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# Scatter matrix analysis of plane waves in a layered media

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SCATTER MATRIX ANALYSIS OF PLANE
WAVES IN A LAYERED MEDIA

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A

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## ABSTRACT

The problem was the application of matrix theory to the solution of the reflection and transmission coefficients for a plane wave incident on a layered media. Background material relating to the problem was reviewed. Equations that were used in the solution of the problem were either derived or explained. The derivations for the reflection and transmission coefficients were carried out in order, starting with the simple case of normal incidence on a junction of two infinite media and progressing through the more difficult case of oblique incidence on a layer of finite thickness separating two infinite media. Scatter matrix analysis for electrical networks was reviewed and was applied to the layered media. Possible simplifications of the resulting equations were considered.

## ACKNOWLEDGEMENT

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# LIST OF SYMBOLS

$\bar{i}, \bar{j}, \bar{k}$ -- unit vectors in the x, y, and z directions, respectively.

$\omega$ -- angular frequency in radians/second.

$j$ -- $\sqrt{-1}$

$d$ -- thickness of a layer.

$e$ -- base of natural logarithms.

$K_d$ -- dielectric constant of a layer.

$N$ -- any integer.

$\tau$ -- transmission coefficient.

$\rho$ -- reflection coefficient.

$\gamma$ -- complex propagation constant.

$\alpha$ -- real component of $\gamma$.

$\beta$ -- imaginary component of $\gamma$.

$S_{ij}$ -- element of a scatter matrix.

$t_{ij}$ -- element of a transmission matrix.

$T$ -- a transmission matrix.

$S$ -- a scatter matrix.

$\bar{T}$ -- the over-all transmission matrix for networks in cascade.

$\bar{S}$ -- the over-all scatter matrix for networks in cascade.

$\Delta S, \Delta T$ -- the determinants of the scatter and transmission matrices, respectively.

$\mu^*$ -- complex permeability

$\mu'$ -- real part of $\mu^*$.

$\mu''$ -- imaginary part of $\mu^*$.

$\epsilon^*$ -- complex permittivity.

$\epsilon'$ -- real part of $\epsilon^*$.

$\epsilon''$ -- imaginary part of $\epsilon^*$.

$\mu_e^*$ -- effective value of the complex permeability for an anisotropic media.

$\epsilon_e^*$ -- effective value of the complex permittivity for an anisotropic media.

$\bar{E}, \bar{H}$ -- instantaneous total electric and magnetic fields.

$E, H$ -- magnitude of total fields.

$\bar{E}_o, \bar{H}_o$ -- instantaneous incident electric and magnetic fields.

$E_o, H_o$ -- magnitude of incident fields.

$\bar{E}_1, \bar{H}_1$ -- instantaneous reflected electric and magnetic fields.

$E_1, H_1$ -- magnitude of reflected fields.

$\bar{E}_2, \bar{H}_2$ -- instantaneous transmitted electric and magnetic fields.

$E_2, H_2$ -- magnitude of transmitted fields.

$E_t, H_t$ -- magnitude of the total fields tangent to a boundry.

$E_{ot}, H_{ot}$ -- magnitude of the tangential component of the incident field.

$\eta$ -- intrinsic impedance of a layer

$Z$ -- wave impedance of a layer for oblique incidence.

$Z_{LN}$ -- load impedance on the $N^{th}$ layer.

$Z_{INN}$ -- input impedance to the $N^{th}$ layer.

# CHAPTER I

# INTRODUCTION

## I. THE PROBLEM

<u>Statement of the problem</u>. The purpose of this study was (1) to review the present methods of determining the transmission and reflection coefficients for plane wave transmission through a layered media, and (2) to apply matrix theory, the scatter matrix in particular, to the problem. A layered media is shown in Figure 1. For this problem, the layers were oriented with the boundries perpendicular to the z direction. The solution was to apply to layers with any combination of dielectric, magnetic, or conductive losses.

<u>Importance of the study</u>. Many methods of solving the reflection and transmission coefficients for layered media have been presented. These methods, however, require the continued re-use of complicated transmission equations or graphical aids, such as the Smith chart or nomographs. It was the intention of the author to express the solution in matrix form, in order that the matrices could be solved with the aid of computers. This would greatly reduce the work required in layered media calculations.

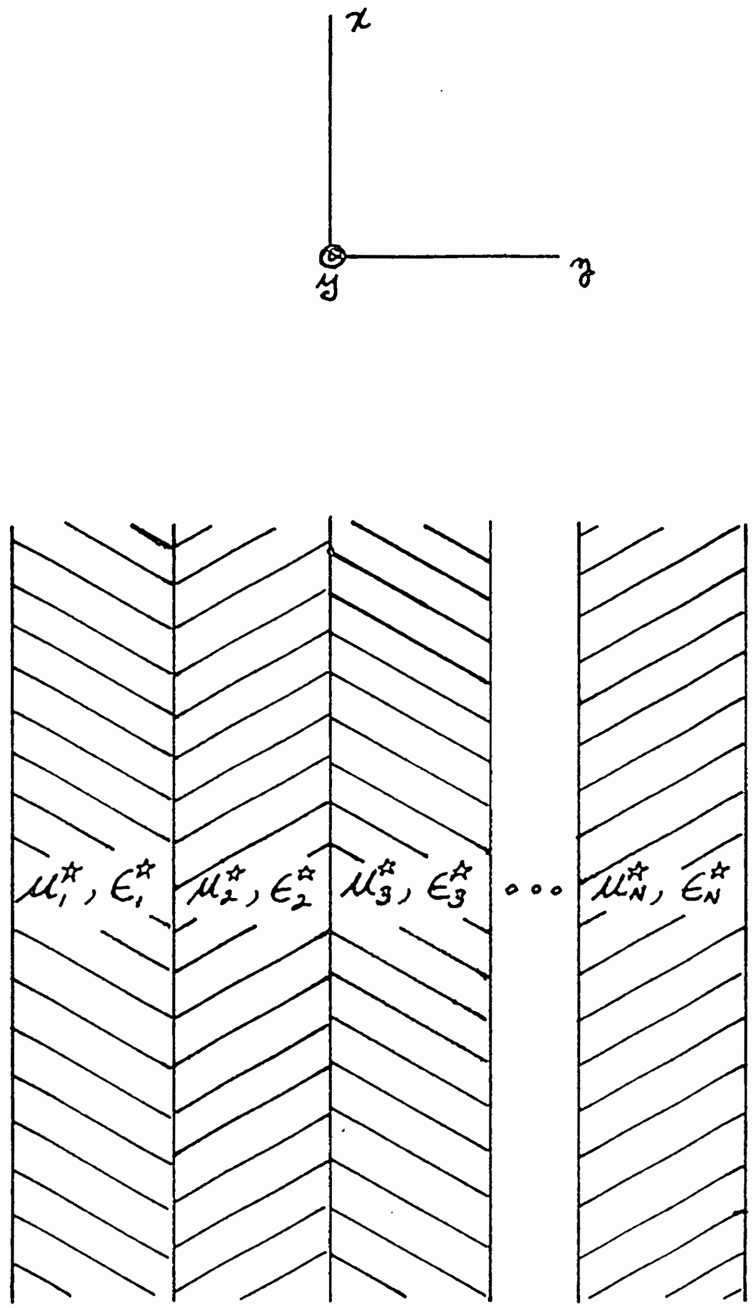


FIGURE 1

A LAYERED MEDIA

## II. SCOPE OF THE INVESTIGATION

The study made was entirely theoretical. Background equations were derived whenever it was believed that a derivation was necessary in the understanding of the use of the equation in the solution of the problem. The application of matrix theory to the problem was limited to the scatter and transmission matrices. Special cases of the layered media problem were considered with the intention of simplifying the resulting equations.

# CHAPTER II

## REVIEW OF LITERATURE

The methods presented, in the literature, for the analysis of wave transmission through a layered media have been varied. Some authors have considered only perfect (lossless) dielectric layers. Some have considered layers which have only conductive dielectric losses (which require the use of complex permittivity). Still others have analyzed layers which have both a complex permittivity and a complex permeability (which accounts for magnetic losses other than those resulting from hysteresis and eddy currents).

Most authors, including Von Hippel[1], Stratton[2], and Brekhovskikh[3], give the reflection and transmission coefficients at the junction of two infinite media in terms of the wave impedances of the media and the angles of incidence and refraction. These equations are well known as Fresnel's equations.

Brekhovskikh and Ramo[4] have found the reflection and transmission coefficients at a layer of finite thickness in terms of the input impedance to the layer. The solution of the input impedance requires the use of a

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[1]All references are in the bibliography

transmission equation in which all terms are complex except the thickness of the layer. The input impedance was recognized as the load on the source side of the layer and the reflection and transmission coefficients could then be calculated.

When many layers are cascaded, the problem of determining the reflection and transmission coefficients increases. Ramo gives a method by which one would start at the last ($n^{th}$) layer of the media and find the input impedance. This impedance was recognized as the load on the $n - 1$ layer. The input impedance to the $n - 1$ layer would be calculated and become the load impedance on the $n - 2$ layer. This process would be repeated until the input impedance to the first layer was found. The reflection and transmission coefficients could then be determined. Each step would require the solution of the transmission equation.

Of course, the steps in the solution as presented by Ramo could be greatly simplified by the use of a Smith chart or a nomograph which can be found in an article by Cafferata[5]. Although Cafferata's nomograph was intended for the purpose of calculating the input impedance of feeders and cables terminated in a complex load, it applies equally well to the case of a layered media.

Brekhovskikh has given one other possible solution to the problem. The equations for the tangential components of the electric and magnetic fields in the $j^{th}$ region are written in terms of the incident and reflected magnitudes. By letting $j$ range from 1 to $n$ (where $n$ is the number of layers) $2n$ equations would be formed. These equations could be solved for the reflected magnitude at the 1st layer and the transmitted magnitude passing through the $n^{th}$ layer. However, when more than two or three layers are present this method becomes so unwieldy as to render it useless.

In the investigation of the problem no material was found on the use of the scatter matrix in layered media calculations.

## CHAPTER III

## PRESENTATION OF BACKGROUND THEORY

In this chapter all equations used in the solution of the problem were either derived or explained. It was believed that the derivations were necessary for a complete understanding of the steps to be taken later in the body of the thesis.

Starting with Maxwell's equations for a media with complex permeability and complex permittivity, the wave equation was derived and a solution of the wave equation was given. The expressions for the reflection and transmission coefficients were derived starting with the simple case of normal incidence on a junction of two infinite media and progressing through the more difficult solution for oblique incidence on a layer of finite thickness separating two infinite media.

### I. DERIVATION OF THE WAVE EQUATION FOR A MEDIUM OF COMPLEX PERMEABILITY AND COMPLEX PERMITTIVITY

The wave equation was derived for propagation in a media which has both conductive and magnetic losses. This was accomplished by starting with Maxwell's equations given by Von Hippel as

$$\nabla \times \bar{E} = -\mu^{*} \frac{\partial \bar{H}}{\partial t}$$

and

$$\nabla \times \bar{H} = \epsilon^{*} \frac{\partial \bar{E}}{\partial t}$$

where

$$\mu^{*} = \mu' - j\,\mu''$$

and

$$\epsilon^{*} = \epsilon' - j\,\epsilon''$$

In the above equations $\mu^{*}$ and $\epsilon^{*}$ are the complex permeability and complex permittivity, respectively. The imaginary components of $\mu^{*}$ and $\epsilon^{*}$ are added so that the magnetic and dielectric and conductive losses can be taken into account in the solution.

For this problem it was assumed that only the x component of electric field and y component of magnetic field were present as shown in Figure 2. This represents a plane wave traveling in the +z direction.

