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2010

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Daniel J. Cross
Haverford College, dcross@haverford.edu

Repository Citation

"A Schwinger Disentangling Theorem," D. J. Cross and R. Gilmore, Journal of Mathematical Physics 51, 103515 (2010).

JOURNAL OF MATHEMATICAL PHYSICS **51**, 103515 (2010)

A Schwinger disentangling theorem

Daniel J. Cross[a] and Robert Gilmore
Department of Physics, Drexel University, Philadelphia, Pennsylvania 19104, USA

(Received 19 April 2010; accepted 22 September 2010; published online 18 October 2010)

Baker–Campbell–Hausdorff formulas are exceedingly useful for disentangling operators so that they may be more easily evaluated on particular states. We present such a disentangling theorem for general bilinear and linear combinations of multiple boson creation and annihilation operators. This work generalizes a classical result of Schwinger. © *2010 American Institute of Physics.*
[doi:10.1063/1.3501027]

In a classic work,[1] Schwinger calculates the expectation value,

$$\langle\Psi_0|e^{A(a,b)}e^{B(a^\dagger,b^\dagger)}|\Psi_0\rangle = \det(I_m + BA)^{-1}. \tag{1}$$

Here, A and B are $m\times m$ skew-symmetric matrices, the m operators $a^\dagger$ and $b^\dagger$ are independent boson creation operators, the m operators a and b are independent boson annihilation operators, $|\Psi_0\rangle$ is the ground state, and $A(a,b)\equiv a^\top Ab = A_{ij}a_i b_j$ and similarly for $B(a^\dagger,b^\dagger)$. The sets of operators a and b mutually commute, and each set obeys the commutation relations $[a_\mu, a_\nu^\dagger] = \delta_{\mu\nu}$. For $m=1$, the usual angular momentum operators are recovered by $J_+ = a^\dagger b$, $J_- = b^\dagger a$, and $J_3 = (a^\dagger a - b^\dagger b)/2$.

Recently, Viskov[2,3] extended Schwinger's result, interpreting it in terms of differential operators acting on the constant function 1. In this paper we generalize both Schwinger and Viskov's results by proving a general operator disentangling theorem using a matrix Baker–Campbell–Hausdorff (BCH) approach.[4] Theorem 1 proves this result for arbitrary bilinear combinations of creation and annihilation operators, a result obtained previously by Hong-yi[5] using the technique of integration within an ordered product. Our main result, Theorem 2, extends Theorem 1 to additionally include arbitrary linear combinations of operators. Along the way we compute ground state expectation values and interpret our results in terms of differential operators in order to compare with Schwinger and Viskov, respectively.

Consider the following general bilinear combinations of creation and annihilation operators:

$$\mathbf{N} = \frac{1}{2}N\{a^\dagger,a\}, \quad \mathbf{R} = R(a^\dagger,a^\dagger), \quad \text{and} \quad \mathbf{L} = L(a,a), \tag{2}$$

where N, R, and L are $n\times n$ matrices, and $N\{a^\dagger,a\} \equiv N(a^\dagger,a) + N(a,a^\dagger)$. The same letter signifies the operator and the associated matrix, the former being in bold. Without loss of generality R and L may be assumed symmetric. These bilinear operators generate a Lie algebra whose commutation relations may be derived from those of the underlying operators.

We generalize the operator appearing in Eq. (1) to

$$\mathcal{Q} = e^{L(a,a)}e^{R(a^\dagger,a^\dagger)}, \tag{3}$$

which is represented by the expression exp $\mathbf{L}$ exp $\mathbf{R}$. This operator is antinormally ordered, meaning that the raising operations act before the lowering operations. A BCH formula may be used to

[a]Present address: Department of Physics, Bryn Mawr College, Bryn Mawr, Pennsylvania, 19010. Electronic mail: djcross@brynmawr.edu.

disentangle this operator, rewriting it in the normally ordered form $\exp \mathbf{R}' \exp \mathbf{N}' \exp \mathbf{L}'$. This is most efficiently executed using a matrix representation of the algebra.[4]

Theorem 1: *With the above definitions,* $\exp \mathbf{L} \exp \mathbf{R} = \exp \mathbf{R}' \exp \mathbf{N}' \exp \mathbf{L}'$, *where* $R' = RD_2'^{-1}$, $L' = D_2'^{-1}L$, $N' = -D_2'^{\top}$, *and* $D_2' = I_n - 4LR$.

