comparing overall performance, MAGGIE becomes the appropriate model to use.

Area 2

Table 3.3. Input data for Area 2 in which the inter-arrival time has the gamma distribution, $\lambda = \frac{1}{5}$; $\mu = 0.10$ (if $k = 3$) $\mu = 0.075$ (if $k = 4$). G=Gamma and T = Triangular

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
13	2	0.35	G	0.15	0.72345
14	2	0.40	G	0.15	0.867227
15	2	0.45	G	0.15	0.983115
16	2	0.50	G	0.15	1.12032
17	2	0.55	G	0.15	1.27526
18	2	0.30	T	0.11	0.422073
19	2	0.35	T	0.11	0.51454
20	2	0.40	T	0.11	0.612067
21	2	0.45	T	0.11	0.733225
22	2	0.50	T	0.11	0.833472
23	2	0.55	T	0.11	1.001011

Table 3.4. Results of MAGGIE, MAR, and K-L, and their errors from simulation

Case	MAGGIE W_q	MAGGIE Error	MAR W_q	MAR Error	K-L W_q	K-L Error
13	0.7384	2.07%	0.9983	37.99%	0.8384	15.89%
14	0.8056	7.11%	1.1181	28.93%	0.9826	13.30%
15	0.8727	11.23%	1.2378	25.91%	1.1271	14.65%
16	0.9398	16.11%	1.3576	21.18%	1.2706	13.41%
17	1.0069	21.04%	1.4774	15.85%	1.4126	10.77%
18	0.4893	15.93%	0.6945	64.54%	0.5149	21.99%
19	0.5393	4.81%	0.7945	54.41%	0.6347	23.35%
20	0.5892	3.74%	0.8944	46.13%	0.7565	23.60%
21	0.6392	12.82%	0.9944	35.62%	0.8787	19.84%
22	0.6892	17.31%	1.0943	31.29%	1.0005	20.04%
23	0.7391	26.16%	1.1943	19.31%	1.1211	12.00%

Similarly to Area 1, it is clear that MAGGIE is the superior model when its performance is compared to MAR, although MAR's error percentage is starting to trend down. When comparing the error percentage of MAGGIE to K-L, there is a similar trend as that of Area 1 and the overall performance is better for MAGGIE.

Area 3

Table 3.5. Input data for Area 3 in which the inter-arrival time has the gamma distribution,$\lambda = \frac{1}{5}$; $\mu = 0.15$ (if $k = 3$) $\mu = 0.10$ (if $k = 3$) $\mu = 0.075$ (if $k = 4$).

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
24	2	0.60	Normal	0.05	1.4424
25	2	0.65	Normal	0.05	1.6156
26	2	0.70	Normal	0.05	1.9426
27	2	0.60	Normal	0.10	1.6149
28	2	0.65	Normal	0.10	1.7789
29	2	0.70	Normal	0.10	2.0797
30	2	0.60	Normal	0.15	1.6937
31	2	0.65	Normal	0.15	1.9307
32	3	0.60	Triangular	0.11	1.2923
33	3	0.60	Gamma	0.15	1.3678
34	3	0.65	Gamma	0.15	1.5598
35	3	0.70	Gamma	0.15	1.7692
36	4	0.60	Triangular	0.11	1.0769
37	4	0.65	Triangular	0.11	1.2286

Table 3.6. Results of MAGGIE, MAR, and K-L, and their errors from simulation

	MAGGIE	MAGGIE	MAR	MAR	K-L	K-L
Case	W_q	Error	W_q	Error	W_q	Error
24	1.3783	4.45%	1.7043	18.16%	1.5968	10.70%
25	1.4339	11.24%	1.8413	13.97%	1.7609	9.00%
26	1.4900	23.30%	1.9783	1.84%	1.9216	1.08%
27	1.5235	5.66%	1.8099	12.07%	1.7297	7.11%
28	1.5831	11.01%	1.9504	9.64%	1.8940	6.47%
29	1.6428	21.01%	2.0908	0.54%	2.0548	1.20%
30	1.6763	1.02%	1.9167	13.17%	1.8627	9.98%
31	1.7382	9.97%	2.0604	6.72%	2.0272	5.00%
32	1.1641	9.92%	1.5201	17.63%	1.4563	12.69%
33	1.3038	4.68%	1.5972	16.77%	1.5523	13.49%
34	1.3418	13.98%	1.7170	10.08%	1.6893	8.30%
35	1.3799	22.00%	1.8368	3.82%	1.8234	3.06%
36	0.9741	9.55%	1.2942	20.17%	1.2398	15.12%
37	0.9999	18.61%	1.3942	13.48%	1.3564	10.40%

The data in Area 3 shows a significant improvement in the MAR model performance and greater competition among the three models.

Area 4

Table 3.7. Input data for Area 4 in which the inter-arrival time has the gamma distribution, $\lambda = \frac{1}{5};\ \mu = 0.15\ (\text{if } k = 3)\ \mu = 0.10\ (\text{if } k = 3)\ \mu = 0.075\ (\text{if } k = 4)$.

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
38	2	0.80	Normal	0.15	2.4871
39	2	0.85	Normal	0.15	2.7473
40	2	0.90	Normal	0.15	2.8023
41	2	0.95	Normal	0.15	3.0979
42	2	1.00	Normal	0.15	3.1409
43	3	0.80	Triangular	0.11	2.0398
44	3	0.85	Triangular	0.11	2.1553
45	3	0.90	Triangular	0.11	2.2597
46	3	0.90	Gamma	0.15	2.3909
47	3	0.95	Gamma	0.15	2.5544
48	3	1.00	Gamma	0.15	2.6709

Table 3.8. Results of MAGGIE, MAR, and K-L, and their errors from simulation

	MAGGIE	MAGGIE	MAR	MAR	K-L	K-L
Case	W_q	Error	W_q	Error	W_q	Error
38	2.4471	1.61%	2.4917	0.19%	2.4980	0.44%
39	2.5014	8.95%	2.6354	4.07%	2.6467	3.66%
40	2.5562	8.78%	2.7792	0.82%	2.7911	0.40%
41	2.6113	15.71%	2.9229	5.65%	2.9311	5.39%
42	2.6667	15.10%	3.0667	2.36%	3.0667	2.36%
43	2.0651	1.24%	1.9897	2.46%	1.9855	2.66%
44	2.0918	2.95%	2.1071	2.24%	2.1094	2.13%
45	2.1196	6.20%	2.2245	1.56%	2.2297	1.33%
46	2.5562	6.91%	2.3160	3.13%	2.3259	2.72%
47	2.5832	1.13%	2.4358	4.64%	2.4426	4.38%
48	2.6111	2.24%	2.5556	4.32%	2.5556	4.32%

In this area, all three models perform well and are interchangeable. Still, due to the computational advantages of MAGGIE, the research suggests to use MAGGIE

Area 5

Table 3.9. Input data for Area 5 in which the inter-arrival time has the gamma distribution, $\lambda = \frac{1}{5};\ \mu = 0.15$ (if $k = 3$) $\mu = 0.10$ (if $k = 3$) $\mu = 0.075$ (if $k = 4$).

