Representation Theory for Strange Attractors

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Repository Citation
"Representation Theory for Strange Attractors," D. J. Cross and R. Gilmore, Physical Review E 80, 056207 (2009).

Representation theory for strange attractors

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(Received 28 October 2008; revised manuscript received 12 October 2009; published 18 November 2009)

Embeddings are diffeomorphisms between some unseen physical attractor and a reconstructed image. Different embeddings may or may not be equivalent under isotopy. We regard embeddings as representations of the attractor, review the labels required to distinguish inequivalent representations for an important class of dynamical systems, and discuss the systematic ways inequivalent embeddings become equivalent as the embedding dimension increases until there is finally only one "universal" embedding in a suitable dimension.

DOI: 10.1103/PhysRevE.80.056207 PACS number(s): 05.45.Xt, 05.45.Ac, 05.45.Gg, 05.45.Ra

I. INTRODUCTION

The first step in the analysis of experimental data generated by chaotic systems is the search for an embedding [1]. An embedding (in the sense of Takens [2]) is used to "reconstruct" the experimental attractor. We usually attempt to construct embeddings in spaces of lowest possible dimension. What we know about experimental attractors is what we learn by studying their embeddings.

In this work we begin to address a question that has never been satisfactorily treated: when we study an embedding, how much of what we learn depends only on the attractor and how much depends on the embedding?

In the case of embeddings of chaotic data into $\mathbb{R}^3$, it is known that many inequivalent embeddings exist [3–5]. For an important class of attractors that includes the Rössler attractor and those created by the periodically driven Duffing and van der Pol oscillators, the labels that are required to distinguish among inequivalent embeddings are also known. We will show that as the embedding dimension is increased, fewer labels are required to distinguish inequivalent embeddings, and by the time the embedding dimension is $N=5$ all embeddings are equivalent. For embeddings of such three-dimensional attractors into $\mathbb{R}^N$, $N\geq 5$, all information that we learn by studying the embedded attractor is embedding independent since all embeddings are equivalent. For $N=3$ and $N=4$ some additional information results from studying the embedding: these are the labels distinguishing inequivalent embeddings.

This work is organized as follows. In Sec. II we review the embedding concepts of Whitney and Takens and discuss the conditions under which two embeddings are considered equivalent or inequivalent. We also introduce three important questions about the use intended for an embedding of chaotic data. In Sec. III we describe the important class of "genus-one" attractors and review the indices that are necessary to distinguish inequivalent embeddings into $\mathbb{R}^3$ of attractors in this class. These indices are parity, oriented knot type, and global torsion. While the necessity of these indices is known [4], we prove their sufficiency, i.e., that they form a complete set. In Sec. IV we show that if the embedding space is enlarged from $\mathbb{R}^3$ to $\mathbb{R}^4$, embeddings with different parity and/or knot type are equivalent. Surprisingly, we find that global torsion continues to distinguish embeddings in this dimension to a certain extent. We also show that if the embedding space is further enlarged from $\mathbb{R}^4$ to $\mathbb{R}^5$ all embeddings become equivalent. In Sec. V we outline the program for a representation theory of strange attractors in more general situations, such as three-dimensional attractors of higher genus and higher dimensional attractors. Our results are summarized in Sec. VI.

II. WHITNEY AND TAKENS EMBEDDINGS

The phase space of dynamical system is a differentiable manifold. An embedding (in the sense of Whitney [6]) of this manifold into $\mathbb{R}^N$ for some N is a diffeomorphism onto its image—it is one to one and everywhere differentiable with a differentiable inverse. An embedding of the phase-space manifold induces an embedding (in the sense of Takens [2]) of the attractor, but the latter is not usually a manifold. In this way the two notions of embedding are related.

Embeddings of either type are not unique: many possible mappings exist. Typically, changing the Whitney embedding of the phase space will alter the Takens embedding of the attractor. We are primarily interested in the spectrum of Takens embeddings of the attractor, but we shall study this indirectly by considering the spectrum of Whitney embeddings of its phase-space manifold. Hereafter the word embedding shall refer exclusively to Whitney embeddings.