Proof: The general bilinear operator $\mathbf{N}+\mathbf{L}+\mathbf{R}$ may be mapped onto the $2n \times 2n$ matrix,

$$\begin{pmatrix} N & 2R \\ -2L & -N^{\top} \end{pmatrix}. \tag{4}$$

This mapping provides a faithful representation of the algebra of bilinear operators in terms of the defining matrix representation of the symplectic algebra $\mathfrak{sp}(2n, \mathbb{R})$. We now solve for the primed operators, which satisfy

$$\exp\begin{pmatrix} 0 & 0 \\ -2L & 0 \end{pmatrix} \exp\begin{pmatrix} 0 & 2R \\ 0 & 0 \end{pmatrix} = \exp\begin{pmatrix} 0 & 2R' \\ 0 & 0 \end{pmatrix} \exp\begin{pmatrix} N' & 0 \\ 0 & -N'^{\top} \end{pmatrix} \exp\begin{pmatrix} 0 & 0 \\ -2L' & 0 \end{pmatrix}. \tag{5}$$

The nilpotent operators are easily exponentiated. Writing $D_1' = \exp N'$ and $D_2' = (D_1'^{\top})^{-1}$, we have

$$\begin{pmatrix} I_n & 0 \\ -2L & I_n \end{pmatrix} \begin{pmatrix} I_n & 2R \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_n & 2R' \\ 0 & I_n \end{pmatrix} \begin{pmatrix} D_1' & 0 \\ 0 & D_2' \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -2L' & I_n \end{pmatrix}, \tag{6}$$

where I_n denotes the $n \times n$ identity matrix. This yields

$$\begin{pmatrix} I_n & 2R \\ -2L & I_n - 4LR \end{pmatrix} = \begin{pmatrix} D_1' - 4R'D_2'L' & 2R'D_2' \\ -2D_2'L' & D_2' \end{pmatrix}, \tag{7}$$

from which we easily obtain

$$D_2' = I_n - 4LR \quad R' = RD_2'^{-1}, \tag{8}$$

$$D_1' = (I_n - 4RL)^{-1} \quad L' = D_2'^{-1}L. \tag{9}$$

We emphasize that these manipulations depend only on the algebraic properties of the operators and the group properties of their exponentials. These results do not depend at all on the invariant vector space on which these operators may act.[6] □

As an immediate corollary, the ground state expectation value of the operator in the Hilbert space of number states is easily evaluated.

Corollary 1: The ground-state expectation value of $\mathcal{Q}$ *is* $\langle \Psi_0 | \mathcal{Q} | \Psi_0 \rangle = \det(I_n - 4RL)^{-1/2}$.

Proof: The exponential of the lowering operator acts as the identity on $|\Psi_0\rangle$. Likewise, the exponential of the raising operator acts as the identity on $\langle \Psi_0|$. Thus,

$$\langle \Psi_0 | \mathcal{Q} | \Psi_0 \rangle = \langle \Psi_0 | \exp N'\{a^{\dagger}, a\} | \Psi_0 \rangle, \tag{10}$$

where $N' = -\ln(I_n - 4RL)$. The operator in the exponential may be normally ordered, yielding $N'(a^{\dagger}, a) + \operatorname{tr} N'/2$. These two terms commute so they may be exponentiated separately. Since the first term is normally ordered, it acts as the identity on the ground state, while the second term is a scalar. The expectation value is just this scalar,

$$\exp \operatorname{tr} N'/2 = \det(I_n - 4RL)^{-1/2}. \tag{11}$$

□

To compare with Schwinger's result, Eq. (1), set $n = 2m$, rewrite the bilinear form $A_{ij} a_i b_j$ by relabeling $b_j \to a_{j+m}$, and similarly changing the second index of A. This new A has a block structure where only the upper right block is nonzero. This should correspond to the matrix L, except that it is not symmetric. It may be symmetrized by setting

$$L=\frac{1}{2}\begin{pmatrix}0 & A\\ A^{\top} & 0\end{pmatrix}. \tag{12}$$

Define R in terms of B in the same way. Then, since

$$(I_{2m}-4RL)=\begin{pmatrix}I_m-BA^{\top} & 0\\ 0 & I_m-B^{\top}A\end{pmatrix}, \tag{13}$$

the expectation value is

$$(\det(I_m-BA^{\top})\det(I_m-B^{\top}A))^{-1/2}=\det(I_m-BA^{\top})^{-1}. \tag{14}$$

If we now assume that A and B are antisymmetric ($A^{\top}=-A$), then this immediately yields Schwinger's result. Note that Eq. (14) is valid for arbitrary A and B.

We may also evaluate the operator $\mathcal{Q}$ on the ground state $|\Psi_0\rangle$.