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
49	3	0.10	Gamma	0.20	0.2878
50	3	0.15	Gamma	0.20	0.3741
51	3	0.20	Gamma	0.20	0.4726
52	3	0.25	Gamma	0.20	0.5723
53	4	0.10	Normal	0.20	0.1858
54	4	0.15	Normal	0.20	0.2674
55	4	0.20	Normal	0.20	0.3517

Table 3.10. Results of MAGGIE, MAR, and K-L, and their errors from simulation

	MAGGIE	MAGGIE	MAR	MAR	K-L	K-L
Case	W_q	Error	W_q	Error	W_q	Error
49	0.2584	10.20%	0.4626	60.76%	0.2710	5.82%
50	0.3244	13.29%	0.5850	56.37%	0.3909	4.49%
51	0.3868	18.15%	0.7075	49.71%	0.5215	10.35%
52	0.4464	22.00%	0.8299	45.01%	0.6592	15.19%
53	0.1745	6.09%	0.3939	111.98%	0.2308	24.21%
54	0.2246	16.02%	0.4981	86.24%	0.3328	24.43%
55	0.2720	22.67%	0.6024	71.27%	0.4440	26.23%

This area behaves similarly to Area 1, where MAR performs poorly and MAGGIE outperforms K-L, but K-L is still acceptable in some areas. In this area, the results show that MAGGIE's error percentage is below 25%.

Area 6

Table 3.11. Input data for Area 6 in which the inter-arrival time has the gamma distribution, $\lambda = \frac{1}{5};\ \mu = 0.15\ (\text{if k } = 3)\ \mu = 0.10\ (\text{if k } = 3)\ \mu = 0.075\ (\text{if k } = 4)$.

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
56	3	0.30	Normal	0.25	0.7710
57	3	0.35	Normal	0.25	0.8983
58	3	0.40	Normal	0.25	1.0368
59	3	0.45	Normal	0.25	1.1721
60	3	0.50	Normal	0.25	1.3400
61	3	0.55	Normal	0.25	1.4760
62	3	0.60	Normal	0.25	1.6127
63	3	0.30	Gamma	0.20	0.7097
64	3	0.35	Gamma	0.20	0.8143
65	3	0.40	Gamma	0.20	0.9456
66	3	0.45	Gamma	0.20	1.0877
67	3	0.50	Gamma	0.20	1.2331
68	3	0.55	Gamma	0.20	1.3490
69	3	0.60	Gamma	0.20	1.4591
70	3	0.30	Gamma	0.25	0.7826
71	3	0.35	Gamma	0.25	0.9029
72	3	0.40	Gamma	0.25	1.0420

Table 3.12. Results of MAGGIE, MAR, and K-L, and their errors from simulation

	MAGGIE	MAGGIE	MAR	MAR	K-L	K-L
Case	W_q	Error	W_q	Error	W_q	Error
56	0.9236	19.80%	1.0278	33.31%	0.9082	17.80%
57	0.9965	10.93%	1.1528	28.33%	1.0544	17.37%
58	1.0694	3.15%	1.2778	23.25%	1.2009	15.83%
59	1.1424	2.54%	1.4028	19.68%	1.3469	14.91%
60	1.2153	9.31%	1.5278	14.01%	1.4914	11.30%
61	1.2882	12.73%	1.6528	11.98%	1.6339	10.69%
62	1.3611	15.60%	1.7778	10.24%	1.7740	10.00%
63	0.7953	12.06%	0.9524	34.20%	0.8015	12.94%
64	0.8655	6.28%	1.0748	31.98%	0.9461	16.18%
65	0.9357	1.05%	1.1973	26.61%	1.0916	15.44%
66	1.0058	7.53%	1.3197	21.32%	1.2369	13.71%
67	1.0760	12.74%	1.4422	16.96%	1.3810	12.00%
68	1.1462	15.03%	1.5646	15.98%	1.5232	12.91%
69	1.2164	16.63%	1.6871	15.63%	1.6631	13.98%
70	0.9236	18.02%	1.0278	31.33%	0.9082	16.05%
71	0.9965	10.36%	1.1528	27.68%	1.0544	16.78%
72	1.0694	2.63%	1.2778	22.63%	1.2009	15.25%

This area follows a similar pattern as that of Area 2. The MAR model improves, and though K-L and MAGGIE are comparable, yet MAGGIE performs better.

Area 7

Table 3.13. Input data for Area 7 in which the inter-arrival time has the gamma distribution, $\lambda = \frac{1}{5}$; $\mu = 0.15$ (if $k = 3$) $\mu = 0.10$ (if $k = 3$) $\mu = 0.075$ (if $k = 4$).

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
73	3	0.60	Triangular	0.22	1.5200
74	3	0.65	Triangular	0.22	1.7133
75	3	0.70	Triangular	0.22	1.8972
76	3	0.60	Normal	0.20	1.4599
77	3	0.65	Normal	0.20	1.6051
78	3	0.70	Normal	0.20	1.9256
79	3	0.60	Normal	0.25	1.6127
80	3	0.65	Normal	0.25	1.7852
81	3	0.70	Normal	0.25	2.0817
82	3	0.60	Gamma	0.20	1.4591
83	3	0.65	Gamma	0.20	1.6832
84	3	0.70	Gamma	0.20	1.8421
85	3	0.60	Gamma	0.25	1.6379
86	3	0.65	Gamma	0.25	1.7366
87	3	0.70	Gamma	0.25	1.9918
88	4	0.60	Triangular	0.22	1.2496
89	4	0.65	Triangular	0.22	1.3855
90	4	0.70	Triangular	0.22	1.5961
91	4	0.60	Normal	0.20	1.2745
92	4	0.65	Normal	0.20	1.3831

Table 3.14. Results of MAGGIE, MAR, and K-L, and their errors from simulation

	MAGGIE	MAGGIE	MAR	MAR	K-L	K-L
Case	W_q	Error	W_q	Error	W_q	Error
73	1.5146	0.35%	1.7137	12.74%	1.6957	11.56%
74	1.5535	9.33%	1.8369	7.21%	1.8329	6.98%
75	1.5924	16.06%	1.9601	3.32%	1.9671	3.69%
76	1.4666	0.46%	1.6871	15.57%	1.6631	13.92%
77	1.5053	6.22%	1.8095	12.73%	1.8003	12.16%
78	1.5440	19.82%	1.9320	0.33%	1.9344	0.45%
79	1.6304	1.10%	1.7778	10.24%	1.7740	10.00%
80	1.6695	6.48%	1.9028	6.59%	1.9113	7.06%
81	1.7087	17.92%	2.0278	2.59%	2.0455	1.74%
82	1.4666	0.51%	1.6871	15.63%	1.6631	13.98%
83	1.5053	10.57%	1.8095	7.50%	1.8003	6.96%
84	1.5440	16.18%	1.9320	4.88%	1.9344	5.01%
85	1.6304	0.46%	1.7778	8.54%	1.7740	8.31%
86	1.6695	3.86%	1.9028	9.57%	1.9113	10.06%
87	1.7087	14.21%	2.0278	1.81%	2.0455	2.70%
88	1.3330	6.67%	1.4595	16.80%	1.4444	15.59%
89	1.3594	1.89%	1.5644	12.91%	1.5612	12.68%
90	1.3860	13.16%	1.6694	4.59%	1.6754	4.97%
91	1.2832	0.68%	1.4364	12.70%	1.4161	11.11%
92	1.3096	5.32%	1.5407	11.39%	1.5328	10.82%

In this area, all three models are acceptable and competitive with one another. The difference between the models shows MAGGIE slightly outperforming MAR and K-L thereby providing a slight advantage to MAGGIE.

