2015

The Physical Origin of Torque and of the Rotational Second Law

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(Received 17 March 2014; accepted 12 September 2014)

We derive the rotational form of Newton's second law $\tau = I\alpha$ from the translational form $\vec{F} = m\vec{a}$ by performing a force analysis of a simple body consisting of two discrete masses. Curiously, a truly rigid body model leads to an incorrect statement of the rotational second law. The failure of this model is traced to its violation of the strong form of Newton's third law. This leads us to consider a slightly modified non-rigid model that respects the third law, produces the correct rotational second law, and makes explicit the importance of the product of the tangential force with the radial distance: the torque.
[http://dx.doi.org/10.1119/1.4896574]

I. INTRODUCTION

Newton's laws of motion express how a mass will move under the influence of known forces, whether that mass is considered as an object in its own right or as a part of some larger object. In particular, the motion of a body consisting of many internal parts can be determined by applying Newton's laws to all of those parts individually. Introductory texts consider this situation and show that the same second law describes the motion of the body as a whole (more properly, its center of mass) in response to the net external force. The essential assumption is that the internal forces obey the third law, allowing them to drop out of the equations of motion for the center of mass. It is the purpose of this manuscript to determine in a similar manner the equation of motion for a rigid body about a fixed pivot (or about its center of mass)—the rotational second law.

Existing approaches to obtaining the rotational second law fall into three classes: (i) elementary: analyzing the motion of a rigid body consisting of a *single* mass in rotational motion (reproduced below); (ii) intermediate: considering rotational kinetic energy and work (real[1] or virtual[2]); and (iii) advanced: formal analysis of many-body systems utilizing vector torque and tensor moment of inertia.[3,4] The third approach is the most rigorous and general, but it is inappropriate for an introductory discussion, and it presupposes the definition of torque rather than producing it. The second approach is very nice because it is fairly elementary and manifestly model independent, but it does require familiarity with energy (out of sequence for some courses) and by completely ignoring the body's internal structure, we are prevented from tracing out how the external force affects the internal parts, somewhat obscuring the physical significance of the torque. The first approach is the simplest, but it is too simple: it neither leads unambiguously to the rotational second law nor does it demonstrate why torque is a relevant physical quantity (this is explained in more detail in the following section). However, analyzing a two-body system, and properly eliminating the internal forces from the equations of motion, unambiguously produces the rotational second law and demonstrates the physical importance of the torque.

II. THE ONE-BODY MODEL

First consider the elementary model: a single mass m is attached by a rigid massless rod of length r to a fixed pivot (see Fig. 1). The rod is essentially a model for the internal force between the mass and pivot (keeping the body rigid) and can be considered to be a very stiff spring. A force $\vec{F}$ is applied tangentially to m. The connecting rod (or spring) provides the centripetal acceleration necessary to prevent radial movement. The external force provides only tangential acceleration, and we can write $F = ma_t$, where the subscript t stands for tangential. If we multiply through with r and express the acceleration in terms of the angular acceleration $a_t = r\alpha$, we find the rotational second law

$$rF = mr^2\alpha, \tag{1}$$

where we identify $rF = \tau$ as the torque and $mr^2 = I$ as the moment of inertia.

Though it leads to the correct formula, this derivation is not terribly convincing because the final result is somewhat arbitrary. In particular, why should we multiply both sides of $F = mr\alpha$ by r to form the quantities rF and mr^2? Why isn't the torque just F and the inertia just mr? Indeed, with a single mass the translational second law suffices to describe the dynamics, so it is not at all obvious why torque even needs to be defined or why a new equation of motion needs to be constructed. At issue is how to properly generalize the equation of motion of one mass to many masses. Evidently, the proper generalization requires weighting the force by the radial distance, but this is not at all obvious in the present model. To find the proper generalization, we should consider a multi-mass model from the start.