For this problem, then,

$$\bar{E} = \bar{i}\,E_x$$

$$\bar{H} = \bar{j}\,H_y$$

where

$$E_x = f(z, t)$$

$$H_y = g(z, t)$$

Taking the curl of $\bar{E}$ gives

$$\nabla \times \bar{E} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \bar{j}\,\frac{\partial E_x}{\partial z} - \bar{k}\,\frac{\partial E_x}{\partial y}$$

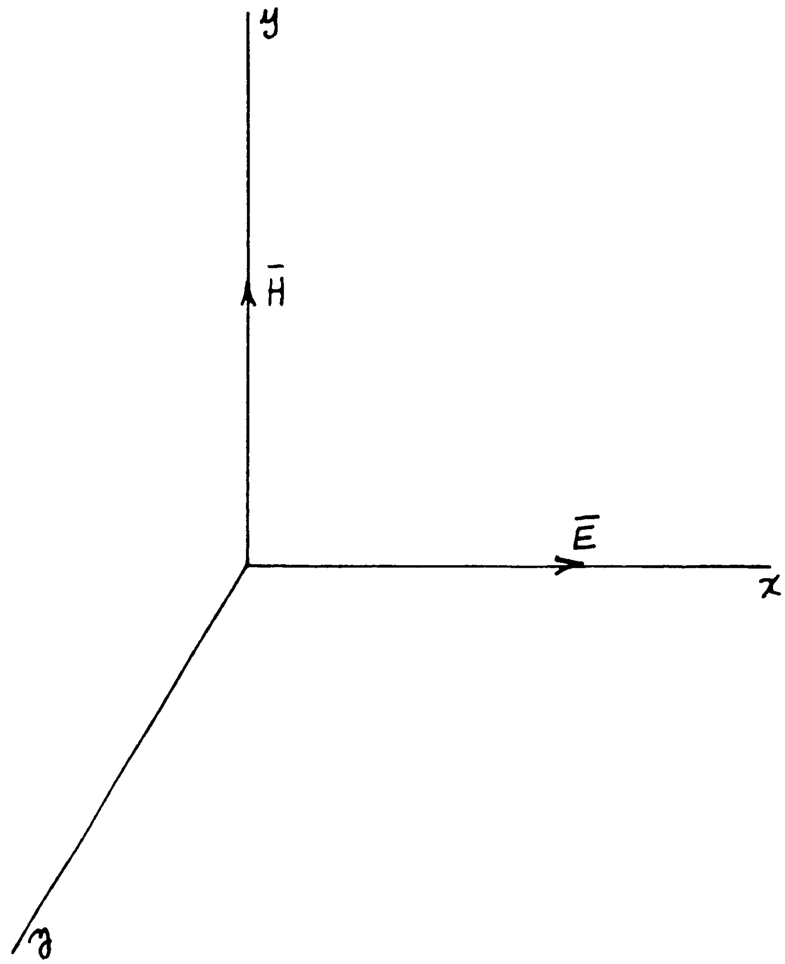


FIGURE 2

ELECTRIC AND MAGNETIC FIELD COMPONENTS
OF PLANE WAVE TRAVELING IN THE +Z DIRECTION

But

$$\nabla \times \bar{E} = -\mu^{*} \frac{\partial \bar{H}}{\partial t} = -\mu^{*} \frac{\partial}{\partial t} \left( \bar{i}\, H_x + \bar{j}\, H_y + \bar{k}\, H_z \right)$$

Equating expressions

$$\bar{j} \frac{\partial E_x}{\partial z} - \bar{k} \frac{\partial E_x}{\partial y} = -\mu^{*} \frac{\partial}{\partial t} \left( \bar{i}\, H_x + \bar{j}\, H_y + \bar{k}\, H_z \right)$$

But $E_x$ does not vary with y and $H_x = H_y = 0$. Then

$$\bar{j} \frac{\partial E_x}{\partial z} = -\bar{j} \mu^{*} \frac{\partial H_y}{\partial t}$$

or

$$\frac{\partial E_x}{\partial z} = -\mu^{*} \frac{\partial H_y}{\partial t} \qquad (1)$$

From the curl of $\bar{H}$

$$\nabla \times \bar{H} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} = -\bar{i} \frac{\partial H_y}{\partial z} + \bar{k} \frac{\partial H_y}{\partial x}$$

also

$$\nabla \times \bar{H} = \epsilon^{*} \frac{\partial \bar{E}}{\partial t} = \epsilon^{*} \frac{\partial}{\partial t} \left( \bar{i}\, E_x + \bar{j}\, E_y + \bar{k}\, E_z \right)$$

But $H_y$ does not vary with x and $E_x = E_z = 0$. Therefore,

$$-\bar{i} \frac{\partial H_y}{\partial z} = \bar{i} \epsilon^{*} \frac{\partial E_x}{\partial t}$$

or

$$\frac{\partial H_y}{\partial z} = -\epsilon^{*} \frac{\partial E_x}{\partial t} \qquad (2)$$

From (1)

$$\frac{\partial}{\partial z} \left( \frac{\partial E_x}{\partial z} \right) = -\mu^{*} \frac{\partial}{\partial z} \left( \frac{\partial H_y}{\partial t} \right)$$

or

$$\frac{d^2 E_x}{dy^2} = -\mu^* \frac{d^2 H_y}{dt\, dy} \qquad (3)$$

From (2)

$$\frac{d}{dt}\left(\frac{dH_y}{dy}\right) = -\epsilon^* \frac{d}{dt}\left(\frac{dE_x}{dt}\right)$$

or

$$\frac{d^2 H_y}{dt\, dy} = -\epsilon^* \frac{d^2 E_x}{dt^2} \qquad (4)$$

Equating the left side of (4) to the right side of (3) yields

$$\frac{d^2 E_x}{dy^2} = \mu^* \epsilon^* \frac{d^2 E_x}{dt^2} \qquad (5)$$

This is the wave equation for the electric field. A similar procedure would give the wave equation for the magnetic field in the form

$$\frac{d^2 H_y}{dy^2} = \mu^* \epsilon^* \frac{d^2 H_y}{dt^2} \qquad (6)$$

## II. SOLUTION OF THE WAVE EQUATIONS IN A SINGLE LAYER CONTAINING COMPLEX PERMEABILITY AND COMPLEX PERMITTIVITY

The solution for the electric and magnetic field components for a plane wave were obtained from wave equations (5) and (6). It was assumed the field components had a time variation given by $e^{j\omega t}$.

Since the time variation of the field is known, wave equation (5) for the electric field can be written as

$$\frac{\partial^2 E_x}{\partial z^2} = -\omega^2 \mu^* \epsilon^* E_x \tag{7}$$

This can be rewritten as

$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0 \tag{8}$$

where

$$\gamma = j\omega\sqrt{\mu^* \epsilon^*}$$

The solution to (8) can easily be determined as

$$E_x = E_0 e^{-\gamma z} + E_1 e^{\gamma z} \tag{9}$$

It is understood that $E_x$ varies with time through $e^{j\omega t}$.

Equation (9) contains two terms, the first representing an "incident" wave traveling in the +z direction and the second term representing a "reflected" wave traveling in the -z direction.

The term, $\gamma$, is seen to be complex and therefore affects the mangitude and phase of the wave, the magnitude of the effect being dependent on the angular frequency $\omega$ and the characteristics of the media $\mu^*$ and $\epsilon^*$. This term has been named the "propagation constant".

If there were no change in the media through which the wave is propagating, there would be no reflected wave and $E_x = E_0 e^{-\gamma z}$. (10)

A solution for $H_x$ can be found by the use of (10) and (1) as shown in the steps below.

$$\frac{dE_x}{dz} = -\mu^* \frac{dH_y}{dt}$$

$$\frac{d}{dz}\left(E_o e^{-\gamma z}\right) = -j\omega\mu^* H_y$$

$$-\gamma E_o e^{-\gamma z} = -j\omega\mu^* H_y$$

$$\gamma E_x = j\omega\mu^* H_y$$

$$H_y = \frac{\gamma E_x}{j\omega\mu^*} = \frac{j\omega\sqrt{\mu^*\epsilon^*}\, E_x}{j\omega\mu^*}$$

or

$$H_y = \sqrt{\epsilon^*/\mu^*}\; E_x \qquad (11)$$

## III. SOLUTION FOR INTRINSIC IMPEDANCE

Since the ratio of electric field intensity to magnetic field intensity has the units of impedance, the intrinsic impedance of the media is defined as

$$\eta = \frac{E_x}{H_y} = \frac{E_x}{\sqrt{\epsilon^*/\mu^*}} = \sqrt{\mu^*/\epsilon^*} \qquad (12)$$

## IV. DERIVATION OF REFLECTION AND TRANSMISSION COEFFICIENTS AT THE JUNCTION OF TWO INFINITE MEDIA FOR NORMAL INCIDENCE OF A PLANE WAVE

Figure 3 shows the conditions for a plane wave incident normally on the junction of two infinite media.

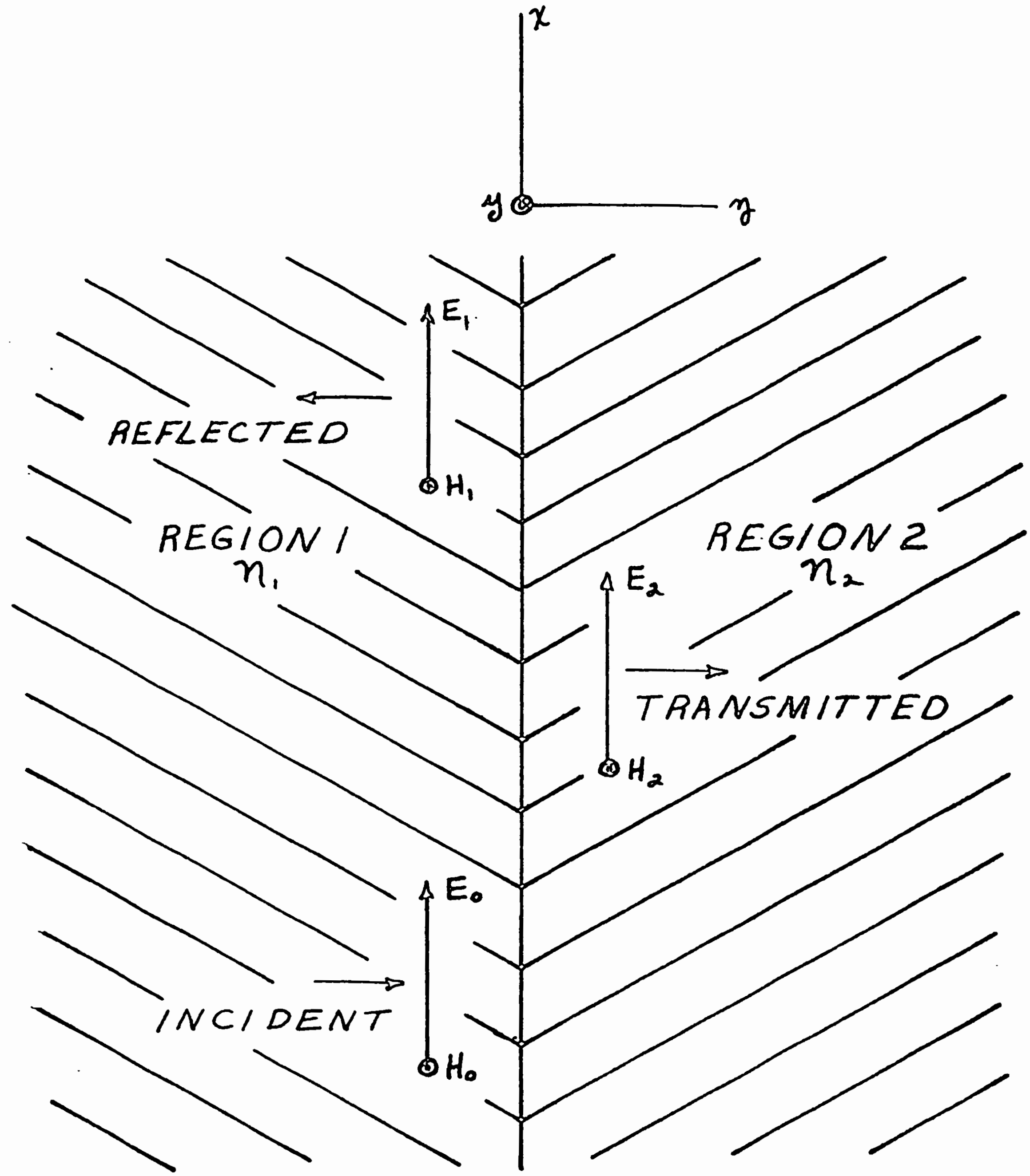


FIGURE 3

PLANE WAVE INCIDENT NORMALLY ON BOUNDRY

In this figure the subscripts 0, 1, and 2 indicate the incident, reflected and transmitted waves, respectively. The electric fields are shown in the x direction and the magnetic fields are shown in the y direction. From the continuity of tangential components at a boundry, the following equations may be obtained.

$$E_0 + E_1 = E_2 \tag{13}$$

and

$$H_0 + H_1 = H_2 \tag{14}$$

But

$$\frac{E_0}{H_0} = \eta_1 \,, \quad \frac{E_1}{H_1} = -\eta_1 \,, \quad \frac{E_2}{H_2} = \eta_2 \tag{15}$$

From (14)

$$H_2 = H_0 + H_1$$

$$\frac{E_2}{\eta_2} = \frac{E_0}{\eta_1} - \frac{E_1}{\eta_1} \tag{16}$$

Multiplying both sides of (16) by $\eta_2$ gives

$$E_2 = \frac{\eta_2}{\eta_1} E_0 - \frac{\eta_2}{\eta_1} E_1 \tag{17}$$

Multiplying (13) by $\frac{\eta_2}{\eta_1}$ yields

$$\frac{\eta_2}{\eta_1} E_2 = \frac{\eta_2}{\eta_1} E_0 + \frac{\eta_2}{\eta_1} E_1 \tag{18}$$

Adding (17) and (18) gives

$$E_2\left(1 + \frac{\eta_2}{\eta_1}\right) = \frac{2\eta_2}{\eta_1} E_0$$

or

$$E_2 = \frac{2\eta_2}{\eta_2+\eta_1} E_0 = \tau E_0 \tag{19}$$

where $\tau$ is the transmission coefficient and is given by

$$\tau = \frac{2\eta_2}{\eta_2+\eta_1} \tag{20}$$

Subtracting (17) from (18)

$$\left(\frac{\eta_2}{\eta_1} - 1\right) E_2 = \frac{2\eta_2}{\eta_1} E_1 \tag{21}$$

Substituting $E_2$ from (19) into (21) gives

$$\left(\frac{\eta_2}{\eta_1} - 1\right) \frac{2\eta_2}{\eta_2+\eta_1} E_0 = \frac{2\eta_2}{\eta_1} E_1 \tag{22}$$

Equation (22) reduces easily to the form

$$E_1 = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} E_0 = \rho E_0 \tag{23}$$

where $\rho$ is the reflection coefficient and is given by

$$\rho = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} \tag{24}$$

## V. DERIVATION OF THE INPUT IMPEDANCE TO A MEDIA OF FINITE THICKNESS

The electric and magnetic field equations at the back (load) side of the layer were found and transferred back towards the front (source) side of the layer, a distance equal to the thickness of the layer. The input impedance was then found by forming the ratio of electric and magnetic fields at that point.