Area 8

Table 3.15. Input data for Area 8 in which the inter-arrival time has the gamma distribution, $\lambda = \frac{1}{5};\ \mu = 0.15\ (\text{if k } = 3)\ \mu = 0.10\ (\text{if k } = 3)\ \mu = 0.075\ (\text{if k } = 4)$.

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
93	2	0.80	Normal	0.25	2.8357
94	2	0.85	Normal	0.25	2.9558
95	2	0.90	Normal	0.25	3.1088
96	2	0.95	Normal	0.25	3.3247
97	2	1.00	Normal	0.25	3.5314
98	3	0.80	Triangular	0.22	2.3044
99	3	0.85	Triangular	0.22	2.3711
100	3	0.90	Triangular	0.22	2.6670
101	3	0.95	Triangular	0.22	2.6792
102	3	1.00	Triangular	0.22	2.8203
103	3	0.80	Gamma	0.20	2.2170
104	3	0.85	Gamma	0.20	2.3400
105	3	0.90	Gamma	0.20	2.4866
106	3	0.95	Gamma	0.20	2.6018

Table 3.16. Results of MAGGIE, MAR, and K-L, and their errors from simulation

	MAGGIE	MAGGIE	MAR	MAR	K-L	K-L
Case	W_q	Error	W_q	Error	W_q	Error
93	2.5082	11.55%	2.7333	3.61%	2.7647	2.50%
94	2.6123	11.62%	2.8833	2.45%	2.9134	1.43%
95	2.7152	12.66%	3.0333	2.43%	3.0578	1.64%
96	2.8166	15.28%	3.1833	4.25%	3.1978	3.82%
97	2.9167	17.41%	3.3333	5.61%	3.3333	5.61%
98	2.2487	2.42%	2.2065	4.25%	2.2255	3.42%
99	2.3235	2.01%	2.3297	1.75%	2.3494	0.92%
100	2.3974	10.11%	2.4529	8.03%	2.4697	7.40%
101	2.4704	7.79%	2.5761	3.85%	2.5864	3.46%
102	2.5424	9.85%	2.6993	4.29%	2.6993	4.29%
103	2.1531	2.88%	2.1769	1.81%	2.1928	1.09%
104	2.2273	4.82%	2.2993	1.74%	2.3167	1.00%
105	2.3006	7.48%	2.4218	2.61%	2.4370	1.99%
106	2.3730	8.79%	2.5442	2.21%	2.5537	1.85%

In Area 8, all three models perform acceptably, with K-L and MAR performing slightly better in some areas. Still the computational advantage of MAGGIE and the acceptable performance and consistency with the other areas leads the research to continue with MAGGIE as the preferred method.

Area 9

Table 3.17. Input data for Area 9 in which the inter-arrival time has the gamma distribution, $\lambda = \frac{1}{5}$; $\mu = 0.06$ (if $k = 5$) $\mu = 0.05$ (if $k = 6$) $\mu = 0.043$ (if $k = 7$) $\mu = 0.038$ (if $k = 8$) $\mu = 0.03$ (if $k = 9$).

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
107	5	0.05	Triangular	0.11	0.0380
108	5	0.10	Triangular	0.11	0.0777
109	5	0.15	Triangular	0.11	0.1263
110	5	0.20	Normal	0.05	0.1318
111	5	0.25	Normal	0.05	0.1867
112	5	0.30	Normal	0.05	0.2552
113	5	0.05	Normal	0.10	0.0360
114	5	0.10	Normal	0.10	0.0702
115	5	0.15	Normal	0.10	0.1217
116	5	0.20	Normal	0.10	0.1759
117	5	0.25	Normal	0.10	0.2397
118	6	0.20	Gamma	0.05	0.0979
119	6	0.25	Gamma	0.05	0.1497
120	6	0.30	Gamma	0.05	0.2133
121	6	0.35	Gamma	0.05	0.2638
122	9	0.20	Triangular	0.11	0.0691
123	9	0.25	Triangular	0.11	0.0993
124	9	0.30	Triangular	0.11	0.1374
125	9	0.35	Triangular	0.11	0.1824

Table 3.18. Results of MAGGIE, MAR, and K-L, and their errors from simulation

	MAGGIE	MAGGIE	MAR	MAR	K-L	K-L
Case	W_q	Error	W_q	Error	W_q	Error
107	0.0401	5.42%	0.1681	341.93%	0.0375	1.41%
108	0.0738	4.97%	0.2544	227.59%	0.0914	17.69%
109	0.1076	14.77%	0.3407	169.86%	0.1640	29.90%
110	0.1371	3.99%	0.3728	182.78%	0.1739	31.91%
111	0.1706	8.63%	0.4567	144.60%	0.2623	40.48%
112	0.2041	20.04%	0.5406	111.80%	0.3585	40.46%
113	0.0397	10.27%	0.1625	351.38%	0.0330	8.34%
114	0.0734	4.53%	0.2485	253.90%	0.0847	20.62%
115	0.1071	11.97%	0.3345	174.93%	0.1558	28.05%
116	0.1408	19.95%	0.4205	139.08%	0.2406	36.79%
117	0.1745	27.19%	0.5065	111.32%	0.3346	39.60%
118	0.1137	16.14%	0.3250	231.96%	0.1516	54.85%
119	0.1416	5.40%	0.3981	165.97%	0.2286	52.72%
120	0.1695	20.54%	0.4712	120.91%	0.3125	46.51%
121	0.1974	25.18%	0.5444	106.35%	0.4005	51.81%
122	0.0765	10.66%	0.2562	270.60%	0.1499	116.83%
123	0.0952	4.13%	0.3079	210.06%	0.2066	108.05%
124	0.1138	17.19%	0.3597	161.76%	0.2667	94.08%
125	0.1324	27.42%	0.4115	125.58%	0.3287	80.19%

This area shows a significant advantage for MAGGIE over the conventional models, MAR and K-L. Particularly, MAR performs poorly and K-L has a large error percentage in some cases. MAGGIE has a few cases where the error percentage exceeds 25%, but the error percentage is still significantly less than that of MAR or K-L.

Area 10

Table 3.19. Input data for Area 10 in which the inter-arrival time has the gamma distribution, $\lambda = \frac{1}{5};\ \mu = 0.06$ (if $k = 5$) $\mu = 0.05$ (if $k = 6$) $\mu = 0.043$ (if $k = 7$) $\mu = 0.038$ (if $k = 8$) $\mu = 0.03$ (if $k = 9$).