Each embedding is a *representation* of the original phase space and its attractor. We introduce a topological notion of equivalence (or inequivalence) of representations. Two embeddings are equivalent if they are smoothly deformable into each other (stretching and bending are allowed, cutting and gluing are not). The intuitive notion of a smooth deformation is made precise by the notion of isotopy. Two embeddings $f_0(x)$ and $f_1(x)$ of a manifold M into $\mathbb{R}^N$ are isotopic if there is a smooth map $F(x,s)$ defined on $M\times[0,1]$ such that $F(x,0)=f_0(x)$, $F(x,1)=f_1(x)$, and $F(x,s)=f_s(x)$ is a embedding for each fixed s. One thinks of this as a one-parameter family of embeddings indexed by s. Nonisotopic embeddings provide distinct or inequivalent representations of an attractor, as one may not be deformed into another without self-intersection. Generally, as N increases the number of inequivalent embeddings decreases. (There are local exceptions to this rule. For example, while there are many knots in $\mathbb{R}^3$, there is only one knot type in $\mathbb{R}^2$.)

These considerations raise three very important questions:

(1) for any analysis methodology, which results depend on the representation and which are representation independent;

(2) for an experimental attractor, what is its spectrum of inequivalent representations and how are they distinguished; and

(3) as the embedding dimension increases, which representations remain inequivalent?

Question (1) has a precise answer. All geometric measures (e.g., spectrum of fractal dimensions) are diffeomorphism invariants. All dynamical measures (e.g., spectrum of Lyapunov exponents) are also embedding invariants, except that care must be taken to account for the extra or "spurious" Lyapunov exponents. As a result, these measures should be independent of the particular representation used to compute them. This means that the second and third questions are not important when the objectives of an analysis are restricted to computation of geometric and dynamical invariants. Unfortunately, such analyses do not provide information about the *mechanism* responsible for generating chaotic behavior [4,5]. By mechanism we mean an understanding of how different regions of the phase space are brought together and compressed under the flow. In terms of cardboard models, or the branched manifolds describing a three-dimensional flow, mechanism describes how the separate branches, each coded by a separate symbol, are joined together at branch lines rather than how they wrap around each other when situated in $\mathbb{R}^3$.

Topological measures depend on the chosen representation [7]. In three-dimensional representations, the spectrum of the linking numbers of unstable periodic orbits depends in a well-defined way on the representation. However, the mechanism responsible for generating chaotic behavior is representation independent [4,5]. For such analyses the second and third questions are important.

III. REPRESENTATION LABELS OF GENUS-ONE ATTRACTORS

We address the second and third questions for a specific but widely occurring class of attractors: those whose three-dimensional phase space is a solid torus $\mathcal{T}=D^2\times S^1$, where D^2 is a disk and S^1 is a circle [8]. This class includes periodically driven two-dimensional nonlinear oscillators such as the Duffing and van der Pol oscillators as well as autonomous three-dimensional dynamical systems such as the Rössler attractor. More generally it includes all attractors whose formation is due to the repetition of stretching and folding processes alone. It does not include the Lorenz attractor, or for that matter any other attractor created by tearing mechanisms. These are attractors of higher genus $g\geq 3$ [8]. The primary result of this paper is a complete answer to questions (2) and (3) for dynamical systems whose phase space is the solid torus.

To answer question (2) we first identify all the indices required to distinguish orientation-preserving diffeomorphisms of $\mathcal{T}$ into itself. This is the intrinsic part of the problem. Then we describe all the inequivalent ways the torus can be embedded into $\mathbb{R}^3$. This is the extrinsic part of the problem. While the problem seems to naturally divide into intrinsic and extrinsic parts, these two aspects are not entirely independent as we shall show.