Figure 4 was used in the following derivation of this input impedance.

The electric field in region 2 is determined from (9) as

$$E_x = E_0 e^{-\gamma_2 z} + E_1 e^{\gamma_2 z} \tag{25}$$

From (15)

$$\frac{E_0}{H_0} = \eta_2$$

$$\frac{E_1}{H_1} = -\eta_2$$

By the use of (15) and (25) the expression for the magnetic field in region 2 is found to be

$$H_y = \frac{E_0}{\eta_2} e^{-\gamma_2 z} - \frac{E_1}{\eta_2} e^{\gamma_2 z} \tag{26}$$

But from (23)

$$E_1 = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} E_0$$

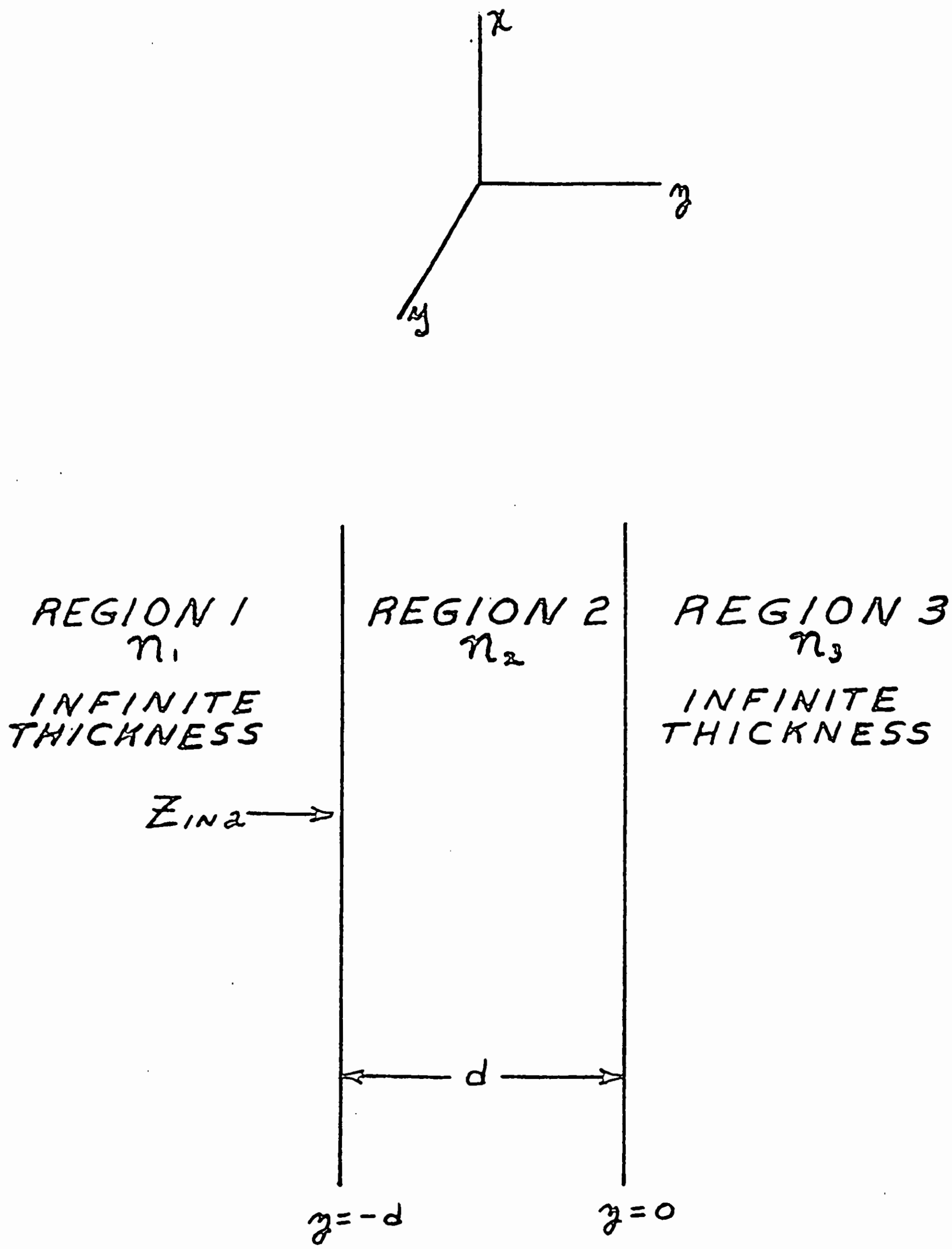


FIGURE 4

FINITE LAYER SEPARATING TWO INFINITE MEDIA

Equations (25) and (26) can now be written as

$$E_x = E_0\left[e^{-\gamma_2 z} + \left(\frac{\eta_3-\eta_2}{\eta_3+\eta_2}\right)e^{\gamma_2 z}\right] \tag{27}$$

and

$$H_y = \frac{E_0}{\eta_2}\left[e^{-\gamma_2 z} - \left(\frac{\eta_3-\eta_2}{\eta_3+\eta_2}\right)e^{\gamma_2 z}\right] \tag{28}$$

To find the expressions for $E_x$ and $H_y$ at the junction of regions 1 and 2, z is set equal to - d in (27) and (28).

$$E_x\Big|_{z=-d} = E_0\left[e^{\gamma_2 d} + \left(\frac{\eta_3-\eta_2}{\eta_3+\eta_2}\right)e^{-\gamma_2 d}\right] \tag{29}$$

$$H_y\Big|_{z=-d} = \frac{E_0}{\eta_2}\left[e^{\gamma_2 d} - \left(\frac{\eta_3-\eta_2}{\eta_3+\eta_2}\right)e^{-\gamma_2 d}\right] \tag{30}$$

Since the tangential components are continuous at a boundry, the above expressions hold, also, immediately to the left of the 1 - 2 junction. The ratio of $E_x$ to $H_y$ at z = - d is the impedance the wave "sees" at this junction and is, therefore, the input impedance to region 2.

$$Z_{IN2} = \frac{E_x}{H_y}\Big|_{z=-d} = \frac{e^{\gamma_2 d} + \left(\frac{\eta_3-\eta_2}{\eta_3+\eta_2}\right)e^{-\gamma_2 d}}{e^{\gamma_2 d} - \left(\frac{\eta_3-\eta_2}{\eta_3+\eta_2}\right)e^{-\gamma_2 d}} \tag{31}$$

This reduces to the form

$$Z_{IN2} = \eta_2 \left[ \frac{\eta_3 \left( e^{\gamma_2 d} + e^{-\gamma_2 d} \right) + \eta_2 \left( e^{\gamma_2 d} - e^{-\gamma_2 d} \right)}{\eta_2 \left( e^{\gamma_2 d} + e^{-\gamma_2 d} \right) + \eta_3 \left( e^{\gamma_2 d} - e^{-\gamma_2 d} \right)} \right] \qquad (32)$$

Dividing top and bottom of (32) by 2 gives

$$Z_{IN2} = \eta_2 \left[ \frac{\eta_3 \left( \frac{e^{\gamma_2 d} + e^{-\gamma_2 d}}{2} \right) + \eta_2 \left( \frac{e^{\gamma_2 d} - e^{-\gamma_2 d}}{2} \right)}{\eta_2 \left( \frac{e^{\gamma_2 d} + e^{-\gamma_2 d}}{2} \right) + \eta_3 \left( \frac{e^{\gamma_2 d} - e^{-\gamma_2 d}}{2} \right)} \right] \qquad (33)$$

From the identities

$$\cosh u = \frac{e^{u} + e^{-u}}{2}$$

$$\sinh u = \frac{e^{u} - e^{-u}}{2}$$

equation (33) can be written in the form

$$Z_{IN2} = \eta_2 \left[ \frac{\eta_3 \cosh \gamma_2 d + \eta_2 \sinh \gamma_2 d}{\eta_2 \cosh \gamma_2 d + \eta_3 \sinh \gamma_2 d} \right] \qquad (34)$$

Recognizing that $\eta_3$ is the load on region 2, the input impedance to the layer can be put in the desired form

$$Z_{IN2} = \eta_2 \left[ \frac{Z_L \cosh \gamma_2 d + \eta_2 \sinh \gamma_2 d}{\eta_2 \cosh \gamma_2 d + Z_L \sinh \gamma_2 d} \right] \qquad (35)$$

## VI. REFLECTION AND TRANSMISSION COEFFICIENTS AT FRONT FACE OF A FINITE LAYER SEPARATING TWO INFINITE MEDIA FOR NORMAL INCIDENCE

With the use of equation (35) it was possible to determine the expressions for the reflection and transmission coefficients under conditions of normal incidence upon a finite layer separating two infinite media.

First, equations (20) and (24) can be rewritten as

$$\tau = \frac{2Z_L}{Z_L + \eta_1} \tag{36}$$

and

$$\rho = \frac{Z_L - \eta_1}{Z_L + \eta_1} \tag{37}$$

Since $\eta_2$ is the load on region 1, this makes the equations more general and will allow the use of the equations whenever the load impedance on region 1 can be determined.

In the previous section, it was explained that $Z_{in2} = Z_{L1}$, since the ratio of the tangential electric field to tangential magnetic field is the same immediately to the right or left of the junction.

Realizing this, the reflection and transmission coefficients can be found by solving for $Z_{in2}$ from (35) and using this result in equations (36) and (37) as shown below.

$$Z_{L1} = Z_{IN2} = \eta_2 \left[ \frac{Z_{L2} \cosh \gamma_2 d + \eta_2 \sinh \gamma_2 d}{\eta_2 \cosh \gamma_2 d + Z_{L2} \sinh \gamma_2 d} \right] \tag{38}$$

$$\tau = \frac{2 Z_{L1}}{Z_{L1} + \eta_1} \tag{39}$$

$$\rho = \frac{Z_{L1} - \eta_1}{Z_{L1} + \eta_1} \tag{40}$$

## VII. REFLECTION AND TRANSMISSION COEFFICIENTS AT THE JUNCTION OF TWO INFINITE MEDIA FOR OBLIQUE INCIDENCE

A new term, the wave impedance for oblique incidence, was defined and proved useful in the determination of the reflection and transmission coefficients for oblique incidence. The reflection and transmission coefficients were derived for two polarizations of the incident wave; that is, with the electric field in the plane of incidence and with the electric field perpendicular to the plane of incidence.

Equations (39) and (40) apply, as written, only for normal incidence. These equations can be modified to apply in the case of oblique incidence by considering the tangential components of electric and magnetic fields only.

First, the wave impedance for oblique incidence is defined as the ratio of electric to magnetic field components parallel to the boundry. The reason for this is the continuity of the tangential components at the boundry and, therefore, the equality of the ratio of the tangential electric field to tangential magnetic field on either side of the boundry. Then, if this ratio is computed as the input impedance to the region on the right of the boundry, it is also the load impedance on the region to the left of the boundry. The wave impedance is defined as

$$Z = \frac{E_t}{H_t}$$

Figure 5(a) shows a plane wave incident obliquely on a boundry. The wave is polarized with the electric field in the plane of incidence. The angle of incidence can be shown to be equal to the angle of reflection as indicated in Figure 5. The tangential components of electric and magnetic fields are

$$E_{ot} = E_o \cos \theta_1$$
$$E_{1t} = E_1 \cos \theta_1$$
$$E_{2t} = E_2 \cos \theta_2$$
$$H_{ot} = H_o$$
$$H_{1t} = H_1$$
$$H_{2t} = H_2$$

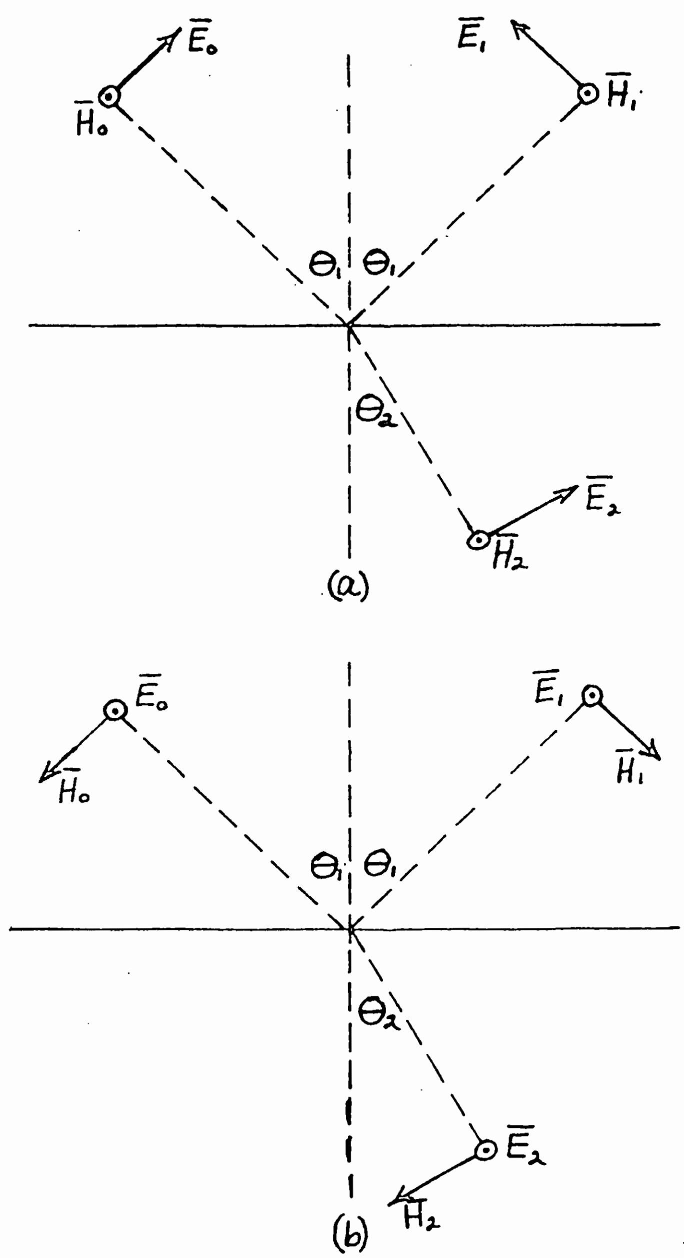


FIGURE 5

PLANE WAVE INCIDENT ON A BOUNDRY

(a) Polarized with the electric field in the plane of incidence
(b) Polarized with the electric field perpendicular to the plane of incidence