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
126	5	0.40	Normal	0.10	0.5015
127	5	0.45	Normal	0.10	0.5899
128	5	0.50	Normal	0.10	0.7074
129	5	0.55	Normal	0.10	0.7877
130	7	0.45	Triangular	0.11	0.4224
131	7	0.50	Triangular	0.11	0.4969
132	7	0.55	Triangular	0.11	0.5720
133	7	0.60	Triangular	0.11	0.6793
134	7	0.65	Triangular	0.11	0.7631
135	8	0.50	Normal	0.10	0.4243
136	8	0.55	Normal	0.10	0.4976
137	8	0.60	Normal	0.10	0.5852
138	8	0.65	Normal	0.10	0.6427
139	9	0.40	Gamma	0.05	0.1949
140	9	0.45	Gamma	0.05	0.2542
141	9	0.50	Gamma	0.05	0.3068
142	9	0.55	Gamma	0.05	0.3700
143	9	0.60	Gamma	0.05	0.4639

Table 3.20. Results of MAGGIE, MAR, and K-L, and their errors from simulation

	MAGGIE	MAGGIE	MAR	MAR	K-L	K-L
Case	W_q	Error	W_q	Error	W_q	Error
126	0.5091	1.51%	0.7645	52.44%	0.6424	28.09%
127	0.5500	6.77%	0.8506	44.18%	0.7479	26.77%
128	0.5909	16.47%	0.9366	32.40%	0.8529	20.57%
129	0.6318	19.79%	1.0226	29.82%	0.9570	21.50%
130	0.4883	15.61%	0.6571	55.57%	0.5807	37.48%
131	0.5195	4.54%	0.7231	45.52%	0.6612	33.06%
132	0.5507	3.73%	0.7892	37.96%	0.7408	29.50%
133	0.5819	14.34%	0.8552	25.90%	0.8193	20.61%
134	0.6132	19.65%	0.9213	20.72%	0.8964	17.46%
135	0.4734	11.56%	0.6332	49.22%	0.5767	35.91%
136	0.5011	0.70%	0.6914	38.95%	0.6470	30.02%
137	0.5287	9.66%	0.7495	28.07%	0.7163	22.40%
138	0.5564	13.43%	0.8077	25.67%	0.7843	22.02%
139	0.2685	37.73%	0.4250	118.01%	0.3377	73.23%
140	0.2909	14.44%	0.4753	86.98%	0.4005	57.55%
141	0.3133	2.13%	0.5256	71.33%	0.4632	50.99%
142	0.3356	9.30%	0.5760	55.68%	0.5254	42.00%
143	0.3580	22.83%	0.6263	35.00%	0.5868	26.49%

Similar to Area 9, the conventional models of MAR and K-L perform beyond an acceptable level of error percentage, although not as badly as in Area 9. Meanwhile, MAGGIE performs well in this area; again, a few data points show error that is higher than desired, but it remains overall in acceptable levels.

Area 12

Table 3.21. Input data for Area 12 in which the inter-arrival time has the gamma distribution, $\lambda = \frac{1}{5}$; $\mu = 0.06$ (if k = 5) $\mu = 0.05$ (if k = 6) $\mu = 0.043$ (if k = 7) $\mu = 0.038$ (if k = 8) $\mu = 0.03$ (if k = 9).

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
144	5	0.05	Normal	0.20	0.0894
145	5	0.10	Normal	0.20	0.1445
146	5	0.15	Normal	0.20	0.2056
147	5	0.20	Normal	0.20	0.2832
148	6	0.05	Gamma	0.20	0.0766
149	6	0.10	Gamma	0.20	0.1169
150	6	0.15	Gamma	0.20	0.1698
151	6	0.20	Gamma	0.20	0.2212
152	6	0.25	Gamma	0.20	0.2830
153	8	0.05	Triangular	0.22	0.0485
154	8	0.10	Triangular	0.22	0.0750
155	8	0.15	Triangular	0.22	0.1128
156	8	0.20	Triangular	0.22	0.1508
157	8	0.25	Triangular	0.22	0.1988
158	8	0.30	Triangular	0.22	0.2575

Table 3.22. Results of MAGGIE, MAR, and K-L, and their errors from simulation

	MAGGIE	MAGGIE	MAR	MAR	K-L	K-L
Case	W_q	Error	W_q	Error	W_q	Error
144	0.0871	2.60%	0.2500	179.57%	0.1226	37.10%
145	0.1303	9.85%	0.3400	135.24%	0.1992	37.82%
146	0.1707	16.99%	0.4300	109.11%	0.2873	39.72%
147	0.2089	26.24%	0.5200	83.61%	0.3833	35.34%
148	0.0672	12.29%	0.2179	184.39%	0.1069	39.52%
149	0.1034	11.51%	0.2964	153.65%	0.1737	48.65%
150	0.1372	19.19%	0.3748	120.75%	0.2504	47.48%
151	0.1692	23.50%	0.4533	104.96%	0.3341	51.06%
152	0.1997	29.42%	0.5317	87.91%	0.4224	49.28%
153	0.0454	6.38%	0.1783	267.68%	0.0940	93.84%
154	0.0723	3.60%	0.2395	219.33%	0.1476	96.80%
155	0.0976	13.51%	0.3007	166.48%	0.2084	84.68%
156	0.1216	19.35%	0.3620	140.10%	0.2741	81.80%
157	0.1445	27.30%	0.4232	112.92%	0.3431	72.62%
158	0.1665	35.34%	0.4844	88.13%	0.4141	60.82%

In this area, there is a significant error percentage in MAR and a high error percentage in K-L. MAGGIE does contain a few high-error data points, but still outperforms the conventional models and is therefore the best choice in this area.

Area 13

Table 3.23. Input data for Area 13 in which the inter-arrival time has the gamma distribution, $\lambda = \frac{1}{5};\ \mu = 0.06$ (if $k = 5$) $\mu = 0.05$ (if $k = 6$) $\mu = 0.043$ (if $k = 7$) $\mu = 0.038$ (if $k = 8$) $\mu = 0.03$ (if $k = 9$).

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
159	5	0.45	Triangular	0.22	0.7681
160	5	0.50	Triangular	0.22	0.8723
161	5	0.55	Triangular	0.22	0.9622
162	5	0.60	Triangular	0.22	1.0854
163	5	0.65	Triangular	0.22	1.2382
164	6	0.45	Gamma	0.20	0.5872
165	6	0.50	Gamma	0.20	0.7102
166	6	0.55	Gamma	0.20	0.7821
167	6	0.60	Gamma	0.20	0.8913
168	6	0.65	Gamma	0.20	1.0130
169	7	0.60	Triangular	0.22	0.7884
170	7	0.65	Triangular	0.22	0.8534
171	7	0.70	Triangular	0.22	0.9990
172	8	0.60	Normal	0.20	0.6848
173	8	0.65	Normal	0.20	0.7520
174	8	0.70	Normal	0.20	0.9046

Table 3.24. Results of MAGGIE, MAR, and K-L, and their errors from simulation

	MAGGIE	MAGGIE	MAR	MAR	K-L	K-L
Case	W_q	Error	W_q	Error	W_q	Error
159	0.8685	13.07%	0.9882	28.66%	0.9334	21.52%
160	0.9153	4.93%	1.0788	23.68%	1.0394	19.16%
161	0.9621	0.01%	1.1694	21.53%	1.1440	18.89%
162	1.0089	7.05%	1.2599	16.08%	1.2469	14.88%
163	1.0557	14.74%	1.3505	9.07%	1.3477	8.85%
164	0.7838	33.49%	0.8456	44.02%	0.7925	34.97%
165	0.8237	15.98%	0.9240	30.10%	0.8848	24.58%
166	0.8635	10.40%	1.0025	28.17%	0.9759	24.77%
167	0.9034	1.36%	1.0809	21.27%	1.0656	19.56%
168	0.9432	6.89%	1.1594	14.45%	1.1535	13.86%
169	0.9070	15.05%	0.9645	22.34%	0.9544	21.06%
170	0.9427	10.46%	1.0338	21.13%	1.0316	20.88%
171	0.9784	2.06%	1.1031	10.42%	1.1071	10.82%
172	0.8302	21.23%	0.8384	22.43%	0.8265	20.69%
173	0.8618	14.60%	0.8992	19.57%	0.8946	18.96%
174	0.8935	1.22%	0.9601	6.14%	0.9613	6.27%

This area provides shows a substantial improvement on the conventional models, MAR and K-L, when compared to previous areas but is still beyond acceptable levels. MAGGIE has a few high-error data points, but still outperforms MAR and K-L.