The inequivalent orientation-preserving diffeomorphisms of flows on the torus are described by the "mapping class group." This group is well known and isomorphic to the abelian group $\mathbb{Z}\oplus\mathbb{Z}_2$ [9]. The subgroup $\mathbb{Z}$ describes rigid rotations about the long axis (longitude) of the torus through an integer number n of full turns. This subgroup is generated by a single such rotation, which is known as a Dehn twist [9]. The subgroup $\mathbb{Z}_2$ parametrizes inversion operations. These operations reverse the directions of both the meridian and the longitude without changing parity.

It is useful to parametrize points in the torus $D^2\times S^1$ by $(re^{i\phi},s)$, with $0\leq r\leq 1$, $0\leq\phi<2\pi$ and $\phi=0$ is identified with $\phi=2\pi$, and S^1 is the unit circle with $0\leq s<2\pi$ and $s=0$ is identified with $s=2\pi$. The two subgroups act on the torus as follows:

$$\mathbb{Z}\ \text{Global Torsion}\ (re^{i\phi},s)\to(re^{i(\phi+ns)},s),\quad n\in\mathbb{Z},$$

$$\mathbb{Z}_2\ \text{Inversion}\ (re^{i\phi},s)\to(re^{-i\phi},2\pi-s).$$

The global torsion n from the mapping class group represents one of the topological indices distinguishing the inequivalent representations of flows on a torus in $\mathbb{R}^3$. It is invariant under isotopy of embeddings into $\mathbb{R}^3$ since it may be calculated as the Gauss linking number of the core ($r=0$ in the parametrization above) of the solid torus with a longitude in its boundary and an isotopy will preserve their link type. In practice, global torsion appears as a systematic change in linking numbers between pairs of periodic orbits of an attractor [4,7,10].

The two extrinsic indices, knot type and parity, describe how the torus sits in $\mathbb{R}^3$ under the embedding. The torus can be mapped into $\mathbb{R}^3$ so that its core follows any smooth closed curve. Each different knot in $\mathbb{R}^3$ provides a different embedding of the torus in $\mathbb{R}^3$. In fact, each knot provides two embeddings which may or may not be equivalent. The argument is as follows. Position along a knot can be described by a real scalar parameter d that is periodic: d and $d+2\pi m$ describe the same point on the knot ($m\in\mathbb{Z}$). The torus can be mapped along any knot in two opposite senses, with $s=d$ or $s=2\pi-d$.

This degree of freedom is clearly related to the inversion degree of freedom ($\mathbb{Z}_2$) in the mapping class group, since $(2\pi-s=d)\simeq(s=2\pi-d)$. When the two oriented knots obtained in this way from a closed loop are isotopic they are called inversion symmetric [11]. Inversion symmetric knots provide equivalent representations of a torus in $\mathbb{R}^3$. Inversion asymmetric knots (8_{17} is the simplest [11]) provide two inequivalent representations of a torus in $\mathbb{R}^3$. Oriented knot type provides the second topological index distinguishing inequivalent representations of a torus in $\mathbb{R}^3$. Note the interaction between the intrinsic (inversion) and extrinsic (knot type) parts of the problem which leads to the absorption of the former into the latter. The intrinsic inversion index merges with extrinsic knot type to produce oriented knot type.

The third index is parity, obtained under the isometry $(x^1,x^2,x^3)\to(x^1,x^2,-x^3)$ in $\mathbb{R}^3$. Unlike the previous operations (the $\mathbb{Z}_2$ inversion symmetry) this one reverses the handedness of the torus.

This provides the answer to question (2) for the class of strange attractors whose phase space is a genus-one torus $D^2 \times S^1$. The complete set of representation labels required to distinguish inequivalent embeddings into $\mathbb{R}^3$ are parity, oriented knot type, and global torsion.

IV. EQUIVALENCES IN INCREASING DIMENSION

Question (3) is addressed by embedding inequivalent representations from $\mathbb{R}^3$ into $\mathbb{R}^N$, $N=4,5,\ldots$ and checking, for each N, when they become isotopic. This procedure must terminate by $N=7$, according to a theorem by Wu. He has shown [12] that any two embeddings of an n-manifold into $\mathbb{R}^{2n+1}$ are isotopic when $n \geq 2$.