Corollary 2: $\mathcal{Q}|\Psi_0\rangle=\det(C)e^{CB(a^{\dagger},b^{\dagger})}|\Psi_0\rangle$, *where* $C=(I_m-BA^{\top})^{-1}$.

Proof: With the operator $\mathcal{Q}$ normally ordered, we obtain

$$e^{R'(a^{\dagger},a^{\dagger})}e^{\operatorname{tr}N'/2}e^{N'(a^{\dagger},a)}e^{L'(a,a)}|\Psi_0\rangle, \tag{15}$$

$$=e^{\operatorname{tr}N'/2}e^{R'(a^{\dagger},a^{\dagger})}|\Psi_0\rangle, \tag{16}$$

$$=\det(I_n-4RL)^{-1/2}e^{R(I_n-4LR)^{-1}(a^{\dagger},a^{\dagger})}|\Psi_0\rangle. \tag{17}$$

If we set $n=2m$, use the two sets of operators a and b, and express L and R in terms of A and B as before, then with the help of the matrix identity $X(I-YX)^{-1}=(I-XY)^{-1}X$ this state becomes

$$\det(C)e^{CB(a^{\dagger},b^{\dagger})}|\Psi_0\rangle, \tag{18}$$

where $C=(I_m-BA^{\top})^{-1}$. □

Viskov[2] interprets Eq. (1) in the following way. Replace the Hilbert space operator with a differential operator $\tilde{\mathcal{Q}}$ through the replacements $a_i^{\dagger}\to x_i$, $b_i^{\dagger}\to y_i$, $a_i\to\partial_{x_i}$, and $b_i\to\partial_{y_i}$. Then have $\tilde{\mathcal{Q}}$ act on the constant function 1 and evaluate the result at $x=y=0$. After these replacements, Schwinger's result becomes

$$\tilde{\mathcal{Q}}1|_0=e^{A(\partial_x,\partial_y)}e^{B(x,y)}1|_0=\det(I_m+BA)^{-1}, \tag{19}$$

where the notation indicates that the expression is to be evaluated at $x=y=0$. We note that acting on 1 with this operator is equivalent to operating on $|\Psi_0\rangle$ in Schwinger's case, and that evaluating at $x=y=0$ is equivalent to operating on $\langle\Psi_0|$.

Viskov uses analytical methods to generalize this result by removing the evaluation at zero and letting A and B be arbitrary. He finds

$$e^{A(\partial_x,\partial_y)}e^{B(x,y)}1=\det(C)e^{CB(x,y)}, \tag{20}$$

where $C=(I_m-BA^{\top})^{-1}$. Notice that this is precisely Eq. (18) with the replacements $a^{\dagger}\to x$, $b^{\dagger}\to y$, and $|\Psi_0\rangle\to 1$.

Our present result generalizes Viskov's. We have the general bilinear differential operator

$$\tilde{\mathcal{Q}}=e^{L(\partial_x,\partial_x)}e^{R(x,x)}. \tag{21}$$

Since x and ∂_x obey the same commutation relations as $a^{\dagger}$ and a, $\tilde{\mathcal{Q}}$ may be disentangled in exactly the same way as before. The action of $\tilde{\mathcal{Q}}$ on 1 is therefore given by Eq. (17) by making the replacements $a^{\dagger}\to x$ and $|\Psi_0\rangle\to 1$. As mentioned previously, Viskov's result is recovered from the special case of Eq. (18) through the same replacements.

Schwinger[1] generalized the expectation value of Eq. (1) to include linear combinations of creation and annihilation operators. If we define the new operator

$$\mathcal{Q} = e^{A(a,b)+\alpha(a)+\alpha'(b)}e^{B(a^\dagger,b^\dagger)+\beta(a^\dagger)+\beta'(b^\dagger)}, \tag{22}$$

where $\alpha(a) \equiv \alpha_i a_i$, then Schwinger showed that

$$\langle \Psi_0 | \mathcal{Q} | \Psi_0 \rangle = \det(C) e^{C(\alpha+\alpha',\beta+\beta')}, \tag{23}$$

where $C=(I_m+BA)^{-1}$ and $C(\alpha+\alpha',\beta+\beta')=C_{ij}(\alpha_i+\alpha_i')(\beta_j+\beta_j')$. We keep the same definitions of Eq. (2) for the bilinear operators and introduce the linear operators $\mathbf{l}=l(a)$ and $\mathbf{r}=r(b^\dagger)$, as well as the multiple of identity operator $\boldsymbol{\delta}=\delta I$, $\delta \in \mathbb{C}$. We now prove our main disentangling result.