III. THE RIGID TWO-BODY MODEL

In order to properly determine the equation of motion for a multi-body system, there should be at least two masses. So

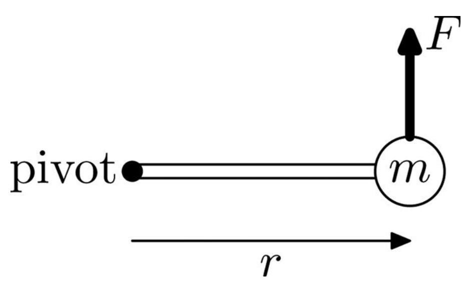

Fig. 1. A mass m attached to a pivot by a rigid massless rod (or stiff spring) of length r. The force $\vec{F}$ is applied tangentially to m, that is, perpendicular to the position vector $\vec{r}$ (displaced for clarity). Vectors are labeled by their magnitudes.

consider a body consisting of a pair of masses m_1 and m_2, at respective distances r_1 and r_2 from a pivot, connected with rigid massless rods (see Fig. 2). A force $\vec{F}$ is applied tangentially to m_2. For the body to remain rigid m_1 must also accelerate tangentially, which requires a tangential force of magnitude T from m_2, transmitted along the rod, and by the third law there must be a reaction force on m_2, which is also tangential and of magnitude T.

Writing out the second law in the tangential direction for both masses, we then find

$$m_1: \quad T = m_1 a_{1t}, \tag{2a}$$

$$m_2: \quad F - T = m_2 a_{2t}. \tag{2b}$$

Summing these equations to eliminate the internal force T and writing $a_1 = r_1\alpha$ and $a_2 = r_2\alpha$, we obtain

$$F = (m_1 r_1 + m_2 r_2)\alpha, \tag{3}$$

or, upon multiplying through by r_2,

$$r_2 F = (m_1 r_1 r_2 + m_2 r_2^2)\alpha. \tag{4}$$

Not only is the multiplication by r_2 again completely arbitrary, this equation is actually incorrect! In fact, Eq. (3) suggests that the "torque" is just F and the "inertia" mr. Something has gone horribly wrong.

IV. FAILURE OF THE RIGID MODEL: THE ROLE OF THE STRONG THIRD LAW

The idealized model employed here seems reasonable enough—just a simple generalization of the original one-body model—yet it gives an incorrect result. Somehow the forces in this model are unphysical. To see why, consider two small masses connected by a spring (see Fig. 3). Assuming that the masses start in equilibrium, it is only when they are pulled away from equilibrium that a force between them develops, and the force on each mass is directed toward (or away from) the other, along the spring axis. More precisely, if the locations of the two masses in some coordinate system are $\vec{r}_1$ and $\vec{r}_2$, then the force on each is always collinear with the relative position vector $\vec{r}_2 - \vec{r}_1$. (If in addition the force magnitude is linear in the displacement from equilibrium we obtain Hooke's law.)

If we model the interatomic forces in a rigid body as (nonlinear) springs, then in order to obtain a tangential component of force some tangential displacement of the masses is

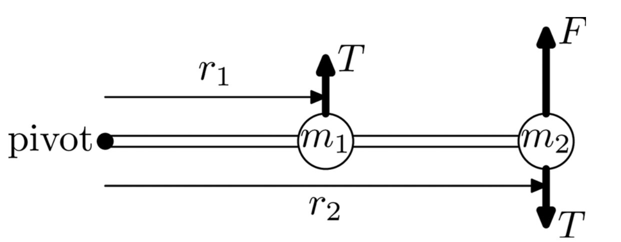

Fig. 2. A rigid body consisting of two collinear masses m_1 and m_2 at distances r_1 and r_2, respectively, from a fixed pivot, held together by rigid massless rods. An external force $\vec{F}$ is applied tangentially to m_2 (perpendicular to $\vec{r}_2$), and there is a reaction pair of tangential forces of magnitude T between the masses. Radial forces are also present but are not shown.