A. Parity

First we show that if the torus is "lifted" into $\mathbb{R}^4$, representations with different parity or knot type are isotopic in this larger space. Thus these indices are no longer obstructions to isotopy, and the corresponding representations become equivalent in $\mathbb{R}^4$.

We change parity by lifting from $\mathbb{R}^3$ into $\mathbb{R}^4$, performing a rotational isotopy on the x^3-x^4 axes, then projecting back down into $\mathbb{R}^3$ using the first three coordinates,

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \xrightarrow{\text{Inject}} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ 0 \end{pmatrix} \xrightarrow{\text{Isotopy}} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \cos\theta \\ x^3 \sin\theta \end{pmatrix} \underset{\theta=\pi}{\xrightarrow{\text{Project}}} \begin{pmatrix} x^1 \\ x^2 \\ -x^3 \end{pmatrix}.$$

B. Knot type

Next we consider equivalence in $\mathbb{R}^4$ of representations with different oriented knot type. It is well known that knots "fall apart" in $\mathbb{R}^4$. However, the analogous result for thickened knots (solid tori) is not well known, so we demonstrate it here. First consider the knot defined by the core of the solid torus. By perturbing the embedding [13] it is possible to ensure that under planar projection $\pi:\mathbb{R}^3 \to \mathbb{R}^2$, the image has only a finite number of double points, each representing a single transverse intersection of the projected knot with itself. An intersection is transverse if it cannot be pulled apart by an arbitrarily small perturbation [14]. These double points are called crossings. Choose any one of these crossings. Above it are two sections of the embedded torus, one above the other. We will show that the handedness of the crossing can be changed by isotopy in $\mathbb{R}^4$.

In the neighborhood of this projected crossing the flows in the upper (U) and lower (L) tubular regions can be parametrized as follows:

$$U:(x,y,z) = [s_1, y_1, z_1 + g(s_1)]$$

$$L:(x,y,z) = [x_2, s_2, z_2 - g(s_2)].$$

The variables s_1 and s_2 parametrize the flow in the upper and lower tubes. The flow in the upper tubular region is in the x direction with $y_1^2+z_1^2 \leq 1$ and the core of this tube is defined by $y_1=z_1=0$. The flow in the lower tubular region is in the y direction with $x_2^2+z_2^2 \leq 1$ and the core of this tube is defined by $x_2=z_2=0$. The cores cross in projection into $\mathbb{R}^2$ at $(x,y)=(0,0)$. The two tubular regions miss by a large margin because of the offset in the z direction. The function $g(s)$ is a "bump function" that is +2 in the neighborhood of the crossing (at $s_1=s_2=0$) and drops to 0 before other double points are reached.

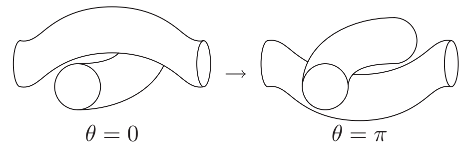

FIG. 1. Projection of the knot-type isotopy into $\mathbb{R}^3$.

We embed into $\mathbb{R}^4(x,y,z,w)$ as follows:

$$U:(s_1, y_1, z_1 + g(s_1)\cos\theta, + g(s_1)\sin\theta),$$

$$L:(x_2, s_2, z_2 - g(s_2)\cos\theta, - g(s_2)\sin\theta).$$

This mapping is an isotopy in $\mathbb{R}^4$ and has the effect of replacing right handed crossings in $\mathbb{R}^3$ by left handed crossings in $\mathbb{R}^3$ as θ varies from 0 to π and the first three coordinates are projected down to $\mathbb{R}^3$, as indicated in Fig. 1. By swapping appropriate crossings through this process every embedded knotted torus can be isotoped in $\mathbb{R}^4$ to a torus that projects to the standard unknotted torus in $\mathbb{R}^3$, which is inversion symmetric.