The relationships between the tangential components of electric and magnetic fields in the incident, reflected and transmitted are shown by Ramo to be

$$\left.\begin{aligned} \frac{E_{o}t}{H_{o}t} &= Z_1 = \eta_1 \cos\theta_1 \\ \frac{E_{1}t}{H_{1}t} &= -Z_1 = -\eta_1 \cos\theta_1 \\ \frac{E_{2}t}{H_{2}t} &= Z_2 = \eta_2 \cos\theta_2 \end{aligned}\right\} \qquad (41)$$

Figure 5(b) shows a plane wave incident upon a boundry. The wave is polarized with the electric field perpendicular to the plane of incidence. The tangential components of electric and magnetic field are

$$\begin{aligned} E_{o}t &= E_o \\ E_{1}t &= E_1 \\ E_{2}t &= E_2 \\ H_{o}t &= H_o \cos\theta_1 \\ H_{1}t &= H_1 \cos\theta_1 \\ H_{2}t &= H_2 \cos\theta_2 \end{aligned}$$

The relationships between the tangential components of the incident, reflected and transmitted waves are

$$\left.\begin{aligned} \frac{E_{o}t}{H_{o}t} &= Z_1 = \eta_1 \sec\theta_1 \\ \frac{E_{1}t}{H_{1}t} &= -Z_1 = -\eta_1 \sec\theta_1 \\ \frac{E_{2}t}{H_{2}t} &= Z_2 = \eta_2 \sec\theta_2 \end{aligned}\right\} \qquad (42)$$

With the wave impedance for oblique incidence so defined, the reflection and transmission coefficients for oblique incidence can be derived.

Figure 6 represents a plane wave incident obliquely on a boundry. No directions for the electric field are shown so that the solution will be valid for either type of polarization considered. The sum of the incident and reflected waves in media 1 may be written as

$$\bar{E} = \bar{E}_o e^{-\gamma_1 \delta} + \bar{E}_1 e^{-\gamma_1 \delta'} \tag{43}$$

where $\bar{E}_o$ and $\bar{E}_1$ are reference values at the orgin. However, it is desired to express equation (43) in terms of x, y, z coordinates. The following conversions are seen to apply:

$$\delta = x \sin\theta_1 + z \cos\theta_1$$

and

$$\delta' = x \sin\theta_1 - z \cos\theta_1$$

Equation (43) may now be written as

$$\bar{E} = \bar{E}_o e^{-\gamma_1 (x \sin\theta_1 + z \cos\theta_1)} + \bar{E}_1 e^{-\gamma_1 (x \sin\theta_1 - z \cos\theta_1)} \tag{44}$$

The tangential component of the electric field in media 1 is

$$E_t = E_{ot} e^{-\gamma_1 (x \sin\theta_1 + z \cos\theta_1)} + \rho E_{ot} e^{-\gamma_1 (x \sin\theta_1 - z \cos\theta_1)}$$

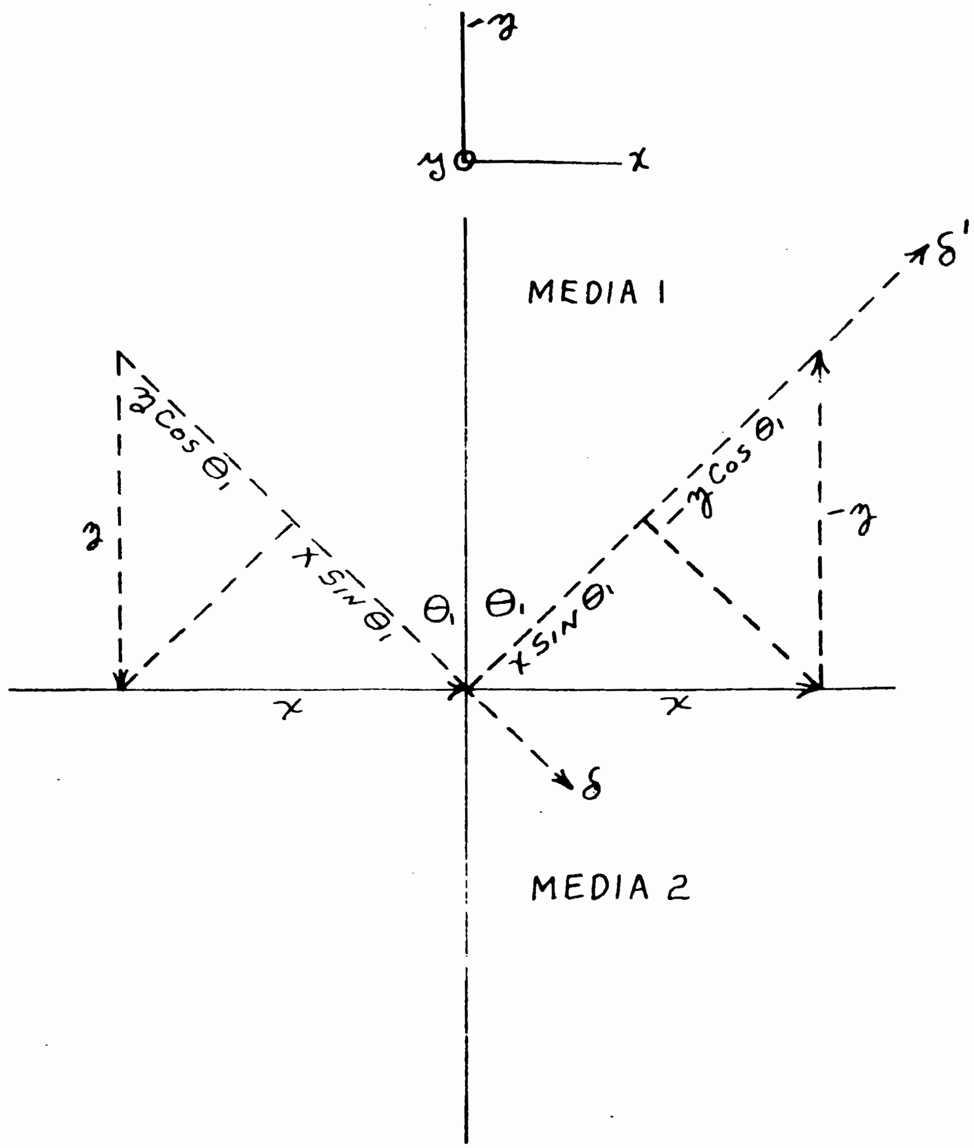


FIGURE 6

OBLIQUE INCIDENCE ON A BOUNDRY

or

$$E_t = E_{ot}\left[e^{-\gamma_1 z \cos\theta_1} + \rho e^{\gamma_1 z \cos\theta_1}\right] e^{-\gamma_1 x \sin\theta_1} \qquad (45)$$

where $E_{ot}$ will equal $E_o$ or $E_o \cos\theta_1$ depending on the polarization of the wave. At z = 0 (45) becomes

$$E_t = E_{ot}\left(1 + \rho\right) e^{-\gamma_1 x \sin\theta_1} \qquad (46)$$

The expression for $H_y$ at z = 0 is written, with reference to equations (41), and (42) as

$$H_y = H_t = \frac{E_{ot}}{Z_1}\left(1 - \rho\right) e^{-\gamma_1 x \sin\theta_1} \qquad (47)$$

The ratio of $E_t$ to $H_t$ at z = 0 becomes

$$\frac{E_t}{H_t} = Z_1\left(\frac{1+\rho}{1-\rho}\right) \qquad (48)$$

This ratio is continuous across the boundry and must also equal $Z_2$, which will equal either $\eta_2 \cos\theta_2$ or $\eta_2 \sec\theta_2$ depending on the polarization of the wave. Then

$$Z_1\left(\frac{1+\rho}{1-\rho}\right) = Z_2 \qquad (49)$$

This equation, when solved for $\rho$, gives

$$\rho = \frac{Z_2 - Z_1}{Z_2 + Z_1} \qquad (50)$$

And since the reflection and transmission coefficients are related by $\tau = 1 + \rho$ the transmission coefficient can be solved in the form

$$\tau = \frac{2 Z_2}{Z_2 + Z_1} \tag{51}$$

Thus, the transmission and reflection coefficients for oblique incidence on a boundry can be determined by solving for the wave impedances in each media and using equations (50) and (51).

## VIII. REFLECTION AND TRANSMISSION COEFFICIENTS AT FRONT FACE OF A LAYER SEPARATING TWO INFINITE MEDIA FOR OBLIQUE INCIDENCE

The input impedance looking into a finite layer separating two infinite media was determined. This was recognized as the load impedance presented to the tangential components in the region on the source side of the layer. Equations (50) and (51) were then used to solve for the reflection and transmission coefficients.

Consider Figure 7 which shows the transmission of a plane wave through such a layer. With reference to equations (45), (47), and (50), the equations for the tangential electric and magnetic fields in region 2, can

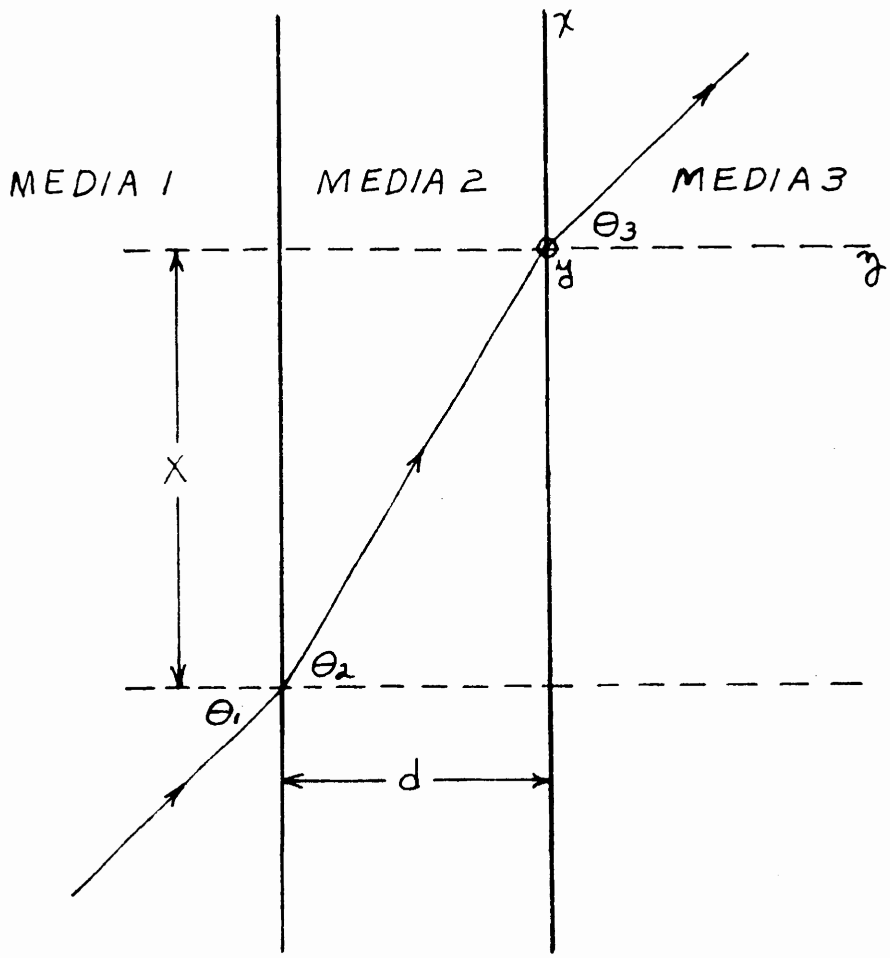


FIGURE 7

PLANE WAVE TRANSMISSION THROUGH

A LAYER OF FINITE THICKNESS

be written as

$$E_t = E_{ot}\left[e^{-\gamma_2 z \cos\theta_2} + \left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right) e^{\gamma_2 z \cos\theta_2}\right] e^{-\gamma_2 x \sin\theta_2} \qquad (52)$$

and

$$H_t = \frac{E_{ot}}{Z_2}\left[e^{-\gamma_2 z \cos\theta_2} + \left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right) e^{\gamma_2 z \cos\theta_2}\right] e^{-\gamma_2 x \sin\theta_2} \qquad (53)$$