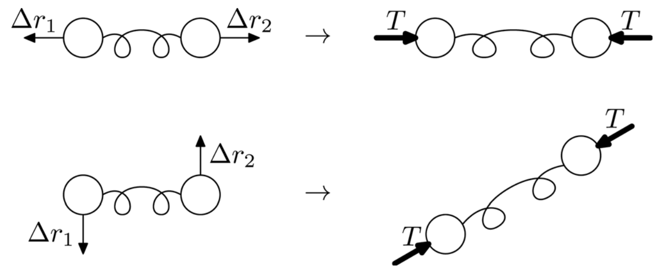

Fig. 3. Two masses connected by a spring as a model for interatomic forces. Top: if the masses are displaced directly away from each other, the resulting forces between them are directed opposite to these displacements. Bottom: if the masses are displaced tangentially, their orientation in space changes as they separate, but the resulting forces are still directed along the line joining them.

required: the masses cannot remain collinear with the pivot, and thus the body cannot remain rigid! A more physical model must allow for this kind of rearrangement of the internal parts. This complicates the analysis, but it is an essential complication.[5]

Modeling interatomic forces as (nonlinear) springs is not as arbitrary as it may seem. The essential feature of the forces in the spring model is that they obey the strong form of Newton's third law. The weak (or usual) form of Newton's third law states that when two masses interact, the forces they exert on each other are equal and opposite. This constraint on interactions is essential to the conservation of linear momentum and is, for that reason, essential to showing that the second law governs the center-of-mass motion of many-body systems.

The strong form of the third law states that when two masses interact, the forces they exert on each other, in addition to being equal and opposite, both lie along the line joining the masses.[6] This constraint is essential to the conservation of angular momentum[3,4] and, as we will see below, to obtaining the law of rotational motion for many-body systems. In fact, because the internal forces in the rigid-body model do not obey the strong third law, they form a couple that generates a self-torque and thus angular momentum.

Finally, it is worth pointing out that these constraints on forces are fundamental,[7] being intimately related to conservation laws, and thus to the translational and rotational symmetry of space, as expressed by Noether's theorem.[8] These connections, which the author will explore in greater detail in a separate manuscript, are summarized in Table I.

We conclude that the rigid two-body model is fundamentally flawed: it is not possible for the masses to simultaneously rotate, remain collinear, and respect the strong third law (or, equivalently, conserve angular momentum). That the body cannot actually remain rigid, even as an idealization, is an interesting observation in its own right. It is well known that special relativity rules out the existence of truly

Table I. Connection between symmetries, conserved quantities, and Newton's third law.

Symmetry	Conserved quantity	Third law
Translation	Linear momentum	Weak
Rotation	Angular momentum	Strong

rigid bodies: rigidity requires that the effect of a localized perturbation (an applied force) be transmitted instantaneously throughout the entire object, but relativity limits this transmission speed to the speed of light.[9,10] We stress that the present argument against rigid bodies is completely classical and independent of relativity, as it considers the spatial behavior of forces rather than their temporal behavior (in fact, we are tacitly assuming instantaneous transmission of forces).

V. THE NON-RIGID TWO-BODY MODEL

Now let us revise the model by allowing some non-rigidity, in accordance with the foregoing analysis. When the force $\vec{F}$ is first applied to m_2 it will be the sole force, so m_2 alone will accelerate, moving tangentially relative to m_1, and the two masses will no longer be collinear. It is at this point that forces develop between the masses, and they will be directed along the line joining them. Depending on the details of the force $\vec{F}$, the dynamics of the masses can be quite complicated, oscillating in both their radial and angular positions. We will assume that $\vec{F}$ is applied gently enough that excitation of internal degrees of freedom is negligible and consider the masses to be in a steady-state configuration with their angular motions characterized by a common angular velocity and acceleration (this is as close to a rigid-body model as we can get).

This steady-state configuration is illustrated in Fig. 4, with m_2 inclined at angle θ_1 relative to m_1. More precisely, if the position vectors of the masses relative to the pivot are $\vec{r}_1$ and $\vec{r}_2$, respectively, then θ_1 is the angle between $\vec{r}_1$ and $\Delta\vec{r} \equiv \vec{r}_2 - \vec{r}_1$. Each rigid rod connecting the masses can thought of as a very stiff spring that supports forces only along its axis. We will assume that the external force $\vec{F}$ remains tangential to m_2, that is, perpendicular to $\vec{r}_2$.