C. Global torsion

1. Global torsion in $\mathbb{R}^3$ revisited

Before discussing global torsion in four and five dimensions we return to the situation in three dimensions in order to introduce a technique that will be useful in higher dimensions. We previously detected global torsion in $\mathbb{R}^3$ by computing the Gauss linking number of the core of $\mathcal{T}$ with a longitude in the boundary. This approach is difficult to generalize. Another way to describe global torsion is to calculate how many times the longitude in the boundary wraps around the core by utilizing group theory. These two approaches are essentially equivalent in $\mathbb{R}^3$, but the latter permits a straightforward generalization to higher dimensions. We will now describe this latter approach in detail.

Let $\mathcal{T}$ be a solid torus embedded in the standard way into $\mathbb{R}^3$ (centered at the origin with rotational symmetry about the z axis). Denote by γ the core of $\mathcal{T}$ and by δ the standard longitude in the boundary defined by the intersection of the xy plane with $\mathcal{T}$. Then δ and γ do not link. If we apply a single Dehn twist to $\mathcal{T}$ then the image of δ is still a longitude in the boundary, but now it rotates at a uniform rate around the core, completing one full rotation. If n Dehn twists are applied, the image of δ will make n full rotations. This n represents the global torsion.

We now make this idea more precise and in the process cast it in a form that applies to general embeddings. Again, start with the standard embedding of $\mathcal{T}$ with core γ and longitude δ in the boundary lying radially outward from γ. Instead of considering the longitude directly, we construct a vector field along γ that indicates how δ is twisting about γ. Begin with the unit vector field t that is tangent to γ at each point. Next, adjoin a unit vector field u that points radially outward from γ to δ. Finally, add a third unit vector field v orthogonal to both t and u so that we have a positively oriented orthonormal basis of $\mathbb{R}^3$ at each point along γ.

Now consider an arbitrary embedding h of $\mathcal{T}$ into $\mathbb{R}^3$. The embedding will carry the triad (t,u,v) at each $x\in\gamma$ onto a new triad $(\tilde{t},\tilde{u},\tilde{v})$ at $h(x)$ that, while still linearly independent, is not generally orthonormal. This may be remedied by applying the Gram-Schmidt process to the triad. First, normalize $\tilde{t}$ to obtain t'. This vector is still tangent to $h(\gamma)$. Next, remove the projection of $\tilde{u}$ onto t' and normalize to obtain u'. This only removes any shearing of $\tilde{u}$ into $\tilde{t}$ and not any twisting of $\tilde{u}$ about $\tilde{t}$. Finally, v' is obtained by removing the shearing of $\tilde{v}$ into u' and t' and normalizing. While this process is abrupt as described, it is possible to smoothly deform the triad $(\tilde{t},\tilde{u},\tilde{v})$ into (t',u',v') [15].

Now we have three orthonormal vector fields along $h(\gamma)$ that describe the twisting of $h(\delta)$ about $h(\gamma)$. Notice that the vector field u' need not "point" directly to $h(\delta)$ because of distortions induced by the embedding. However, it does indicate the location of $h(\delta)$ in a more general sense which can be seen as follows. In the original standard embedding of $\mathcal{T}$ one can connect γ and δ with a ribbon or annulus so that the vector fields t and u are tangent to the ribbon and v is normal. The image of this ribbon under h is a ribbon connecting $h(\gamma)$ to $h(\delta)$ with tangents t' and u' and normal v'. Therefore u' describes the direction one would start out on in the ribbon to reach $h(\delta)$. The twisting of $h(\delta)$ about $h(\gamma)$ is equivalent to the twisting of this attached ribbon.