The tangential components at the front face of the layer can be found by letting z = -d and x = -X in equations (52) and (53). Performing this step gives

$$E_t = E_{ot}\left[e^{\gamma_2 d \cos\theta_2} + \left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right) e^{-\gamma_2 d \cos\theta_2}\right] e^{\gamma_2 X \sin\theta_2} \qquad (54)$$

and

$$H_t = \frac{E_{ot}}{Z_2}\left[e^{\gamma_2 d \cos\theta_2} - \left(\frac{Z_3 - Z_2}{Z_3 + Z_2}\right) e^{-\gamma_2 d \cos\theta_2}\right] e^{\gamma_2 X \sin\theta_2} \qquad (55)$$

The ratio of $E_t$ to $H_t$ is the input impedance to region 2. Taking this ratio results in

$$Z_{IN2} = \frac{E_t}{H_t} = Z_2 \left[ \frac{Z_3\left(e^{\gamma_2 d \cos\theta_2} + e^{-\gamma_2 d \cos\theta_2}\right) + Z_2\left(e^{\gamma_2 d \cos\theta_2} - e^{-\gamma_2 d \cos\theta_2}\right)}{Z_2\left(e^{\gamma_2 d \cos\theta_2} + e^{-\gamma_2 d \cos\theta_2}\right) + Z_3\left(e^{\gamma_2 d \cos\theta_2} - e^{-\gamma_2 d \cos\theta_2}\right)} \right] \quad (56)$$

Dividing top and bottom of (56) by 2 gives

$$Z_{IN2} = Z_2 \left[ \frac{Z_3\left(\frac{e^{\gamma_2 d \cos\theta_2} + e^{-\gamma_2 d \cos\theta_2}}{2}\right) + Z_2\left(\frac{e^{\gamma_2 d \cos\theta_2} - e^{-\gamma_2 d \cos\theta_2}}{2}\right)}{Z_2\left(\frac{e^{\gamma_2 d \cos\theta_2} + e^{-\gamma_2 d \cos\theta_2}}{2}\right) + Z_3\left(\frac{e^{\gamma_2 d \cos\theta_2} - e^{-\gamma_2 d \cos\theta_2}}{2}\right)} \right] \quad (57)$$

By the use of the identities

$$\cosh u = \frac{e^{u} + e^{-u}}{2}$$

and

$$\sinh u = \frac{e^{u} - e^{-u}}{2}$$

equation (57) can be written in the form

$$Z_{IN2} = Z_2 \left[ \frac{Z_3 \cosh \gamma_2 d \cos\theta_2 + Z_2 \sinh \gamma_2 d \cos\theta_2}{Z_2 \cosh \gamma_2 d \cos\theta_2 + Z_3 \sinh \gamma_2 d \cos\theta_2} \right] \quad (58)$$

Since $Z_{in2} = Z_{L1}$, the reflection and transmission coefficients at the front face of a layer may be determined with the use of equations (50) and (51) as

$$\tau = \frac{2 Z_{IN2}}{Z_{IN2} + Z_1} \quad (59)$$

and

$$\rho = \frac{Z_{IN2} - Z_1}{Z_{IN2} + Z_1} \qquad (60)$$

The wave impedance in equations (58), (59), and (60) will be the intrinsic impedance of the particular region multiplied by either $\cos\theta_N$ or $\sec\theta_N$ , depending on the polarization of the electric with respect to the boundry.

## IX. SNELL'S LAW

The use of the idea of wave impedance to determine the input impedance to a region and, therefore, the reflection and transmission coefficients, requires the use of the physical angles in the regions concerned. Snell's law may be used to complete this step in the analysis.

Consider a wave, incident at angle $\theta_1$ and refracted at angle $\theta_2$ , as shown in Figure 8. The sine of $\theta_2$ may be found from Snell's law as

$$\frac{\sin\theta_2}{\sin\theta_1} = \sqrt{\frac{\mu_1^* \epsilon_1^*}{\mu_2^* \epsilon_2^*}} \qquad (61)$$

With a real angle of incidence, $\theta_1$ , it is seen that, in general, sin $\theta_2$ and, consequently, cos $\theta_2$ will be complex.

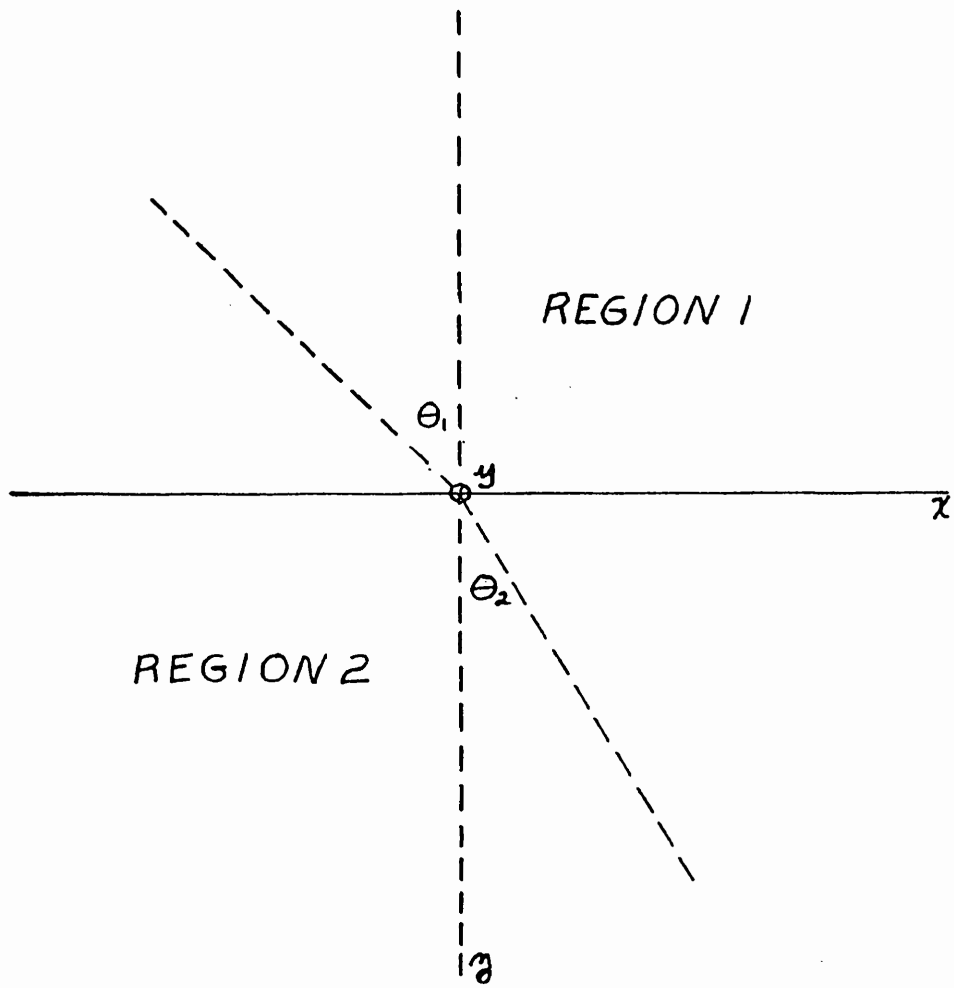


FIGURE 8

INCIDENCE AND REFRACTION AT A BOUNDRY

One interesting point that is easily shown by the use of Snell's law is that for any number layers, if the entrance media to the layers is the same as the exit media, the physical exit angle is the same as the initial angle of incidence.

By the use of Snell's law and the idea of wave impedance it is possible to determine the reflection and transmission coefficients for oblique incidence.

It should be mentioned at this point that the solution to the problem will hold only when conditions are such that Snell's law holds. That is, the arrangement of the layers and the angles of incidence must be such that the sine of the angle of refraction in any layer will have a magnitude no greater that unity.

CHAPTER IV

# APPLICATION OF MATRIX THEORY TO THE PROBLEM

## I. THE SCATTER MATRIX FOR A TWO-PORT NETWORK

The application of matrices to electrical networks is not a new idea. Dunn and Ross[6] have presented a very concentrated introduction to the application of the scatter matrix to electric circuit analysis. In this section an introduction following that of Dunn and Ross is presented.

Consider the general four terminal two-port network of Figure 9. The incident and reflected waves at port 1 are represented by $a_1$ and $b_1$, respectively, and $a_2$ and $b_2$ are the incident and reflected waves, respectively, at port 2.

The reflected waves are related to the incident waves by the following

$$b_1 = S_{11} a_1 + S_{12} a_2 \tag{62}$$

$$b_2 = S_{21} a_1 + S_{22} a_2 \tag{63}$$

From (62) $S_{11} = b_2/a_1$ with $a_2 = 0$ or simply is the reflection coefficient at port 1 with a perfect termination on port 2.

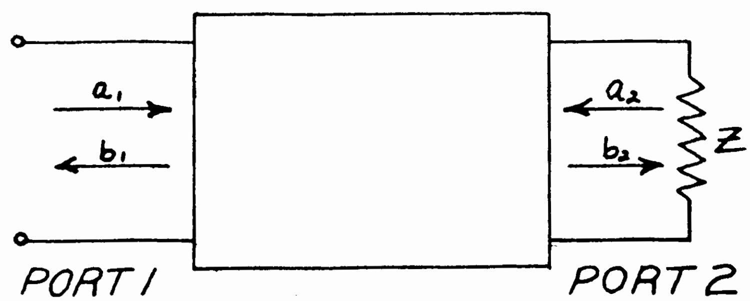


FIGURE 9

GENERAL FOUR TERMINAL NETWORK

From (63), $S_{21} = b_2/a_1$ with $a_2 = 0$ or $S_{21}$ is the transmission coefficient at port 1 with a perfect termination on port 2.

From (63), $S_{22} = b_2/a_2$ with $a_1 = 0$ and is the reflection coefficient at port 2 with port 1 perfectly terminated.

From (62), $S_{12} = b_1/a_2$ with $a_1 = 0$ and is the transmission coefficient at port 2 with port 1 perfectly terminated.

Equations (62) and (63) can be written in matrix form as $[B] = [S][A]$ or

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

where the matrix $\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ is termed a scatter matrix.

Whenever the equivalent circuit of the network is available or can be determined, the elements of the scatter matrix may be determined, The reader is referred to Appendix A for an example of this procedure.

## II. THE OVER-ALL TRANSMISSION MATRIX FOR N CASCADED NETWORKS

When networks are cascaded, as shown in Figure 10, it is convenient to define a new matrix, called the T matrix. The T matrix for the $n^{th}$ network is defined as

$$\begin{bmatrix} b_{2N} \\ a_{2N} \end{bmatrix} = \begin{bmatrix} T_N \end{bmatrix} \cdot \begin{bmatrix} a_{2N-1} \\ b_{2N-1} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} a_{2N-1} \\ b_{2N-1} \end{bmatrix} \tag{64}$$

The reason for defining the T matrix in this fashion will become clear in the following steps.

From Figure 10, it is clear that for the first network $b_2 = a_3$ and $a_2 = b_3$ so that

$$\begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} = T_1 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \tag{65}$$

For the second network $b_4 = a_5$ and $a_4 = b_5$ so that

$$\begin{bmatrix} b_4 \\ a_4 \end{bmatrix} = \begin{bmatrix} a_5 \\ b_5 \end{bmatrix} = T_2 \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} \tag{66}$$

But from (65)

$$\begin{bmatrix} a_3 \\ b_3 \end{bmatrix} = T_1 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

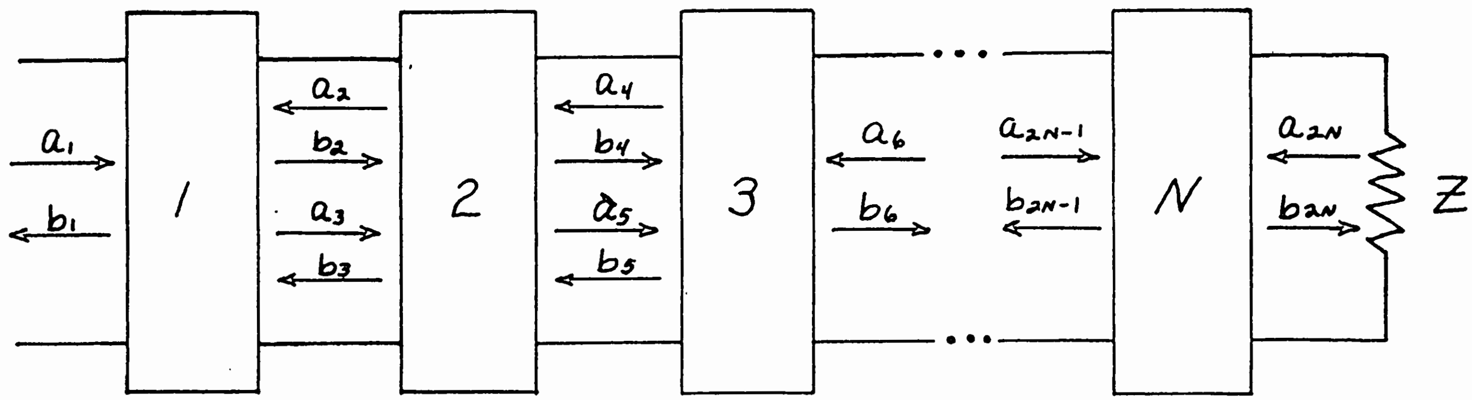


FIGURE 10

FOUR TERMINAL NETWORKS IN CASCADE

so that

$$\begin{bmatrix} b_4 \\ a_4 \end{bmatrix} = \begin{bmatrix} a_5 \\ b_5 \end{bmatrix} = T_2 T_1 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \tag{67}$$

For the third network $a_6 = b_7$ and $b_6 = a_7$ so that

$$\begin{bmatrix} b_6 \\ a_6 \end{bmatrix} = \begin{bmatrix} a_7 \\ b_7 \end{bmatrix} = T_3 \begin{bmatrix} a_5 \\ b_5 \end{bmatrix} \tag{68}$$

But from (67)

$$\begin{bmatrix} a_5 \\ b_5 \end{bmatrix} = T_2 T_1 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} a_7 \\ b_7 \end{bmatrix} = T_3 T_2 T_1 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \tag{69}$$

The pattern can now be seen and for n networks the output waves and input waves are related by

$$\begin{bmatrix} b_{2N} \\ a_{2N} \end{bmatrix} = \begin{bmatrix} T_N T_{N-1} \cdots T_1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \tag{70}$$

The over-all transmission matrix, $\overline{T}$, is defined as $\overline{T} = T_N T_{N-1} \cdots T_1$ . Now (70) can be written in the form

$$\begin{bmatrix} b_{2N} \\ a_{2N} \end{bmatrix} = \overline{T} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \tag{71}$$

Equation (71) is seen to give the output waves in terms of the input waves for a chain of networks.