A force diagram for this configuration is shown in Fig. 5. The geometry is the same as in Fig. 4, with angle θ_1 between $\vec{r}_1$ and $\Delta\vec{r}$ and angle θ_2 between $\vec{r}_2$ and $\Delta\vec{r}$, though the angles are exaggerated for clarity. The force $\vec{F}$ is applied tangentially to m_2 (perpendicular to $\vec{r}_2$) and there is a reaction pair of forces between the masses collinear with $\Delta\vec{r}$ (thick arrows, unlabeled). Interestingly, even though these internal forces obey the strong form of the third law, the tangential components T_1 (on m_1) and T_2 (on m_2) do not. The reason is that, owing to the non-rigidity, $\vec{r}_1$ and $\vec{r}_2$ are no longer collinear, so the "tangential direction" is a different direction for each mass.

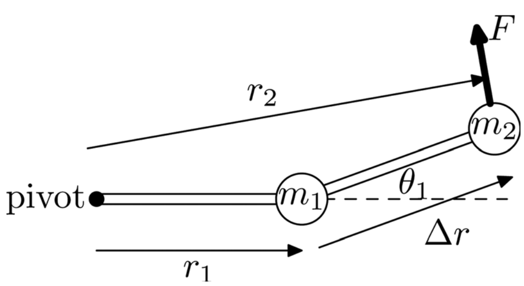

Fig. 4. In the non-rigid model, masses m_1 and m_2 are located at $\vec{r}_1$ and $\vec{r}_2$, respectively. The force $\vec{F}$ is applied tangentially to m_2 ($\vec{F} \perp \vec{r}_2$), and m_2 is inclined at angle θ_1 relative to m_1 (θ_1 is the angle between $\vec{r}_1$ and $\Delta\vec{r} \equiv \vec{r}_2 - \vec{r}_1$).

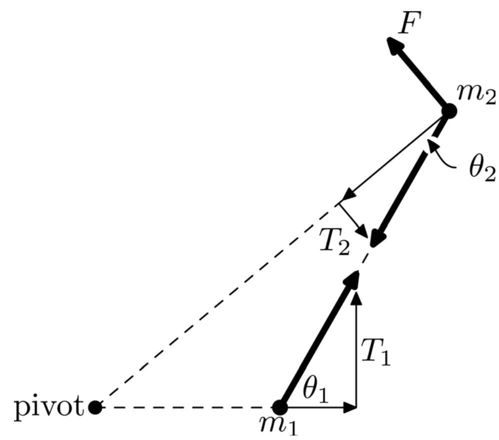

Fig. 5. Force diagram for the non-rigid two-mass body (angles exaggerated for clarity). Here, θ_1 is the angle between $\vec{r}_1$ and $\Delta\vec{r}$ and θ_2 is the angle between $\vec{r}_2$ and $\Delta\vec{r}$. The external force $\vec{F}$ and the internal forces between masses (unlabeled) are shown as bold arrows, and the latter forces are collinear with $\Delta\vec{r}$. Notice that although these internal forces obey the strong third law, the tangential components T_1 (on m_1) and T_2 (on m_2) do not.

As is evident in the diagram, not only are the tangential directions for the two masses different, the tangential force T_1 on m_1 is greater (in magnitude) than the tangential force T_2 on m_2. We now determine the exact ratio of these forces. Denote the magnitude of the internal forces by T. Then, since the components T_1 and T_2 are opposite the angles θ_1 and θ_2, we have $T_1 = T\sin\theta_1$ and $T_2 = T\sin\theta_2$, or

$$\frac{T_1}{T_2} = \frac{\sin\theta_1}{\sin\theta_2}. \tag{5}$$

Now, in the triangle formed by the pivot and the masses r_1 is opposite θ_2, while r_2 is opposite $\pi - \theta_1$, so by the law of sines

$$\frac{r_2}{r_1} = \frac{\sin(\pi - \theta_1)}{\sin\theta_2} = \frac{\sin\theta_1}{\sin\theta_2}. \tag{6}$$

Combining Eqs. (5) and (6) then yields

$$\frac{T_1}{T_2} = \frac{r_2}{r_1}, \quad \text{or} \quad T_1 r_1 = T_2 r_2. \tag{7}$$

So when the internal forces obey the strong third law, the tangential components are in inverse proportion to their radial distances from the pivot: the farther mass exerts a larger tangential force on the nearer mass. In this way, forces exerted at a greater distance from the pivot are more effective at producing rotation. These considerations show that the torque $\tau = rF\sin\theta$ (for an arbitrary force $\vec{F}$ at angle θ relative to the position vector $\vec{r}$) is an important quantity for rotational motion. Moreover, in light of Eq. (7), when two objects interact they exert equal and opposite torques on each other (conserving angular momentum), again illustrating the utility of this quantity.