We now desire a way to extract the global torsion from these vector fields. This can be done by calculating the total accumulated twisting in the fields as one traverses $h(\gamma)$. To do this one needs a way of comparing the triads at different points in a standard way. This is accomplished by "parallel transport" along $h(\gamma)$, which is a way to push vectors along $h(\gamma)$ that keeps initially parallel vectors parallel and normal vectors normal, but otherwise does not alter them. Of course, t' is always tangent by construction so we need not consider it further. Now let n be some normal vector field along $h(\gamma)$. It is parallel transported if it obeys the equation $\dot{n}-\langle\dot{n},t'\rangle t'=0$, where the dot indicates differentiation with respect to arc length and the angle brackets indicate the innder product in the ambient Euclidean space. Essentially, to keep the vector field normal one must remove any tangential component during the translation. This construction makes sense in any dimension. For further details see [16].

Now that we have a means of moving triads around $h(\gamma)$ it is possible to compare the frames at different points. Pick a reference point $x_0\in h(\gamma)$. Parallel translate the triad at x_0 along $h(\gamma)$ to x. As mentioned above the tangent vectors t' always coincide. Thus we need only compare the pair of vectors (u',v') in the space normal to t' which is a copy of $\mathbb{R}^2$. The transformation between a pair of orthonormal bases is an orthogonal transformation or element of $SO(2)$. The transformation at x_0 is the identity and the transformations vary continuously with x. So, for each $x\in h(\gamma)$ an element of $SO(2)$ is determined, which describes a closed path in $SO(2)$ starting from and ending at the identity element.

We have associated with an embedding h of $\mathcal{T}$ a closed path through the identity in $SO(2)$ which encodes the global torsion, but this path is not unique. We know that an embedding h_1 isotopic to $h_0=h$ has the same global torsion, but this new embedding will determine a different curve in $SO(2)$. However, these two curves are related as follows. Denote by h_s the isotopy from h_0 to h_1. Fix a point $p\in\gamma$ and let the image point $x_s=h_s(p)$ under each embedding be the reference point for comparing frames in each embedding h_s. We thus obtain a family of loops in $SO(2)$ through the identity that vary continuously in s, but this just says the curves are homotopic. We conclude that isotopic embedding determine homotopic loops in $SO(2)$ so that global torsion depends only on the homotopy class of the loops.

The set of homotopy class of loops in a space X is called the fundamental group, denoted $\pi_1(X)$. It is known that $\pi_1[SO(2)]\simeq\mathbb{Z}$. Assuming that each of these classes is realized by some embedding we have recovered the global torsion of $\mathbb{Z}$ in $\mathbb{R}^3$. To show that each class has some realization by an embedding it is sufficient to consider the standard embedding of $\mathcal{T}$ into $\mathbb{R}^3$ with n Dehn twists applied. The original triad field on the untwisted $\mathcal{T}$ is given by (e_ϕ,e_r,e_z) and it is easy to see that these fields are parallel transported along γ. Thus the corresponding path in $SO(2)$ is the constant path at the identity which represents the trivial element $0\in\mathbb{Z}\simeq\pi_1[SO(2)]$. If n Dehn twists are applied the normal frame fields may be written $e^{in\phi}(e_r,e_z)$, which twist around the core at a constant rate. The corresponding path in $SO(2)$ makes n full rotations and represents the element $n\in\mathbb{Z}$. We conclude that global torsion is represented by an integer $(\mathbb{Z})$ in $\mathbb{R}^3$.

2. Global torsion in $\mathbb{R}^4$

We now apply the method described in the previous section in order to determine the global torsion of embeddings in $\mathbb{R}^4$. Begin again with the standard $\mathcal{T}$ in $\mathbb{R}^3$ with core γ, standard longitude δ, and frame fields (t,u,v). Now let h be an arbitrary embedding of $\mathcal{T}$ into $\mathbb{R}^4$. This carries the orthonormal triad (t,u,v) onto a a nonorthonormal triad $(\tilde{t},\tilde{u},\tilde{v})$ and Gram-Schmidt may be used again to obtain an orthonormal triad (t',u',v'). Now adjoin the unique unit vector field w' which completes this triad to a positively oriented orthonormal basis of $\mathbb{R}^4$.