## III. THE OVER-ALL SCATTER MATRIX $\overline{S}$ FOR N CASCADED NETWORKS

No attempt was made to prove the matrix conversions given in this section. The reader is referred to Appendix B for the proof of these conversions.

The relationship between the transmission, T, matrix and the scatter, S, matrix is given by

$$\begin{bmatrix} T \end{bmatrix} = \frac{1}{S_{12}} \begin{bmatrix} -\Delta S & S_{22} \\ -S_{11} & 1 \end{bmatrix} \tag{72}$$

where

$$\Delta S = S_{11} S_{22} - S_{12} S_{21}$$

and

$$\begin{bmatrix} S \end{bmatrix} = \frac{1}{t_{22}} \begin{bmatrix} -t_{21} & 1 \\ \Delta T & t_{12} \end{bmatrix} \tag{73}$$

where

$$\Delta T = t_{11} t_{22} - t_{12} t_{21}$$

Therefore, once the over-all transmission matrix, $\overline{T}$, has been found, the conversion to the over-all scatter matrix, $\overline{S}$, may be carried out with the use of (73).

## IV. SUMMARY OF SCATTER MATRIX ANALYSIS OF CASCADED NETWORKS

The solution of a chain of networks by the use of the scatter and transmission matrices is summarized in in the following steps:

1. Find the scatter matrix of each network.

2. Convert the scatter matrix to a transmission matrix by (72).

3. Multiply all transmission matrices in the order given in (70) to obtain an over-all transmission matrix, $\overline{T}$.

4. Convert the $\overline{T}$ matrix to an over-all scatter matrix, $\overline{S}$, by (73).

5. The elements $S_{11}$ and $S_{21}$ of $\overline{S}$ will be the reflection and transmission coefficients when the network is driven (wave incident) at port 1.

## V. THE SCATTER MATRIX FOR A LAYERED MEDIA

In this section the scatter matrix was used to solve for the reflection and transmission coefficients for a plane wave incident upon a layered media.

Figure 11 illustrates a plane wave transmission through a media consisting of n layers. The effects of the layers on the wave are a function of $\mu^*$ and $\epsilon^*$ of the layers only, since $\cos\theta = f(\mu^*, \epsilon^*)$. Equation (61) indicates that $\cos\theta$ is a function of $\mu^*, \epsilon^*$.

If the entrance and exit media are free space, $\theta_{N+1} = \theta_0$ as has been pointed out previously.

It is assumed that in Figure 11 the entrance and exit media are the same. The layers may be thought of as separate individual networks as shown in

Figure 12, the characteristics of which depend upon $\mu^*$ and $\epsilon^*$. The scatter matrix of these "networks" may be found and the over-all system analyzed as in the case of simple electric circuits in cascade.

To determine the scatter matrix of the $n^{th}$ layer, it is helpful to first review the definitions of the scatter elements.

The element, $S_{11}$, is the reflection coefficient at the left face of layer n with a perfect termination on the right side of the layer.

The element, $S_{12}$, is the transmission coefficient at the right face of layer n with a perfect termination on the left face.

The element, $S_{21}$, is the transmission coefficient at the left face of layer n with a perfect termination on the right face.

The element, $S_{22}$, is the reflection coefficient at the right face of layer n with a perfect termination on the left face.

Since, in this problem, the transmission system is free space, a perfect termination on the right or left face requires that a region of free space of infinite extent be to the right or left of the face.

To determine the scatter matrix of the $n^{th}$ layer, refer to Figure 12. The angles in the various layers

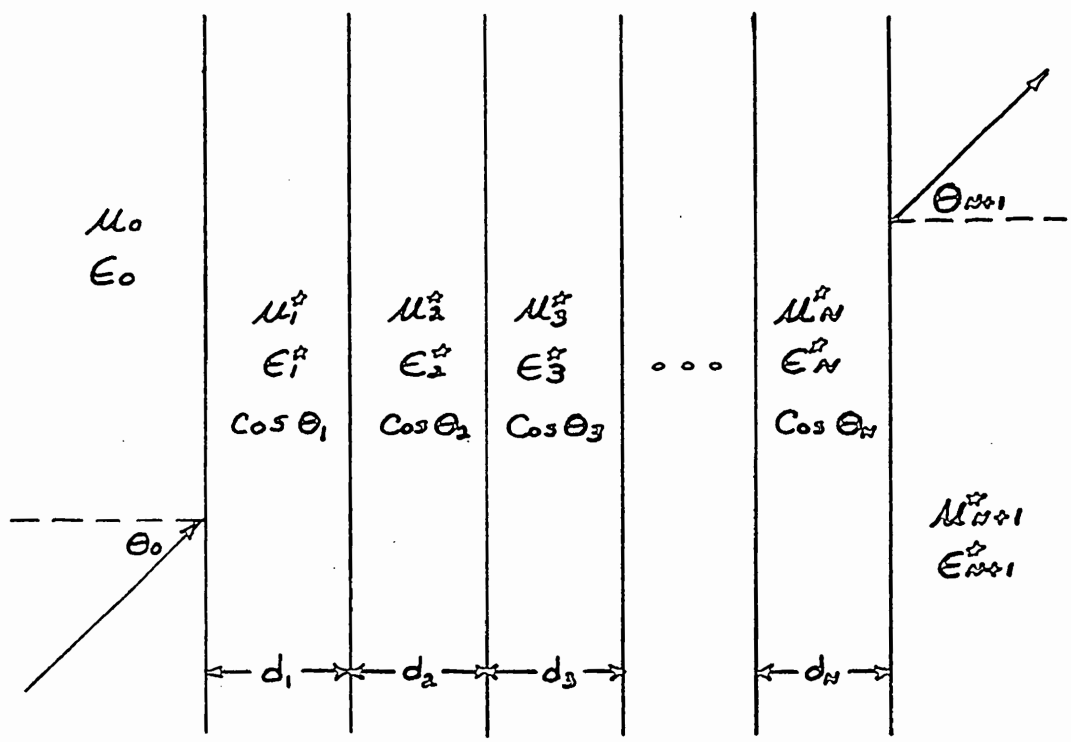


FIGURE 11

PLANE WAVE TRANSMISSION THROUGH

A LAYERED MEDIA

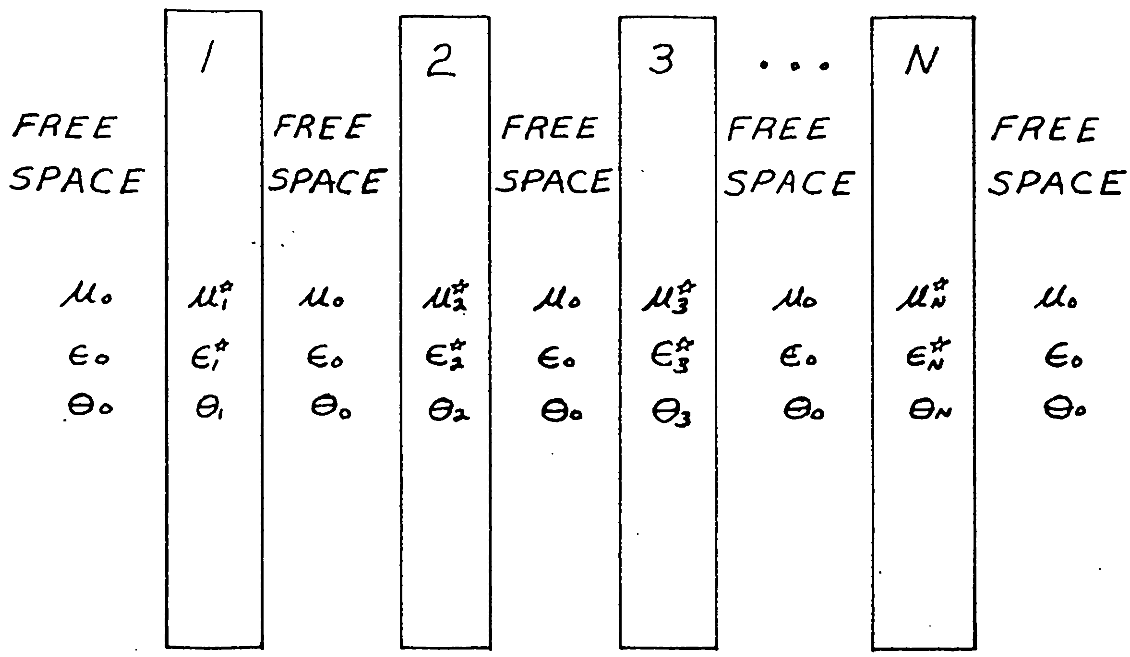


FIGURE 12

LAYERS TREATED AS INDIVIDUAL NETWORKS

may be determined easily by the use of Snell's law. Then the input impedance at the left face of the layer can be found from (58) as

$$Z_{in\,N} = Z_N \left[ \frac{Z_0 \cosh \gamma_N d_N \cos\theta_N + Z_N \sinh \gamma_N d_N \cos\theta_N}{Z_N \cosh \gamma_N d_N \cos\theta_N + Z_0 \sinh \gamma_N d_N \cos\theta_N} \right] \quad (74)$$

where

$$Z_N = \eta_N \cos\theta_N \quad \text{or} \quad \eta_N \sec\theta_N$$

and

$$Z_0 = \eta_0 \cos\theta_0 \quad \text{or} \quad \eta_0 \sec\theta_0$$

The reflection coefficient, $S_{11}$, can be calculated from (60) and the transmission coefficient, $S_{21}$, can be calculated from (59).

It is apparent that the network is symetrical and that $S_{12} = S_{21}$, and that $S_{22} = S_{11}$. The elements of the scatter matrix of the $n^{th}$ layer have been found. Each scatter matrix is found by the same procedure; that is, solving for $Z_{in\ n}$ and calculating $\rho$ and $\tau$ from equations (59) and (60).

## VI. SUMMARY OF SCATTER MATRIX ANALYSIS OF LAYERED MEDIA

The steps for the determination of the reflection and transmission coefficients for a media of n layers are given below:

1. Determine $\cos\theta$ in each layer by the use of Snell's law.

2. Determine the scatter matrix elements by the use of equations (74), (60), and (59).

3. Convert each scatter matrix to a transmission matrix by equation (72).

4. Multiply all transmission matrices in the order given in equation (70).

5. Convert the resulting over-all transmission matrix to an over-all scatter matrix by equation (73).

6. The elements $S_{11}$ and $S_{21}$ of the over-all scatter matrix will be the reflection and transmission coefficients, respectively, for the multi-layer system.

# CHAPTER V

## ATTEMPTS AT SIMPLIFICATION

In this chapter special cases were considered in an attempt to reduce the work involved in the solution of the problem. A simplification was defined as either a reduction in the number of steps to be performed or any process which makes the completion of an individual step easier.

### I. EFFECTS OF THE CHARACTERISTICS OF THE LAYERS

First consider a perfect (lossless) non-magnetic layer. For this layer both the permittivity and the permeability would be purely real. In the determination of the scatter matrix for this layer, the layer would be separated from the system and considered a separate element with air on either side, as was explained in the previous chapter. Most dielectrics have a dielectric constant greater than unity. From Snell's law

$$\sin \theta_d = \sin \theta_o \sqrt{\frac{\mu_o \epsilon_o}{\mu_d^* \epsilon_d^*}}$$

where the subscripts o and d refer to air and the dielectric layer, respectively. Since the layer under consideration has $\mu_d^* = \mu_o$ and $\epsilon_d^* = \epsilon_d' = K_d \epsilon_o$,

$$\sin\theta_d = \sin\theta_0\sqrt{\frac{1}{K_d}}$$

where Kd is the dielectric constant for the layer. It is seen that $\sin\theta_d$ will be real and less than unity. Therefore, since

$$\cos\theta_d = \sqrt{1-\sin^2\theta_d}$$

the $\cos\theta_d$ will also be real and less than unity.