Now, the tangential part of Newton's second law for the two masses reads

$$m_1: \quad T_1 = m_1 a_{1t}, \tag{8a}$$

$$m_2: \quad F - T_2 = m_2 a_{2t}. \tag{8b}$$

In order to eliminate the internal force from the equations of motion we must, in light of Eq. (7), form the quantities r_1T_1 and r_2T_2. Multiplying the first equation by r_1 and the second by r_2, and writing $a_1 = r_1\alpha$ and $a_2 = r_2\alpha$, these equations become

$$T_1r_1 = m_1r_1^2\alpha, \tag{9a}$$

$$Fr_2 - T_2r_2 = m_2r_2^2\alpha. \tag{9b}$$

Summing these equations eliminates the internal force, resulting in the correct rotational second law

$$Fr_2 = (m_1r_1^2 + m_2r_2^2)\alpha, \tag{10}$$

including the correct rotational inertia $\sum mr^2$.

VI. DISCUSSION OF THE NON-RIGID MODEL

In contrast to the one-body model, in the present case multiplication of the equations of motion by the radial distance is necessary to eliminate the internal forces, on account of Eq. (7). It is worth noting that there is nothing particularly special about having only two masses: the model can be extended to any number of masses and external forces, yielding the correct form of $\tau = I\alpha$ in all cases, with τ the total torque of all external forces and $I = \sum mr^2$ the total moment of inertia.

Finally, note that in the rigid body limit $\theta_1, \theta_2 \to 0$, the tangential components remain finite, but the force magnitude diverges: $T = T_1/\sin\theta_1 \to \infty$. The radial components also diverge. This divergence is another indication of the impossibility of a truly rigid body. It is curious that the actual values of θ_1 and θ_2 are irrelevant for determining the rotational second law, especially since their values are uniquely determined by the radial motion. If we denote by F_0 the radial force on m_1 from the pivot, then we can write down the radial equations of motion:

$$m_1: \quad T\cos\theta_1 - F_0 = m_1r_1\omega^2, \tag{11a}$$

$$m_2: \quad T\cos\theta_2 = m_2r_2\omega^2. \tag{11b}$$

The four equations of motion (two radial and two tangential) are supplemented by the constraint $r_1\sin\theta_1 = r_2\sin\theta_2$. These five equations can be used to determine the angles and forces (θ_1, θ_2, T, F, and F_0) given the other parameters. In particular, it can be shown that $\tan\theta_2 = (m_1/m_2)(r_1/r_2)^2(\alpha/\omega^2)$ and subsequently $\sin\theta_1 = (r_2/r_1)\sin\theta_2$. For $m_1 = m_2$, $r_2 = 2r_1$, and $\alpha = \omega^2$ we find $\theta_2 = 0.24$ rad (14°) and $\theta_1 = 0.51$ rad (29°). It would be an interesting exercise to determine the angle between consecutive masses for an arbitrary number $n > 2$ of masses, and in the continuum limit $n \to \infty$ and then to compare the results to experiments with masses connected by stiff springs and with thin ropes, respectively.

VII. CONCLUSION

The rotational second law is an invaluable principle, but it is not a first principle,[11] and it arises from first principles—Newton's three laws—in a somewhat nontrivial manner. To obtain a sufficiently general statement of the rotational second law, one must at least consider the dynamics of a two-mass body, but the assumption of a rigid body is inconsistent with one of these first principles (strong third law) and leads to an incorrect equation of motion [Eq. (4)]. We conclude that rigid bodies are inconsistent with classical (non-relativistic) physics. When we allow for non-rigidity, the model becomes consistent with all of Newton's laws, and the correct equation of motion [Eq. (10)] is then obtained straightforwardly. Moreover, considering the effect of non-rigidity on the tangential components of the internal forces makes evident the physical importance of the torque.