Pick a base point $x_0\in h(\gamma)$ and parallel translate the frame at x_0 to every other $x\in h(\gamma)$ to compare frames. As before, the tangents are always identical so we need only compare the vectors (u',v',w') in the space orthogonal to t', which is now a copy of $\mathbb{R}^3$. The transformations between triads is now an element in $SO(3)$, and by comparing the frames at every point along $h(\gamma)$ we obtain a closed path in $SO(3)$ through the identity.

As before, the homotopy class of this path is an invariant under isotopy. The fundamental group in this case is $\pi_1[SO(3)]\simeq\mathbb{Z}_2$, so that there are at most only two classes of global torsion in $\mathbb{R}^4$. It remains to show that each is represented by some embedding. To this end it is sufficient to lift

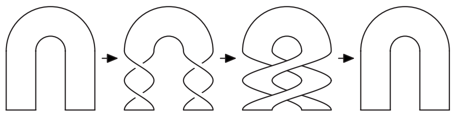

FIG. 2. Isotopy of two Dehn twists to the identity in $\mathbb{R}^4$.

the standard $\mathcal{T}$ in $\mathbb{R}^3$ with n Dehn twists applied into $\mathbb{R}^4$. The frame fields normal to t' are now given by $[e^{in\phi}(e_r, e_z), e_w]$ and the paths determined in $SO(3)$ describe n full rotations about the w-axis which determines the element $n \bmod 2 \in \mathbb{Z}_2$. The homotopy deforming one path to the other is illustrated explicitly in Fig. 5.7 of [17]. We conclude that global torsion is represented by an integer mod 2 ($\mathbb{Z}_2$) in $\mathbb{R}^4$. Since we already know that knot type and parity are no longer obstructions to isotopy in $\mathbb{R}^4$ this shows that there are exactly two representations in this dimension and they differ by a single Dehn twist.

While the above proof was somewhat abstract, it is possible to see directly why two Dehn twists should be isotopic to the identity. Embed $\mathcal{T}$ in the standard way into $\mathbb{R}^3 \subset \mathbb{R}^4$. The xz plane intersects the embedding along two disjoint disks, dividing $\mathcal{T}$ into two cylinders or handles. Cut open the embedding along these two disks and insert two Dehn twists, one at each disk, and reattach (see Fig. 2). Now, spinning the whole handle on one side of the xz-plane through one full turn converts the two Dehn twists into a writhe. Finally, the writhe may be removed by passing one part of the handle through another in $\mathbb{R}^4$ as in section IVB, which results in the trivial embedding. This phenomenon is related to the well-known Dirac belt and Feynman plate tricks, which demonstrate that two rotations about an axis in $\mathbb{R}^3$ are isotopic to the identity [18].

One may wonder whether a single Dehn twist is isotopic to the identity, but the preceding proof demonstrates that this is not the case. If two embeddings are isotopic the curves determined in $SO(3)$ are homotopic, but the curves for zero and one Dehn twist belong to different classes in the fundamental group so cannot be homotopic.

This result is somewhat surprising since γ and δ do not link in $\mathbb{R}^4$ when considered as just curves in $\mathbb{R}^4$. The fact that they are actually embedded within $\mathcal{T}$ provides the additional structure necessary to have them still link in a meaningful way and determine a global torsion. However, the triviality of the extrinsic embeddings does have an influence since it allows pairs of twists to annihilate, leaving a global torsion that is only defined mod 2.

3. Global torsion in $\mathbb{R}^5$

The method utilized in the previous sections essentially fails in $\mathbb{R}^5$. Everything applies verbatim through the part when one arrives at the orthonormal frame (t', u', v') along $h(\gamma)$. At this point there is no unique way to complete this frame to a frame of $\mathbb{R}^5$ because there is a two-dimensional subspace left to span. The pair of vectors may always be chosen so that the corresponding path in $SO(4)$ the frames determine is represented by the trivial element in $\pi_1[SO(4)] \simeq \mathbb{Z}_2$.