Special Areas

As discussed in the previous section, two regions, Areas 11 and 14, do not use the MAGGIE queuing model due to the better performance from the conventional models. This difference in performance is found as the number of servers increase (greater than five) and as the squared coefficient of variation for the service time increases (greater than 0.75); the results are summarized in the tables below.

In the cases shown below, the data shows that the K-L and MAR models outperformed the MAGGIE model, with K-L having the slight edge over MAR. While MAGGIE does perform comparably, even better in some sub-regions, it fails to consistently perform well enough to make it the model of choice. Therefore, in these cases, the research strongly suggests using the K-L model.

Area 11

Table 3.25. Input data for Area 11 in which the inter-arrival time has the gamma distribution, $\lambda = \frac{1}{5};\ \mu = 0.06$ (if $k = 5$) $\mu = 0.05$ (if $k = 6$) $\mu = 0.043$ (if $k = 7$) $\mu = 0.038$ (if $k = 8$) $\mu = 0.03$ (if $k = 9$).

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
175	5	0.90	Triangular	0.11	1.6689
176	5	0.95	Triangular	0.11	1.8533
177	5	1.00	Triangular	0.11	1.9310
178	5	0.75	Normal	0.05	1.2840
179	5	0.80	Normal	0.05	1.3967
180	5	0.85	Normal	0.05	1.6240
181	5	0.75	Normal	0.15	1.4276
182	5	0.80	Normal	0.15	1.5777
183	5	0.85	Normal	0.15	1.7123
184	5	0.90	Normal	0.15	1.8181
185	5	0.95	Normal	0.15	1.8557
186	5	1.00	Normal	0.15	2.0881
187	6	0.75	Triangular	0.11	1.2314
188	6	0.80	Triangular	0.11	1.3244
189	6	0.85	Triangular	0.11	1.4304
190	6	0.90	Triangular	0.11	1.5065
191	6	0.95	Triangular	0.11	1.5953
192	6	1.00	Triangular	0.11	1.7946
193	9	0.75	Gamma	0.10	0.8444
194	9	0.80	Gamma	0.10	0.8558
195	9	0.85	Gamma	0.10	0.9666
196	9	0.90	Gamma	0.10	1.0481
197	9	0.95	Gamma	0.10	1.1746
198	9	1.00	Gamma	0.10	1.2403
199	9	0.75	Gamma	0.15	0.8398
200	9	0.80	Gamma	0.15	0.9137
201	9	0.85	Gamma	0.15	1.0522
202	9	0.90	Gamma	0.15	1.0935
203	9	0.95	Gamma	0.15	1.2343
204	9	1.00	Gamma	0.15	1.2689

Table 3.26. Results of MAGGIE, MAR, and K-L, and their errors from simulation

	MAGGIE	MAGGIE	MAR	MAR	K-L	K-L
Case	W_q	Error	W_q	Error	W_q	Error
175	1.6430	1.55%	1.6350	2.03%	1.6388	1.80%
176	1.6884	8.90%	1.7213	7.12%	1.7245	6.95%
177	1.7333	10.24%	1.8076	6.39%	1.8076	6.39%
178	1.2589	1.96%	1.2956	0.90%	1.2731	0.85%
179	1.3578	2.79%	1.3795	1.23%	1.3667	2.15%
180	1.4683	9.59%	1.4634	9.89%	1.4578	10.23%
181	1.0637	25.49%	1.4381	0.74%	1.4364	0.62%
182	1.9768	25.30%	1.5262	3.26%	1.5301	3.01%
183	2.0251	18.27%	1.6142	5.73%	1.6211	5.33%
184	2.0728	14.01%	1.7023	6.37%	1.7096	5.97%
185	2.1200	14.24%	1.7903	3.52%	1.7953	3.25%
186	2.1667	3.76%	1.8783	10.05%	1.8783	10.05%
187	1.6006	29.99%	1.1996	2.58%	1.1904	3.33%
188	1.6414	23.94%	1.2748	3.74%	1.2721	3.95%
189	1.6818	17.57%	1.3500	5.62%	1.3515	5.52%
190	1.7216	14.28%	1.4252	5.39%	1.4286	5.17%
191	1.7611	10.39%	1.5005	5.94%	1.5033	5.77%
192	1.8000	0.30%	1.5757	12.20%	1.5757	12.20%
193	0.6570	22.19%	0.8199	2.90%	0.8128	3.74%
194	0.6656	22.23%	0.8715	1.83%	0.8689	1.53%
195	0.6746	30.21%	0.9231	4.50%	0.9236	4.45%
196	0.6840	34.74%	0.9747	7.00%	0.9766	6.82%
197	0.6937	40.94%	1.0263	12.63%	1.0281	12.47%
198	0.7037	43.26%	1.0779	13.09%	1.0779	13.09%
199	0.8273	1.49%	0.8628	2.74%	0.8617	2.61%
200	0.8351	8.60%	0.9156	0.21%	0.9179	0.46%
201	0.8434	19.85%	0.9684	7.97%	0.9726	7.57%
202	0.8521	22.08%	1.0212	6.61%	1.0256	6.21%
203	0.8611	30.23%	1.0741	12.98%	1.0771	12.73%
204	0.8704	31.40%	1.1269	11.19%	1.1269	11.19%

Area 14

Table 3.27. Input data for Area 14 in which the inter-arrival time has the gamma distribution, $\lambda = \frac{1}{5};\ \mu = 0.06$ (if k = 5) $\mu = 0.05$ (if k = 6) $\mu = 0.043$ (if k = 7) $\mu = 0.038$ (if k = 8) $\mu = 0.03$ (if k = 9).

Case	Number of Servers (k)	C_a^2	Service Time Distribution	C_s^2	$W_q^{Simulation}$
205	5	0.80	Triangular	0.22	1.6383
206	5	0.85	Triangular	0.22	1.7878
207	5	0.90	Triangular	0.22	1.9530
208	5	0.95	Triangular	0.22	2.1008
209	5	1.00	Triangular	0.22	2.1481
210	7	0.80	Triangular	0.22	1.2467
211	7	0.85	Triangular	0.22	1.3959
212	7	0.90	Triangular	0.22	1.4735
213	7	0.95	Triangular	0.22	1.5865
214	7	1.00	Triangular	0.22	1.6418

Table 3.28. Results of MAGGIE, MAR, and K-L, and their errors from simulation

Case	MAGGIE W_q	MAGGIE Error	MAR W_q	MAR Error	K-L W_q	K-L Error
205	1.3012	20.58%	1.6222	0.98%	1.6362	0.13%
206	1.3214	26.09%	1.7128	4.19%	1.7273	3.38%
207	1.3418	31.29%	1.8034	7.66%	1.8157	7.03%
208	2.7684	31.78%	1.8939	9.85%	1.9015	9.49%
209	2.8167	31.13%	1.9845	7.61%	1.9845	7.61%
210	1.1426	8.35%	1.2418	0.39%	1.2525	0.46%
211	1.1547	17.28%	1.3111	6.07%	1.3222	5.28%
212	1.1671	20.79%	1.3804	6.32%	1.3899	5.67%
213	1.1799	25.63%	1.4498	8.62%	1.4556	8.25%
214	1.9290	27.34%	1.5191	7.47%	1.5191	7.47%

In both of these areas, one can see that MAR and K-L show acceptable results on all data points in the area. Meanwhile, MAGGIE shows unacceptable results in some areas. Due to the consistently positive results in the conventional models, the research suggests to continue to use K-L (or MAR) in this area.