As noted in Sec. VI, the exact amount of deformation of the object (measured by the angles θ_1 and θ_2 in Fig. 5) is irrelevant for obtaining the rotational second law, but the deformation is rather large for the moderate rotational velocity and acceleration considered in the numerical example there. So while the model does give us the rotational second law, it does not approximate a "rigid body" very well. A closer approximation to a rigid body could be obtained by considering a large number of masses distributed two-dimensionally or by replacing the two point masses with concentric cylinders connected by a large number of springs. It would also be instructive in the two-mass model to determine the complete two-body dynamics, investigate the approach to steady-state behavior (with the addition of some damping terms), and put bounds on the applicability of the steady-state approximation.

The analysis of this paper also serves as a cautionary tale about modeling physical systems. Approximations and idealizations are always necessary and appropriate when constructing a model, but sometimes certain approximations run afoul of fundamental physical constraints, leading to unphysical results. The rigid-body model employed here violates the strong third law and angular momentum conservation (via Noether's theorem), leading to a completely incorrect statement of the rotational second law. What is perhaps so striking about this failure is that while the assumption of perfect rigidity leads to an incorrect result, any arbitrarily small amount of non-rigidity (that is, arbitrarily small values of θ_1 and θ_2 in Fig. 5) leads to a perfectly correct result. In this case, approximate rigidity is perfectly good, while perfect rigidity is not even approximately good.

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[1]Frederick Reif, *Understanding Basic Mechanics* (Wiley, New York, 1995).

[2]Richard P. Feynman, Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964).

[3]John R. Taylor, *Classical Mechanics* (University Science Books, Sausalito, CA, 2005).

[4]Herbert Goldstein, Charles Poole, and John Safko, *Classical Mechanics*, 3rd ed. (Addison-Wesley, San Francisco, 2002).

[5]As Einstein has advised us, "make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience," or as it has come to be pithily paraphrased: *make things as simple as possible, but not simpler*. Albert Einstein, "On the Method of Theoretical Physics," The Herbert Spencer Lecture delivered at Oxford, June 10, 1933, reprinted in Philosophy of Science **1**(2), 163–169 (1934).

[6]The strong form of the third law applies to interatomic forces, which act at a distance. At the macroscopic level, extended objects can exert contact forces on each other that are not in line with their individual centers of mass, e.g., surface friction, generating torque. When these forces respect the weak third law they will automatically respect the strong third law because both forces are exerted at the same point in space: the point of contact.

[7]An important caveat is that the electromagnetic force does not respect these symmetries in general, and mechanical momentum need not be conserved. However, including the electromagnetic field momentum restores conservation, and the resulting field theory respects spacetime symmetries and Noether's theorem.

[8]Dwight E. Neuenschwander, "Resource letter NTUC-1: Noether's theorem in the undergraduate curriculum," Am. J. Phys. **82**(3), 183–188 (2014).

[9]Max Born, "Die theorie des starren elektrons in der kinematik des relativitäsprinzips," Ann. Phys. **335**, 1–56 (1909).

[10]Robert R. Hart, "Relativistic rigid bodies and the uncertainty principle," Am. J. Phys. **33**(12), 1006–1007 (1965).

[11]It is therefore possible to analyze rotational motion using only Newton's laws. For a torque-free analysis of gyroscopic motion see Ernest F. Barker, "Elementary analysis of the gyroscope," Am. J. Phys. **28**(9), 808–810 (1960).

Murfee's Resonator

In 1912 Edward Murfee patented a device that could be used to tune pianos and measure the frequency of the sound produced by a vibrating object. It consists of a piston and cylinder device that is used as a quarter-wave resonator. A hollow piston rod leads to a pair of acoustical headphones. In the picture, the frequency of the buzzer on the table is being obtained by noting the depth of the open cylinder that gives the maximum sound; since the pressure-sensitive human ear is being used as the detector, the antinode is at the surface of the piston and the node is slightly outside the open end of the cylinder. The frequency is thus the local speed of sound divided by the wavelength. The "buzzer" is actually the make and break device used to drive a demonstration transformer. See: Thomas B. Greenslade, Jr. and David Keeports, Murfee's Resonator, *eRittenhouse*, 24 (2013) (Notes by Thomas B. Greenslade, Jr., Kenyon College)