Looking now at the expression for the propagation constant, $\gamma$,

$$\gamma_d = j\omega\sqrt{\mu_d^* \epsilon_d^*}$$

which for the ideal layer under consideration reduces to

$$\gamma_d = j\omega\sqrt{K_d \mu_0 \epsilon_0} = j\beta_d$$

The propagation constant is seen to be purely imaginary.

The scatter element for the layer were given in (59) and (60) as

$$\tau = S_{12} = S_{21} = \frac{2Z_{IN d}}{Z_{IN d} + Z_0}$$

$$\rho = S_{11} = S_{22} = \frac{Z_{IN d} - Z_0}{Z_{IN d} + Z_0}$$

where $Z_{ind}$ is

$$Z_{IND} = Z_d \left[ \frac{Z_o \cosh \gamma_d d \cos\theta_d + Z_d \sinh \gamma_d d \cos\theta_d}{Z_d \cosh \gamma_d d \cos\theta_d + Z_o \sinh \gamma_d d \cos\theta_d} \right]$$

as defined in (58).

In the above equation, all Z's will be purely real and $\gamma_d$ will be purely imaginary, making it possible to write the equation in the form

$$Z_{IND} = Z_d \left[ \frac{Z_o + j\, Z_d \tan \beta_d d \cos\theta_d}{Z_d + j\, Z_o \tan \beta_d d \cos\theta_d} \right]$$

With the use of this simplified equation and (59) and (60), the scatter elements of the layer can be determined.

Now consider a layer in which magnetic and/or dielectric losses are present. This requires the use of a complex permeability and/or complex permittivity. From Snell's law $\sin\theta_d$ will be complex and, therefore $\cos\theta_d$ will be complex. The expression for the propagation constant is also complex for the layer. It may be said, then, that whenever a layer with losses is being considered, no simplifications can be made on the original results.

The chances of simplifying the results by taking into account the characteristics of the layer are seen to be small. Even in the ideal case of no losses, each step must be completed as in the case of a layer with losses. However, the act of completing the steps is simplified since some of the calculations would involve purely real numbers, and the hyperbolic functions of (58) could be replaced by trigonometric functions. Whenever a layer has either magnetic or dielectric losses and the losses must be considered, no reduction can be made.

## II. EFFECT OF THICKNESS OF LAYERS - THIN LAYERS

General. It is possible for a system to be composed of thin layers (with respect to the wave length) of different materials. In the solution of the problem, each layer would be analyzed separately. That is, the scatter elements would be determined for each layer, individually. Any possible simplification would have to be carried out through the reduction of (58) which is given as

$$Z_{ind} = Z_d \left[ \frac{Z_o \cosh \gamma_d d \cos\theta_d + Z_d \sinh \gamma_d d \cos\theta_d}{Z_d \cosh \gamma_d d \cos\theta_d + Z_o \sinh \gamma_d d \cos\theta_d} \right]$$

At first glance, it seems that for very thin layers $\sinh \gamma_d d \cos\theta_d$ could be dropped from the equation. If this were done (58) would be simplified as

$$Z_{ind} = Z_o$$

which indicates that for this particular layer, there would be, to good approximation, no reflection, phase shift or attenuation. However, even though the effects of each layer may be small, each must be considered because the system as a whole might have a very great effect on the attenuation or phase shift of the wave.

The case of a thin layer in a system of thick layers is entirely different. In such a system the thick layers will determine, almost entirely, the overall effects on the wave and the thin layer may be omitted in the solution of the problem.

Of course, the possibility does exist, if $\mu_d^*$ and $\epsilon_d^*$ are very great, that even though the layer is thin with respect to the other layers of the system, $\gamma_d d \cos\theta_d$ may be great enough so that the effects of the layer cannot be ignored in the analysis.

The results may be simplified by considering the thickness of the layer and the magnitude of the propagation constant in conjunction with the characteristics of the other layers of the system.

Thin alterating layers of two dielectrics. Many writers such as Brekhovskikh[7], Collin[8], and Kirschbaum and Chen[9] have pointed out the fact that two thin homogeneous isotropic layers will behave on the whole as one homogeneous anisotropic system having different characteristics along different axes. Brekhovskikh has given the effective values for permeability and permittivity for the system as

$$\epsilon_{1e}^{*} = \frac{d_1 \epsilon_1^{*} + d_2 \epsilon_2^{*}}{d_1 + d_2} \tag{75}$$

$$\mu_{1e}^{*} = \frac{d_1 \mu_1^{*} + d_2 \mu_2^{*}}{d_1 + d_2} \tag{76}$$

$$\epsilon_{2e}^{*} = \frac{\epsilon_1^{*} \epsilon_2^{*} (d_1 + d_2)}{d_1 \epsilon_2^{*} + d_2 \epsilon_1^{*}} \tag{77}$$

$$\mu_{2e}^{*} = \frac{\mu_1^{*} \mu_2^{*} (d_1 + d_2)}{d_1 \mu_2^{*} + d_2 \mu_1^{*}} \tag{78}$$

The effective values to be used depends on the direction of propagation and polarization of the field. When the wave is polarized with the electric field parallel to the boundry of the dielectrics and propagation takes place parallel to the boundries, $\mu_{2e}^{*}$ and $\epsilon_{1e}^{*}$ would be used. When the wave is polarized with the electric field perpendicular to the boundry and propagation is parallel to the boundry $\mu_{1e}^{*}$ and $\epsilon_{2e}^{*}$ would be used. When propagation takes place perpendicular to the boundry $\mu_{1e}^{*}$ and $\epsilon_{1e}^{*}$ would be used.

The use of the effective values for permeability and permittivity allows one to treat the two layers as a homogeneous system for propagation in one direction.

Collin has pointed out an interesting point. Although equations (75) through (78) were derived for thin layers (with respect to the free space wave length) the effective values for thicker layers differ only a few per cent from those of thin layers. This has been confirmed by Kirschbaum and Chen who have stated that tests have shown the use of the effective values as given is justified up to frequencies at which $d_1 + d_2 \leq \lambda_0(.3)$ where $\lambda_0$ is the wave length of the radiation in free space.

Realizing the limitations of this method, namely, that $d_1 + d_2 \leq \lambda_0(.3)$ and that propagation must take place in one direction only, the possible reduction in the analysis is obvious.

If a system consisted of 2n layers of dielectrics (n layers of each dielectric), the effective values of $\mu^*$ and $\epsilon^*$ would be found from equations (75) through (78) depending on the polarization of the field and the direction of propagation. After the effective values of $\mu^*$ and $\epsilon^*$ are determined the system may be treated as a single layer of thickness $d = n(d_1 + d_2)$ and having characteristics dependent upon the effective values of $\mu^*$ and $\epsilon^*$. The input impedance to this layer could be determined and the reflection and transmission coefficients determined easily.

# CHAPTER VI

## SUGGESTIONS FOR ADDITIONAL DEVELOPMENT

The problem investigated considered plane waves with two types of polarization - with the electric field in the plane of incidence and with the electric field perpendicular to the plane of incidence. In many cases the electric field may lie neither in the plane of incidence nor perpendicular to the plane of incidence. This problem should be investigated. The solution may prove simple, being a problem of superposition of the two cases considered in this thesis.

In the solution given for the reflection and transmission coefficients for plane wave propagation through a layered media, the effects of an individual layer on the transmission of a wave are lost.

The effects of a layer are lost in the scatter matrix to transmission matrix conversion, in the multiplication of the transmission matrices, and in the conversion from the over-all transmission matrix to the over-all scatter matrix. It would be desirable to find a method of determining, in a simple manner, the effects of one layer as part of the system of layers.

It was found that the problem could be simplified for the case of thin alternating layers by the use of effective values of permeability and permittivity for the two layers treated as a single layer. However, the simplification held only for certain polarizations of the incident wave and for propagation in only one direction. Although it may prove very difficult to do so, it would be desirable to obtain expressions for effective values for the permeability and permittivity for oblique incidence. The effective values would be a function of the angles of incidence, reflection and transmission.

## CHAPTER VII

## SUMMARY

The scatter matrix can be applied in the solution of the reflection and transmission coefficients for the propagation of a plane wave through a layered media. This process consists of determining the elements of the scatter matrix for each layer, converting each scatter matrix to a transmission matrix, multiplying all transmission matrices, and converting the over-all transmission matrix to an over-all scatter matrix. The elements of the over-all scatter matrix will give the reflection and transmission coefficients for the layered system.

The main drawback to the solution as presented is that the effects of an individual layer, as part of the system, are lost in the matrix manipulations. The effects of an individual layer would have to be determined by performing the matrix analysis twice, once with the layer in the system and once with the layer omitted from the system. The results would then be compared to determine the effects of an individual layer on the transmission of the wave.

Although an attempt was made to find ways to simplify or reduce in number the steps to be taken in the solution, very few simplifications could be made. The simplifications were limited to a thin layer in a system of thick layers and to the case of an artificial anisotropic media, in which instance effective values of permeability and permittivity would be used. The effective values to be used were determined by the direction of propagation (either perpendicular or parallel to the boundry of the layers) and polarization of the incident wave.

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# VITA

Bill W. Ashworth was born on August 12, 1935, at Mammoth Springs, Arkansas. He attended public schools in various sections of the state of Missouri from 1941 to 1949. In 1949 he entered Central High School in St. Louis, Missouri, from which he was graduated in January, 1953.

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During the summers of 1958 and 1960 the author was employed by the Bendix Corporation of Kansas City, Missouri.

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## APPENDIX A

## EXAMPLE OF SCATTER MATRIX ANALYSIS FOR A SIMPLE NETWORK

For the purposes of this example, consider the network of Figure 13. The shunt impedances, $Z_1$ and $Z_2$, will be treated as cascaded networks, and the reflection and transmission coefficients at the front of the set of the cascaded networks will be found by use of the scatter matrix.

First, consider $Z_1$ as an individual network. To find the scatter coefficients of this network, it will be driven from port 1 with a perfect termination on port 2, as shown in Figure 14. The load looking into port 1 is

$$Z_L = \frac{2 \times 1}{2+1} = 2/3 \ \Omega$$

The reflection and transmission coefficients can now be calculated as

$$\tau = S_{21} = \frac{2 Z_L}{Z_L + Z_0} = \frac{2 \times 2/3}{2 + 2/3} = 4/5$$

$$\rho = S_{11} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{2/3 - 1}{2/3 + 1} = -1/5$$

The elements $S_{12}$ and $S_{22}$ are the transmission and reflection coefficients, respectively, when the network is driven from port 2 with a perfect termination on port 1. It is easily seen that $S_{22} = S_{11}$ and $S_{12} = S_{21}$.