4. CONCLUSIONS

While current research presents accurate approximations for multiple server queues, MAGGIE provides a model that is computationally easy and accurate. It has the advantage over some of the more complex models and simulation in that it is a closed-form, two-moment approximation which only requires knowledge of the mean and variance. This characteristic saves significant time in the data-gathering process. This is especially significant when compared to simulations, where one would need to know the distributions for the random variables before running the model. Depending on the application, gathering enough data to have statistically significant distributions to apply in the simulation model can take weeks and months. There is also a time-saving benefit from running the models: MAGGIE is a simple equation which takes less than a second to run, whereas a simulation model can take a few minutes to hours depending on computing power, simulation software, number of runs and replications and other parameters. MAGGIE also presents a cost savings over simulation in that it is an equation, therefore it can plugged in excel, programmed in an open-source code or any other platform virtually free. In contrast, simulation packages con cost thousands of dollars for licensing fees.

Finally, MAGGIE provides an advantage in accuracy over conventional theoretical models (MAR and K-L) in many of the studied cases. The evidence in the Numerical Results section above shows that MAGGIE consistently performs under 25% error, which according to the literature is an acceptable error percentage for the studied applications of airport queues, manufacturing cell and other service queues. In contrast, the conventional models show high error percentage; which is a finding in itself. Finally, MAGGIE does not require computation of the steady-state probabilities of an M/M/k queue, which is required with the G/G/k approximations of MAR and K-L and is therefore easier to compute than the existing models.

Therefore, MAGGIE is the recommended model for the applicable areas delineated in this paper due to its simplicity, accuracy and costs savings over the conventional methods and approximations used today.

Proposed Future Work:

- Development of a queueing network based on approximations in Buzacott and Shanthikumar (1993) – Paper 2
- Application of the closed-form model via MAGGIE and other models in an airport system
- Testing of MAR and K-L on G/G/1 queues
- Computational work with the airport system

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II: A MATHEMATICAL MODEL FOR APPROXIMATING AN AIRPORT QUEUING NETWORK

ABSTRACT

Queuing networks (QNs) arise in airports during passenger checking-in. For studying such systems, much of the literature either suggests the use of discrete-event simulation models, which are unfortunately harder to optimize, or the use of models based on the exponential distribution for inter-arrival and/or service times. Mathematical models that work for any given distribution, which are more generally applicable and are easier to optimize, are less frequently studied in the literature. The researcher presents a mathematical approximation procedure, rooted in existing QN approximations, but applicable for any given distribution, to study waiting times and queue lengths in a typical/generic airport QN. The latter usually consists of two queues: the first queue, which is multi-server (G/G/k), is for the ID check, and the second queue, which is single-server (G/G/1), is for the body/X-ray scanner. The main contribution lies in developing an approximation for the squared coefficient of variation for the inter-departure time in a multi-server queue, which is necessary to compute the same for the inter-arrival time to the second queue. Numerical results from the research model show that it approximates results from discrete-event simulation well. The model can be handily incorporated into an optimization framework to determine the optimal number of servers.

1. INTRODUCTION

The literature shows a need for predictive models in queuing systems as there is a significant amount of interest amongst engineers in developing closed-form approximations for queuing systems, because one can plug in values into closed-form formulas to obtain estimates for the performance metrics of interest. In this paper, the research will expand the model from single, multiple server queues to include a network of queues.

Currently, simulation is the main tool used to estimate performance measures from queuing networks (QNs). In fact, the literature review section in this paper will cover some of the best studies as they apply to airports, manufacturing systems and other service queues. While there has been an increase in computer processing power, which reduces the time needed for running simulation models, two problems remain with simulation: (i) the cost of simulation packages and (ii) the time needed to get enough data to determine distributions for inter-arrival and service times for the different servers is still very long. A closed form approximation that evaluates QNs would still be a preferred method due to its simplicity and time savings. After all, due to the dynamic environment, requirements change fast and decision makers need adaptable tools that can help them respond to demands in a feasible manner. These demands can take many forms, from regulators to the passengers who “make a variety of different demands on the capacity offered by an airport, which in turn generate a varied range of different revenues” (Humphreys and Francis, 2002).

2. LITERATURE REVIEW

The paper by Guizzi et al (2009) is a clear baseline for the performance that can be extracted from simulation modeling. It describes an access interface, process area and flight interface for an airport area that can be simulated through Discrete Event Simulation to obtain the performance measures within the airport. This review also covers important case studies for airport terminal applications conducted by Nagoya University, Buffalo International Airport, and the Department of Civil Engineering of Surayabe in Indonesia.

Ray and Claramunt (2003) develops a predictive model based on modeling and simulation principles, a distributed computer environment, and recreating a real-world system, an airport terminal. This is an interesting approach to acquire performance measures and yields benefits of “flexibility and scalability to the system” (Ray and Claramunt, 2003). This is one of the objectives of this paper, but the difference lies in the area of data collection, which is still needed for simulation.

The two models proposed by Brunetta, et. al, (1999) and Brunetta and Jacur (1999) are called SLAM and AIRLAB, respectively. SLAM presents an operations research approach that determines airport capacity; this is a powerful tool, but still relies on simulation. AIRLAB focusses on Levels of Service (LOS), or performance measures. This model is more aligned with the research model in that is tries to evaluate acceptable levels of performance, but again, it relies on simulation which is time consuming and expensive.

The model by Manataki and Zografos (2009) is a detailed decision-making tool that can help airport managers through mesoscopic analysis “that strikes a balance between flexibility and realistic results, adopting a system dynamics approach.” This is a very complete model and provides a holistic approach to the problems an airport will face.

However, the complexity of the model and the data needed to apply the model renders it too narrow in perspective; in contrast, the research model would give a quick solution to the server problem as it relates to performance measures. However, a marriage of both models would be a very interesting topic for future study.

Closed-form mathematical queuing theory, on the other hand, seeks to operate in terms of quick and accurate approximations. However, in the literature on mathematical queueing theory, the research had found that for much of the work, there are limitations – either in terms of its scope or the right level of complexity to address the particular problem of queuing networks. Bertsimas, et al. (1999) seeks to estimate the performance of queuing networks. However, this work is tied to specific distributions.

The dynamic model by Lee and Jacobson (2011) shows a revamp to the "entire screening system paradigm to provide a solution that balances the trade-off between maximizing security and minimizing the expected time it takes to screen passengers" (Lee and Jacobson, 2011). This is also a detailed piece of work, but the complexity in the computation, since it uses steady-state probabilities and the limiting factor that is assumes Poisson processes, makes the research model, in contrast, more powerful in its simplicity and the use of general distributions.

Elyasi and Salmasi (2013) also present a dynamic model, which shows great improvement over established models and evaluates the need for additional servers. Like Lee and Jacobson (2011), it relies on the use of probabilities and assumes the normal distribution for the service time.

