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Optical Statistical Measurements of Free Convection Flow Patterns

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OPTICAL STATISTICAL MEASUREMENTS OF FREE CONVECTION FLOW PATTERNS*

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ABSTRACT

Measurements have been made of the autocorrelation of photographs of the flow patterns produced at high Rayleigh numbers in a horizontal layer of liquid heated from below and cooled on top. The relation of these measurements to temperature measurements in the fluid are considered. **Analogous** situations in other physical systems are also discussed.

INTRODUCTION

The characteristics of fluid flows are most commonly investigated experimentally by means of velocity measurements. For non-isothermal flows the temperature distribution is also a quantity which can provide information on the flow. Such measurements usually (but not always) require the insertion of a probe into the fluid with consequent possible disturbance of the flow. Probing techniques may also have a spatial and/or temporal resolution that is limited by the physical characteristics of the probe. Optical procedures may provide a satisfactory means of overcoming these various disadvantages. For example certain optical methods (such as that described in this paper) are able to provide in a very short time much more information for statistical analysis than could be obtained by making measurements at a finite number of points. Furthermore, this data is obtained from a two-dimensional region in the flow field. This is particularly advantageous in free convection flows (the type discussed in this paper) where spatial averaging may be a more efficient process than temporal averaging.[2] In certain circumstances such as the examination of the flows encountered in astrophysics and meterology, an optical approach is the only one possible.

There are a number of optical flow measurement methods available but of these an instantaneous photograph of the flow pattern (which is related to the fluid velocity) represents one of the simplest of optical measurements. The data which can be obtained most directly from this measurement would be concerned with the characteristic size of the features in the flow pattern. Where the flow situation under investigation is laminar a direct visual examination and measurement should be sufficient to obtain an appropriate characteristic scale. If the flow is turbulent or otherwise spatially and/or temporally irregular a statistical approach would provide the best method of obtaining from the photograph an objective measurement of the scale of the flow pattern. The spatial variation of the autocovariance of the transparency of the photograph (defined in the next section) of the flow pattern provides a suitable statistic. In particular the full width at the half-maximum (FWHM) of the autocovariance curve is a good representation of the smallest spatial scale of the photograph (see Appendix).

In the experiments reported in this paper this approach has been applied for the first time to the flow in a horizontal layer of fluid heated from below and cooled at its upper surface. This hydrodynamic system is of particular interest because it represents a prototype situation from which information relevant to a basic understanding of more general (turbulent) flows might be obtained.[13]

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Convection in horizontal fluid layers would also appear to have similar features to the flow in the outer layers of the solar atmosphere and in the terrestrial atmosphere. This similarity between the laboratory and the natural flow situation has generally only been considered from a qualitative point of view. A comparison of the characteristic scales of the three systems might provide a more systematic indication of their common features.

The method of Kretzmer[7] was used in the experiments reported here to obtain the spatial variation of the autocorrelation function for the various flow patterns which were examined. Kretzmer originally applied this technique to a statistical analysis of television pictures. The same method has also been employed by Kovasznay [6,20] in the examination of shadowgraph pictures of supersonic flows and by Leighton[11,12] to the flow patterns in the solar atmosphere. Autocorrelation functions have also been measured using slightly different techniques by Soo[16] in two phase flows and Vohr[21] with boiling flows. Other applications and methods are discussed in references 1 and 9.

TECHNIQUE AND ANALYSIS*

A photograph may be considered as being made up of a spatially distributed variation of tones between black and white. Where the photograph is on a slide, it may be considered as a spatial variation in the transparency, i.e., the transparency is expressible in polar coordinates as $T(s,\theta)$. Then the autocovariance between two horizontally adjoining elements of the photograph is the spatially averaged product of the two transparencies of each pair of picture elements which appear in the region* over which the spatial averaging is carried out. If the picture elements are separated by a distance Δs then the autocovariance is given by

$$C(\Delta s,\theta) = \frac{1}{A}\int_A T(s,\theta)\, T(s + \Delta s,\theta)\, dA \tag{1}$$

where A is the area of averaging.

The determination of the autocovariance can be realised in practice by measuring the relative optical transparency of two identical slides placed face to face and shifted from register ($\Delta s = 0$) by equal and opposite amounts ($\pm\Delta s/2$). Then according to equation (1) the autocovariance is

$$C(\Delta s,\theta) = \frac{1}{A}\int_A T(+\Delta s/2,\theta)\, T(-\Delta s/2,\theta) dA \tag{2}$$

The autocovariance as defined in equations (1) and (2) is only valid when the mean transparency of the slides is zero; in practice this condition never holds. In these circumstances the following definition (derived in the Appendix) is more appropriate

$$C(\Delta s,\theta) = \frac{1}{A}\int_A T(-\Delta s/2,\theta)T(+\Delta s/2,\theta)dA - \frac{1}{A^2}\int_A T(-\Delta s/2,\theta)dA \int_A T(+\Delta s/2,\theta)dA \tag{3}$$

* The discussion of this section follows that of reference 7.

*The region over which the spatial averaging is carried out should be large enough (say, an order of magnitude larger) to encompass the fine scale structure of the photograph. To show that the averaging area is sufficiently large, the "stop" (Figure 1) used in the apparatus employed in the investigation reported in this paper can be moved around to demonstrate that the variation of the autocovariance with the position of the stop is small.