Theorem 2: $\exp(\mathbf{L}+\mathbf{l})\exp(\mathbf{R}+\mathbf{r})=\exp(\mathbf{R}'+\mathbf{r}')\exp(\mathbf{N}'+\boldsymbol{\delta}')\exp(\mathbf{L}'+\mathbf{l}')$, *where R', L', and N' are just as in Theorem 1, Eqs. (8) and (9), and*

$$r' = D_1'(r + 2R^\top l), \tag{24}$$

$$l' = D_2'^{-1}(l + 2Lr), \tag{25}$$

$$2\delta' = (r^\top + 2l^\top R)D_2'^{-1}(l + 2Lr) + l^\top r. \tag{26}$$

Proof: The general operator $\mathbf{N}+\mathbf{L}+\mathbf{R}+\mathbf{l}+\mathbf{r}+\boldsymbol{\delta}$ may be mapped onto the $2(n+1)\times 2(n+1)$ matrix,

$$\begin{pmatrix} 0 & l^\top & r^\top & -2\delta \\ 0 & N & 2R & -r \\ 0 & -2L & -N^\top & l \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{27}$$

The operator $\exp(\mathbf{L}+\mathbf{l})\exp(\mathbf{R}+\mathbf{r})$ may be put into the normally ordered form following the same procedure as in the purely bilinear case. The details are left to the reader. □

We immediately obtain the ground state expectation value.

Corollary 3: $\langle \Psi_0 | \mathcal{Q} | \Psi_0 \rangle = \det(I_n - 4RL)^{-1/2} e^{\delta'}$.

We want to compare this result with Eq. (23). First, note that with the help of the matrix identity $X(I-YX)^{-1}Y=(I-XY)^{-1}-I$ the number δ' may be rewritten as the quadratic form,

$$D_1'(l,r) + RD_1'^\top(l,l) + D_1'^\top L(r,r). \tag{28}$$

Here, $D_1'(l,r)=(D_1')_{ij}l_i r_j$, and similarly for the other terms. To compare with Eq. (23), set $n=2m$, write L and R in terms of A and B, and write $l=(\alpha,\alpha')=(\alpha_1,\alpha_2,\ldots,\alpha_m,\alpha_1',\alpha_2',\ldots,\alpha_m')$, and similarly write $r=(\beta,\beta')$. If we assume that both A and B are antisymmetric, then the matrices $RD_1'^\top$ and $D_1'^\top L$ are also antisymmetric and their corresponding quadratic forms vanish. The remaining quadratic form becomes

$$C(\alpha,\beta) + C(\alpha',\beta') = C(\alpha + \alpha', \beta + \beta'), \tag{29}$$

with $C=(I_m+BA)^{-1}$. Since we already established that under these assumptions $\det(I_{2m} - 4RL)^{-1/2} = \det C$, we are done.

Again, we evaluate $\mathcal{Q}$ on the ground state $|\Psi_0\rangle$.

Corollary 4: $\mathcal{Q}|\Psi_0\rangle = e^{\mathrm{tr}\, N'/2+\delta'} e^{R(I_n-4LR)^{-1}(a^\dagger,a^\dagger)} e^{(I_n-4RL)^{-1}(r+2Rl)(a^\dagger)}|\Psi_0\rangle$.

If we again set $n=2m$ and write L and R in terms of antisymmetric A and B, this state becomes

$$\det(C) e^{CB(a^\dagger,b^\dagger)} e^{C(\alpha+\alpha',\beta+\beta')} e^{(C\beta+CB\alpha')(a^\dagger)} e^{(C\beta'-CB\alpha)(b^\dagger)}|\Psi_0\rangle, \tag{30}$$

and $C=(I_m+BA)^{-1}$.

We note that Viskov[3] also proved a differential operator identity similar to Corollary 4, but with no linear derivative terms. By making the usual replacements, Viskov's result may be generalized to include these linear operators as well.

This paper has proven an operator disentangling theorem for arbitrary linear and bilinear combinations of antinormally ordered boson creation and annihilation operators using a matrix BCH approach. The ground state expectation values were calculated, generalizing Schwinger's result, Eq. (1). The states obtained by applying the operators to the ground state were also calculated. The foregoing results were then reinterpreted as differential operators applied to the constant function 1, generalizing a result of Viskov, Eq. (20). Finally, using the matrix representation of the operators, Eq. (27), the present methods may be easily extended to reorder any given arrangement of linear and bilinear operators into any other.

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