In summary, while simulation studies cited above are important contributions to the literature, there are many more that provide insights into performance measures, e.g., see

Gatersleben and Van Der Weij (1999), Gillen and Lall (1997), Leone and Liu (2011), and Ovidiu (2012). However, they all run into similar issues that simulation models run into for optimization. As discussed above, closed-form QN models make assumptions about distributions that are best to avoid. The research model offers simplicity and accuracy of a two-moment closed form approximation, works with general distributions, and has the benefit of limited input data required.

Contributions of this paper: this paper seeks to obtain performance measures from a two-stage QN with generally distributed inter-arrival and service times. In particular, this research will study a multiple-server queue followed by multiple single-server queues in parallel. Using the principles in this model, one can potentially perform a complete study of any set of QNs in which some queues are single-server and some are multiple-server.

3. MATHEMATICAL MODEL

The underlying problem here can be modeled as a 2-stage open QN (Ross, 1997), where one has the data for the inter-arrival time to the first queue and knowledge of the following: (i) the number of servers in each stage, (ii) the probability of an entity leaving the first stage to join a queue in the second, and (iii) data for the service times in each queue in the system (as well as the queueing disciplines).

In the above, by "data," it is meant either the distributions of the underlying random variable (i.e., for inter-arrival times and the service times) or the values of the first two moments of the underlying random variables. Further, the assumption is that each queue works on a first in first out (FIFO) discipline. Another assumption is that the travel time from exiting the first stage to joining the queue in the second stage is negligibly small. Usually, for simulation models, the underlying distributions are required, but this mathematical approximation will rest on knowledge of the first two moments.

Figure 3.1 represents the proposed QN model. Customers arrive to the first queue (the ID Check Queue, where identification documents are checked) in Figure 3.1, which is the first stage in the security processing. This queue is a multiple-server, single channel queue with generally distributed inter-arrival times and service times (G/G/k to use standard queueing notation). When customers complete their ID check, they are sent to one of the several parallel queues in the second stage (the Body Scanner/Carryon-Luggage Scanner Queue), shown in Figure 3.1. Each queue in the second stage is a single-server queue with generally distributed inter-arrival times and service times (G/G/1 to us standard queueing notation). The inputs to the research model model are: (i) the first two moments of the inter-arrival time and service time to the first queue (first stage), (ii) the number of servers in the first stage, (iii) the number of queues in the second stage, and (iv) the first two moments of the service time for each queue in the second stage. The outputs from the model will be the mean waiting time and number in each queue in the system. Since it is a mathematical model, a computer program will generate the result instantaneously, and hence can be used to optimize the number of servers in the first stage and the number of queues in the second stage.

The basic methodology adopted applies (i) Marchal's approximation of a G/G/k queue (Marchal 1985) to obtain the performance measures from the first queue in the network, (ii) classical queuing calculus principles (Buzacott and Shanthikumar, 1993) to obtain the first two moments of the inter-arrival time to each queue in the second stage and (iii) Marchal's approximation of a G/G/1 queue (Marchal, 1976) to obtain the performance measures in the second queue in the network.

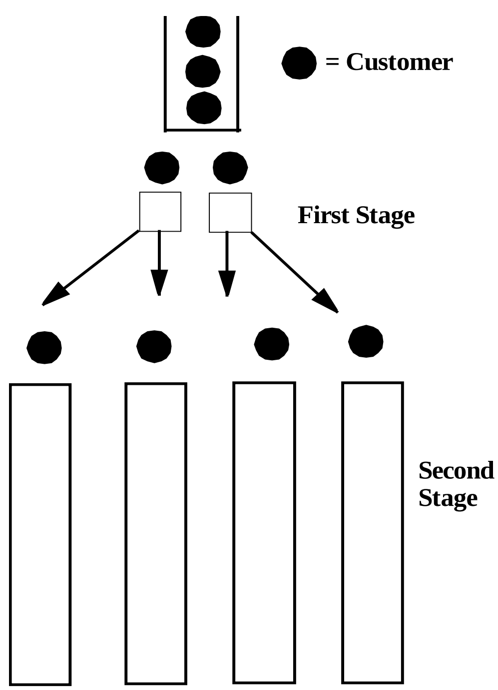

Figure 3.1. A QN in an Airport Security Line with 2 servers in the first stage and 4 queues in the second

3.1. BASIC QUEUING THEORY NOTATION

To ensure consistent communication through this paper, let us begin with some notation:

k: Number of servers in the multi-server queue in the first stage

λ: Mean rate of arrival= $\frac{1}{\mathrm{E(inter-arrival\ time)}}$ to the queue in the first stage

μ: Mean service rate = $\frac{1}{\mathrm{E(service\ time)}}$ of the queue in the first stage

ρ: Utilization in the first stage = $\frac{\lambda}{k\mu}$

$W_q^{G/G/k}$: Mean wait time in the multi-server queue in the first stage

$W_{q,i}^{G/G/1}$: Mean wait time in the ith queue in the second stage

σ_a^2: Variance of the inter-arrival time to the first queue

σ_s^2: Variance of the service time of one server in the first queue

$\mathrm{C_a^2}$: Squared coefficient of variation for the inter-arrival time = $\frac{\sigma_a^2}{\left(\frac{1}{\lambda}\right)^2}$

$\mathrm{C_s^2}$: Squared coefficient of variation for the service time of one server in the first stage

$\mathrm{C_d^2}$: Squared coefficient of variation for the time between successive departures from the first stage

$\mathrm{C_{a,i}^2}$: Squared coefficient of variation for the inter-arrival time in the *i*th queue in the second stage

3.2. MODEL

Currently in the literature, the approximation developed in Marchal (1976) for a generally distributed inter-arrival times and service times, for a single queue (G/G/1) is the accepted and conventional methodology for measuring queue performance. The computation is shown in the equation below:

$$L_q^{G/G/1} \cong \frac{\rho^2\,(1 + C_s^2)(C_a^2 + \rho^2 C_s^2)}{2\,(1 - \rho)(1 + \rho^2 C_s^2)} \tag{1}$$

However, that approximation is for a G/G/1 queue. A few years later, Marchal (1985) sought to exploit the existing M/M/k formula (exact), developed by Lee and Haughton (1959) and shown in Equation (2) below, in his own work:

$$P_0 = \sum_{m=0}^{k-1} \frac{\mathrm{k}\rho^{\mathrm{m}}}{\mathrm{m}!} \tag{2}$$

$$L_q^{M/M/k} = \frac{P_0\left(\frac{\lambda}{\mu}\right)^k \rho}{k!\,(1-\rho)^2} \tag{3}$$

In particular, Marchal (1985) developed a scaling factor that exploits his own G/G/1 approximation in Marchal (1976), which, when used with M/M/k approximation from Equation (2) above, generates an approximation for the G/G/k queue: The scaling factor, SF, is defined in Equation (4) below:

$$SF = \frac{(1+C_s^2)\left(C_s^2+(\rho^2 C_s^2)\right)}{2(\rho^2 C_s^2)} \tag{4}$$

Combining Equations (3) and (4) results in the following approximate formula for the G/G/k queue (Marchal, 1985):

$$L_q^{G/G/k} = SF.L_q^{M/M/k} = \frac{P_0\left(\frac{\lambda}{\mu}\right)^k \rho}{k!\,(1-\rho)^2}\,\frac{(1+C_s^2)\left(C_s^2+(\rho^2 C_s^2)\right)}{2(\rho^2 C_s^2)} \tag{5}$$

For this paper, the researcher uses the above (i.e., Marchal's G/G/k approximation) for the first queue in the network and Marchal's G/G/1 approximation for the queues in the second stage in the network.