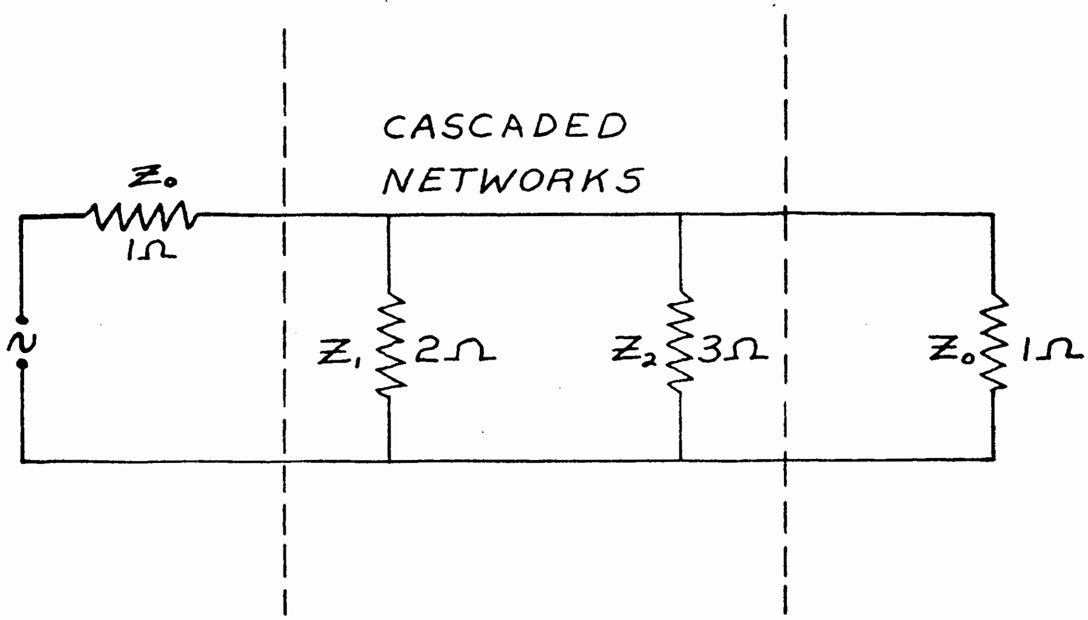


FIGURE 13

A SIMPLE ELECTRIC NETWORK

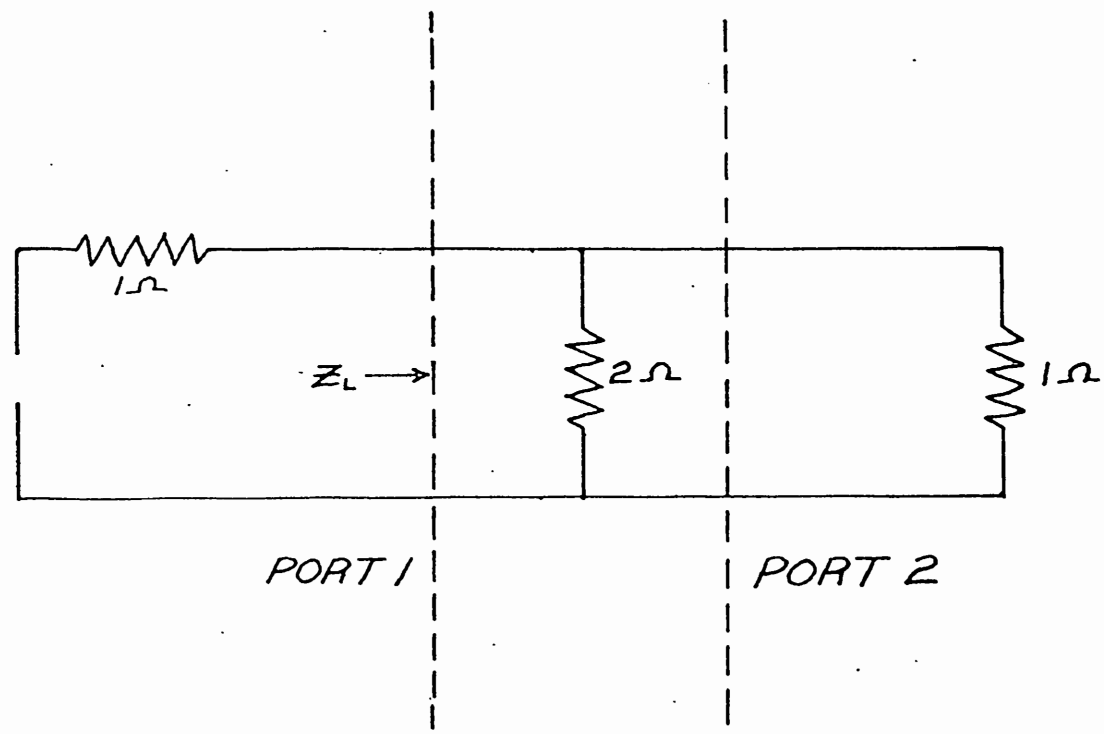


FIGURE 14

SHUNT ELEMENT TREATED AS A NETWORK

The scatter elements for the second network can be determined from Figure 15. In this case, $Z_L = 3/4$ and the scatter elements are

$$\tau = S_{12} = 6/7$$
$$\rho = S_{11} = -1/7$$

The scatter matrix for circuit 1 is

$$\left[S_1\right] = \begin{bmatrix} -1/5 & 4/5 \\ 4/5 & -1/5 \end{bmatrix}$$

with $\Delta S_1 = -3/5$ = determinant of $\left[S_1\right]$

The scatter matrix of circuit 2 is

$$\left[S_2\right] = \begin{bmatrix} -1/7 & 6/7 \\ 6/7 & -1/7 \end{bmatrix}$$

with $\Delta S_2 = -5/7$ = determinant of $\left[S_2\right]$

The $\left[S\right]$ to $\left[T\right]$ matrix conversion is given as

$$\left[T\right] = \frac{1}{S_{12}} \begin{bmatrix} -\Delta S & S_{22} \\ -S_{11} & 1 \end{bmatrix}$$

so that

$$\left[T_1\right] = 5/4 \begin{bmatrix} 3/5 & -1/5 \\ 1/5 & 1 \end{bmatrix} = \begin{bmatrix} 3/4 & -1/4 \\ 1/4 & 5/4 \end{bmatrix}$$

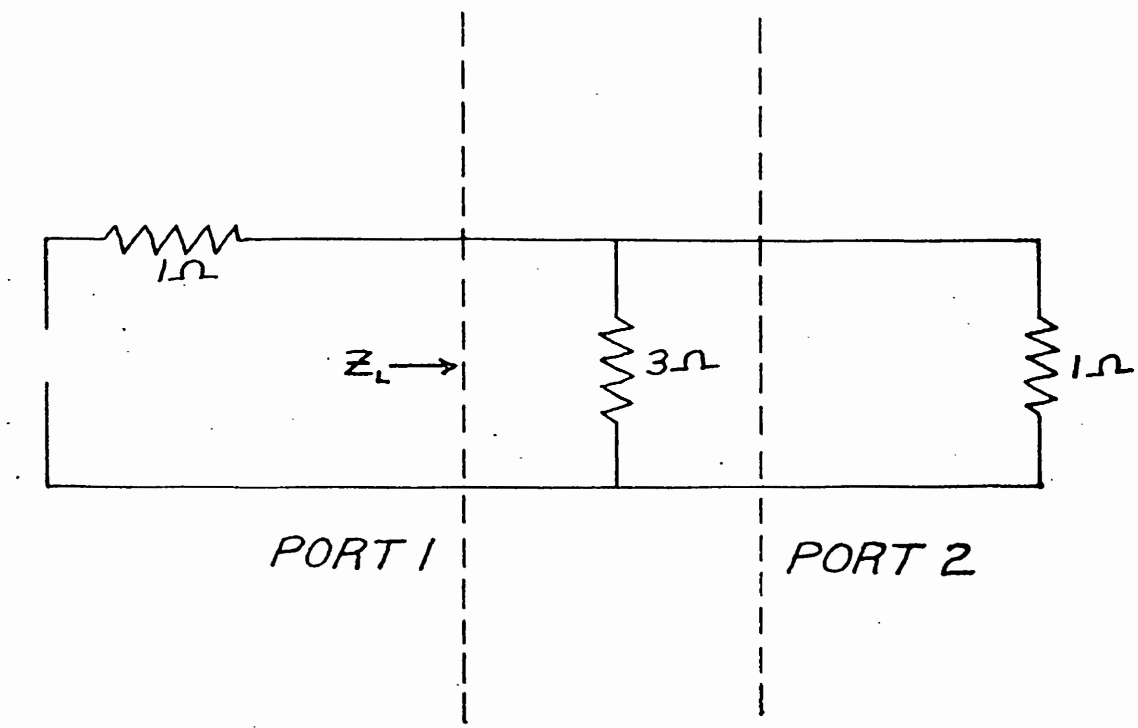


FIGURE 15

SHUNT ELEMENT TREATED AS A NETWORK

and

$$[T_2] = 7/6 \begin{bmatrix} 5/7 & -1/7 \\ 1/7 & 1 \end{bmatrix} = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 7/6 \end{bmatrix}$$

The over-all transmission matrix is defined as

$$\overline{T} = T_N \, T_{N-1} \cdots T_1$$

so that

$$[\overline{T}] = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 7/6 \end{bmatrix} \circ \begin{bmatrix} 3/4 & -1/4 \\ 1/4 & 5/4 \end{bmatrix}$$

Performing the matrix multiplication gives

$$[\overline{T}] = \begin{bmatrix} 7/12 & -5/12 \\ 5/12 & 17/12 \end{bmatrix}$$

with the determinant of $[\overline{T}] = \Delta\overline{T} = 1.$

The $[T]$ to $[S]$ conversion is given as

$$[S] = \frac{1}{t_{22}} \begin{bmatrix} -t_{21} & 1 \\ \Delta T & t_{12} \end{bmatrix}$$

so that

$$[\overline{S}] = \frac{12}{17} \begin{bmatrix} -5/12 & 1 \\ 1 & -5/12 \end{bmatrix} = \begin{bmatrix} -5/17 & 12/17 \\ 12/17 & -5/17 \end{bmatrix}$$

This is the over-all scatter matrix and the elements $S_{11}$ and $S_{21}$ give the reflection and transmission coefficients as

$$S_{11} = \rho = -5/17$$

$$S_{21} = \tau = 12/17$$

This result can be verified by considering the original circuit of Figure 11. The load impedance is

$$Z_L = \frac{1}{1/2 + 1/3 + 1} = 6/11 \ \Omega$$

The reflection and transmission coefficients are calculated in the following steps:

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{6/11 - 1}{6/11 + 1} = -5/17$$

$$\tau = \frac{2 Z_L}{Z_L + Z_0} = \frac{2 \times 6/11}{6/11 + 1} = 12/17$$

# APPENDIX B

## PROOF OF $[S]$ TO $[T]$ AND $[T]$ TO $[S]$ MATRIX CONVERSIONS

The conversions between the $[S]$ and $[T]$ were given by equations (72) and (73) as

$$[T] = \frac{1}{S_{12}} \begin{bmatrix} -\Delta S & S_{22} \\ -S_{11} & 1 \end{bmatrix}$$

with

$$\Delta S = S_{11} S_{22} - S_{12} S_{21}$$

and

$$[S] = \frac{1}{t_{22}} \begin{bmatrix} -t_{21} & 1 \\ \Delta T & t_{12} \end{bmatrix}$$

with

$$\Delta T = t_{11} t_{22} - t_{12} t_{21}$$

For the proof of the $[S]$ to $[T]$ conversion the following equations are given.

$$b_1 = S_{11} a_1 + S_{12} a_2$$

$$b_2 = S_{21} a_1 + S_{22} a_2$$

To prove the $[S]$ to $[T]$ conversion, it is only necessary to show that the resulting matrix will give the same initial equations. For a particular network, the output waves are related to the input waves by the

$[T]$ matrix in the form

$$\begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = T \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

But the $[T]$ matrix in terms of the elements of the scatter matrix is

$$[T] = \frac{1}{S_{12}} \begin{bmatrix} -\Delta S & S_{22} \\ -S_{11} & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{-S_{11}S_{22}+S_{12}S_{21}}{S_{12}} & \dfrac{S_{22}}{S_{12}} \\ -\dfrac{S_{11}}{S_{12}} & \dfrac{1}{S_{12}} \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{-S_{11}S_{22}+S_{12}S_{21}}{S_{12}} & \dfrac{S_{22}}{S_{12}} \\ -\dfrac{S_{11}}{S_{12}} & \dfrac{1}{S_{12}} \end{bmatrix} \circ \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

or

$$\begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} \left(\dfrac{-S_{11}S_{22}+S_{12}S_{21}}{S_{12}}\right)a_1 + \dfrac{S_{22}}{S_{12}} b_2 \\ -\dfrac{S_{11}}{S_{12}} a_1 + \dfrac{b_1}{S_{12}} \end{bmatrix}$$

The equations which this matrix represent are

$$b_2 = \left(\frac{-S_{11}S_{22}+S_{12}S_{21}}{S_{12}}\right)a_1 + \frac{S_{22}}{S_{12}} b_1 .$$

and

$$a_2 = \frac{-S_{11}}{S_{12}} a_1 + \frac{1}{S_{12}} b_1$$

From the latter equation

$$b_1 = S_{11}a_1 + S_{12}a_2$$

Substituting this into the former equation gives

$$b_2 = \left(\frac{-S_{11}S_{22} + S_{12}S_{21}}{S_{12}}\right)a_1 + \frac{S_{22}}{S_{12}}\left(S_{11}a_1 + S_{12}a_2\right)$$

or

$$b_2 = S_{21}a_1 + S_{22}a_2$$

The original equations are obtained which verifies the $[S]$ to $[T]$ matrix conversion.

To show the validity of the $[T]$ to $[S]$ conversion, consider the equations

$$b_2 = t_{11}a_1 + t_{12}b_1$$

$$a_2 = t_{21}a_1 + t_{22}b_1$$

which may be written in matrix form as

$$\begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \circ \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

If the $[T]$ to $[S]$ conversion is valid, the resulting matrix must give the original equations.

For a particular network the input and output waves are related by the scatter matrix in the form

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = S \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

But the scatter matrix in terms of the elements of the [T] matrix is

$$[S] = \frac{1}{t_{22}} \begin{bmatrix} -t_{21} & 1 \\ \Delta T & t_{12} \end{bmatrix} = \begin{bmatrix} -\frac{t_{21}}{t_{22}} & \frac{1}{t_{22}} \\ \frac{t_{11}t_{22}-t_{12}t_{21}}{t_{22}} & \frac{t_{12}}{t_{22}} \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{-t_{21}}{t_{22}} & \frac{1}{t_{22}} \\ \frac{t_{11}t_{22}-t_{12}t_{21}}{t_{22}} & \frac{t_{12}}{t_{22}} \end{bmatrix} \circ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

or

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{-t_{21}}{t_{22}} a_1 + \frac{a_2}{t_{22}} \\ \left(\frac{t_{11}t_{22}-t_{12}t_{21}}{t_{22}}\right) a_1 + \frac{t_{12}}{t_{22}} a_2 \end{bmatrix}$$

The equations which this matrix represents are

$$b_1 = -\frac{t_{21}}{t_{22}} a_1 + \frac{a_2}{t_{22}}$$

and

$$b_2 = \left(\frac{t_{11}t_{22}-t_{12}t_{21}}{t_{22}}\right) a_1 + \frac{t_{12}}{t_{22}} a_2$$

These equations can be solved for $b_2$ and $a_2$ as

$$b_2 = t_{11} a_1 + t_{12} b_1$$

and

$$a_2 = t_{21} a_1 + t_{22} b_1$$

Thus, the validity of the [T] to [S] conversion has been established.