This seems to indicate that the global torsion in $\mathbb{R}^5$ is trivial. This is in fact the case as we now verify. An isotopy between representations that differ by a Dehn twist may be constructed by first lifting $D^2 \times S^1$ into $D^4 \times S^1$,

$$\begin{pmatrix} s \\ re^{i\phi} \end{pmatrix} \mapsto \begin{pmatrix} s \\ re^{i\phi} \\ re^{i(\phi+s)} \end{pmatrix}.$$

Now, define an isotopy by

$$\begin{pmatrix} 1 & \multicolumn{2}{c}{0} \\ 0 & \cos\theta & \sin\theta \\ & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} s \\ re^{i\phi} \\ re^{i(\phi+s)} \end{pmatrix}.$$

This is in fact an isometry. This rotation effectively interchanges the two complex factors between $\theta=0$ and $\theta=\pi/2$, so that the projection onto the first two components goes from an untwisted to a twisted torus. Thus global torsion is no longer an invariant in $D^4 \times S^1$ and, by the natural embedding, in $\mathbb{R}^5$.

All the degrees of freedom have now been exhausted and we have arrived at a "universal" embedding in $\mathbb{R}^5$. Every lower dimensional representation may be regarded as a nonsingular projection of this universal attractor into the appropriate Euclidean space, where these representations are distinguished by the appropriate topological indices.

V. PROGRAM FOR A REPRESENTATION THEORY

In this paper we have taken the first steps in responding to an important question: what information about a strange attractor depends on its embedding and what information is independent of the embedding? The proper tool for responding to this question is a representation theory for strange attractors. Inequivalent representations are distinguished by representation labels of topological origin. As the dimension of the embedding space increases there is more room in which to move about. This extra room lifts obstructions to isotopy so that some representation labels drop away. Finally, at sufficiently high dimension there is so much room that all obstructions are lifted and all representations become isotopic and therefore equivalent.

This general program is summarized by the following steps:

(1) determine a Euclidean space of minimum dimension into which chaotic data and its phase-space manifold can be embedded;

(2) compute the complete set of inequivalent representations (and topological indices) in this space;

(3) embed this manifold into a Euclidean space of one higher dimension. Determine which representations remain inequivalent and which become equivalent because of the additional room available for isotopies;

(4) repeat until all representations become equivalent.

We have carried this problem out for an important class of three-dimensional strange attractors: those of genus one. The extension of this program to three-dimensional systems of higher genus is currently under way. In this case the representation labels that distinguish inequivalent representations are already known [5].

In the higher dimensional case very little is known about the topological indices that are necessary to identify distinct representations of a strange attractor. What can be stated is that if any representation of the attractor is contained in an n-dimensional phase-space manifold, all representations become equivalent when lifted into spaces of dimension $\mathbb{R}^N$, $N \geq 2n+1$ [12]. Every dynamical system possesses a single universal representation in these dimensions.

VI. SUMMARY

In this paper we have taken the first steps in creating a representation theory for dynamical systems. We have carried out this program for a restricted but widely occurring and very important class of dynamical systems: those whose natural phase space is the topological solid torus $D^2 \times S^1$. Three topological indices are required to distinguish inequivalent embeddings into $\mathbb{R}^3$. Parity and knot type are extrinsic indices and global torsion is an intrinsic index. When embeddings are lifted into $\mathbb{R}^4$, parity and knot type are no longer obstructions to isotopy and the global torsion is reduced from $\mathbb{Z}$ to $\mathbb{Z}_2$. When the embedding space is further enlarged to $\mathbb{R}^5$ global torsion is also no longer an obstruction to isotopy. All embeddings into $\mathbb{R}^5$ are equivalent. These results are summarized in Table I.

TABLE I. Representation labels for genus-one systems.

Representation	Obstructions to Isotopy		
Labels	$\mathbb{R}^3$	$\mathbb{R}^4$	$\mathbb{R}^5$
Global torsion	Y	Y	
Parity	Y		
Knot type	Y		

ACKNOWLEDGMENTS

This work was supported in part by the U.S. National Science Foundation under Grant No. PHY-0754081.

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