3.3. QUEUING COMPUTATION

A key element of this research is the computation of the mean rate of arrival into the subsequent queues. The principle used in this paper is to use the inter-arrival rate from the first queue, λ, and multiplying it by the probability that the entity will go to one of the following parallel, single server queues i.

$$\lambda_2 = P_i\lambda_1 \tag{6}$$

The next step is to calculate the squared coefficient of variation of the time between successive departures from the first queue, C_d^2, as shown in Equation (7) below:

$$C_d^2 = \rho^2 C_s^2 + (1-\rho^2)C_a^2 \tag{7}$$

And C_a^2 for the second queue in the network is calculated using the squared coefficient of variation of the departure time and the probability that the entity will go to the single-server queue, i, through the following equation:

$$C_a^2 = 1 - P_i + (P_i * C_d^2) \tag{8}$$

3.4. NUMERICAL RESULTS

The researcher tested the mathematical model on 10 representative cases that vary in terms of: (i) squared coefficients of inter-arrival time in the system, (ii) squared coefficient of variation of service time for the first and second queue in the network, (iii) the number of servers, and (iv) the service time distributions. This research uses simulations in ARENA to benchmark the performance of the model. The error is computed against results from simulations. The error in the mean wait in the queue is calculated as

$$\boldsymbol{Error}\ \% = \frac{|W_q^{Model} - W_q^{Simulation}|}{W_q^{Simulation}}\ \boldsymbol{X\ 100}.$$

Numerical results from all the experiments are presented in Tables 3.1 and 3.2 below. The computer programs were written in MATLAB, and run on an Intel Pentium Processor with a speed of 2.66 GHz on a 64-bit operating system. The computer programs for the mathematical model took about 10 milliseconds; however, the simulation programs took longer (about 1 minute per case), since they involve multiple replications.

Usually, queuing approximations can result in errors of about 25% (see [13-14]). Therefore, the numerical results are quite encouraging: on the low end, the error computed was 0% and on the high-end the error was 26.5%.

Table 3.1. Results for first queue in the network: T (min, mode, max) denotes the triangular distribution, N(mean, variance) denotes the normal distribution, and Gm (mean, variance) denotes the gamma distribution. The inter-arrival time has a gamma distribution whose mean is 5 for each case and whose Ca2 value is specified for each case in the table

Case	k	C_a^2	Service Dist. First Stage	μ	C_s^2	$W_q^{Simulation}$	W_q^{Model}	% Error
1	2	0.45	T(1.33, 3.33, 15.33)	0.15	0.215	1.3952	1.6134	15.64
2	2	0.50	T(1.33, 3.33, 15.33)	0.15	0.215	1.4617	1.7613	20.50
3	3	0.60	N(10,5)	0.10	0.05	1.1643	1.4203	21.99
4	3	0.65	N(10,5)	0.10	0.05	1.2134	1.5344	26.45
5	4	0.75	N(10,5)	0.075	0.05	1.4112	1.5008	6.35
6	4	0.95	N(10,5)	0.075	0.15	2.1968	2.0739	5.59
7	6	0.65	Gm(20,60)	0.05	0.15	0.89842	1.1001	22.45
8	6	0.70	Gm(20,60)	0.05	0.15	1.1029	1.1769	6.71
9	7	0.65	T(4.67,11.87, 53.67)	0.043	0.215	0.9466	1.0338	9.21
10	7	0.70	T(4.67,11.87, 53.67)	0.043	0.215	1.1915	1.1031	7.42

Table 3.2. Results of the second queue in the network which consists of five single-server queues in parallel where service times are normally-distributed. The first four servers have a mean service rate of 1/20 and the same for fifth server is 1/23. Also, Pi = 1/5 for all values of i.

	i= 1:5	Servers 1 - 4			Server 5		
Case	$C_{s,i}^2$	$W_q^{Simulation}$	W_q^{Model}	% Error	$W_q^{Simulation}$	W_q^{Model}	% Error
1	0.10	37.84375	38.5873	1.96	138.59	127.9198	7.70
2	0.15	40.73775	40.7397	0.00	141.08	135.1733	4.19
3	0.10	39.60825	38.67	2.37	156.17	128.1881	17.92
4	0.15	39.635	40.8236	3.00	141.05	135.4432	3.98
5	0.10	45.796	39.3593	14.06	119.73	130.4235	8.93
6	0.15	44.05875	42.5957	3.32	186.78	141.1413	24.43
7	0.10	38.36725	39.2673	2.35	141.95	130.1254	8.33
8	0.15	51.751	41.4298	19.94	186.94	137.3925	26.50
9	0.10	38.0355	39.5063	3.87	117.52	130.9	11.39
10	0.15	48.90425	41.6723	14.79	186.09	138.1723	25.75

4. CONCLUSION

Queuing approximations such as the G/G/1 approximation presented above are now widely used in manufacturing systems for measuring lead times (Askin and Goldberg, 2002). Queueing network (QN) approximations, which are more complex than approximating a single queue, have also been used extensively in modeling production lines (Papadopolous, et al, 1993). In this paper, the research presented a new mathematical model for approximating a (mixed) 2-stage QN in which in one stage there is a G/G/k queue and in another there is a set of parallel queues, each belonging to the G/G/1 family. The contribution is in formulating a novel way to compute the squared coefficient of variation for the time between successive departures from the first queue. The research obtained successful numerical results with the approximation procedure.

There are multiple avenues for future research based on this work. First, the research approximating procedure can be used to optimize the number of servers. Another potential line for further research would measure the variance of the waiting time in each queue using the third moment

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SECTION

2. DISSERTATION CONCLUSION

This research presents a study of queuing theory, its applications and a solution for two important gaps in the literature: an approximation for a multiple server queue with generally distribute inter-arrival and service times in medium traffic, and a methodology for solving queuing networks with multiple servers in series with generally distributed inter-arrival and service times.

The history of queuing theory is indeed rich. This research evaluated the existing models since the inception of the field of queuing theory to the latest research and considered that the approaches by Marchal and Kraemer and Langenbach-Belz were the most appropriate to use as a baseline in developing a solution for an accurate, yet simple closed-form, two-moment approximation for the multiple server queue with generally distributed inter-arrival and service times in medium traffic. The outcome of the first paper in this dissertation is MAGGIE, an approximation which showed an improved performance over the existing models within the boundary conditions define in the paper.

In the case of queuing networks, the existing literature considers series of single server queues for closed-form approximations and mostly simulation for multiple server queues. This research presented a close-form, two moment approximation approach for multiple server queues in a network. This approach, which rests on calculating the squared coefficient of variation for the departure times and then using server transition probabilities, can be easily augmented into as many servers as needed. This augmentation provides a significant advantage over existing models and simulation.

The results from both of these contributions show a significant performance improvement on the benchmarked models and a time and cost savings when compared to simulation, where it is required to know the specific distributions for the system the research wishes to study. The applications for the models developed in this research are wide: airports, manufacturing and service systems, telecommunication, computing and many more. This work is the beginning of what can revolutionize decision models in systems with queues.

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