In some cases it is desirable to use a normalized measure of spatial correlation called the coefficient of autocorrelation (see Appendix)

$$R(\Delta s,\theta) = \frac{\frac{1}{A}\int_A T(-\Delta s/2,\theta)T(+\Delta s/2,\theta)dA - \frac{1}{A^2}\int_A T(-\Delta s/2,\theta)dA \int_A T(+\Delta s/2,\theta)dA}{\frac{1}{A}\int_A T^2(0,\theta)dA - \left[\frac{1}{A}\int_A T(0,\theta)dA\right]^2} \quad (4)$$

The quantities C and R can be plotted as functions of Δs and θ. It is shown in the Appendix that a characteristic scale for the distribution of transparency can be obtained from the FWHM of such a curve.

OPTICAL CORRELATION COMPUTER

The device used to measure the autocovariance is shown schematically in Figure 1; it is an exact copy of the apparatus described in reference 7. A collimated beam of light is passed through two 4 in. x 5 in. glass slides of the photograph under consideration. The area of spatial integration (see preceding section) is defined by the stop (this was of 2 in. diameter in the experiments reported here). The light transmitted by the slides is focused on a ground glass screen. The light transmitted by the ground glass falls on the photo-cathode of a type 931A photomultiplier tube. The source of illumination is a 6 volt filament bulb (GE1493). It is supplied by a regulated DC power supply so that the fluctuations in luminous intensity are a minimum.

The slides are carefully aligned and clamped in two aluminum holders with their emulsions in contact*. The holders can be moved by a micrometer screw relative to one another so that the two slides move by equal amounts in opposite directions. This movement can be measured to within 0.002 in. The holders are held under tension so that residual variations in the back-lash of the driving screw are extremely small. The slides are supported in the holders by two close fitting graduated aluminum rings which permit accurately determined rotation of both slides or one slide.

The photomultiplier tube and light bulb are mounted in light proof boxes. The small gap between the open ends of these boxes and the plate carriers was closed by a black velvet "boot". The measurements were made in a room artificially illuminated with low intensity light so that fluctuations in the ambient light did not affect the measurements. The high voltage was continuously maintained on the photomultiplier in order that the characteristic erratic behavior associated with the initial operation of such tubes be eliminated. The tube was suitably shielded when changing the glass plates in order to avoid damage by exposing it to high ambient light levels.

The output of the photomultiplier is proportional to the autocovariance as defined in equation (2) and is measured by a Hewlett-Packard model 425A microvolt-ammeter (Figure 2).

The micrometer screw which drives the plate holders was adjusted until the reading of the voltmeter was a maximum. This point was taken to correspond to zero relative displacement of the plates ($\Delta s = 0$). This procedure was used to minimize the errors associated with the initial alignment of the plates in the holders, which was done by eye.

A bucking voltage was then inserted into the circuit so that the voltmeter reading corresponding to the maximum correlation was zero. Since the variation of the autocovariance did not usually exceed 10% of its maximum value, it was then possible to make the full scale deflection of the voltmeter correspond to this variation by increasing the voltmeter gain.

The plates are then displaced by an appropriate amount and the new value of the autocovariance is determined. This procedure is repeated until the autocovariance appears to have leveled off to an essentially steady value. The coefficient of autocorrelation (equation (4)) is determined using single plate transparency values determined in the optical correlation computer.

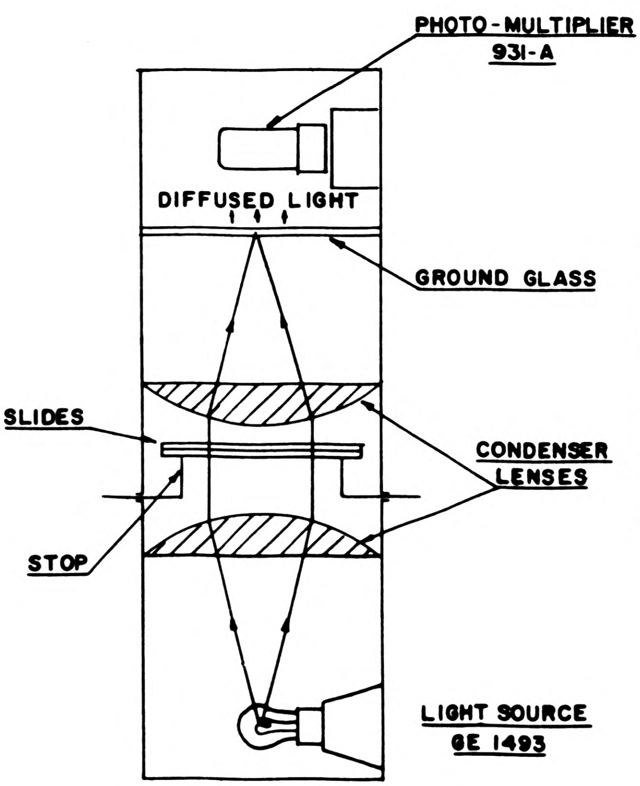

Figure 1. Schematic diagram of optical correlation computer.

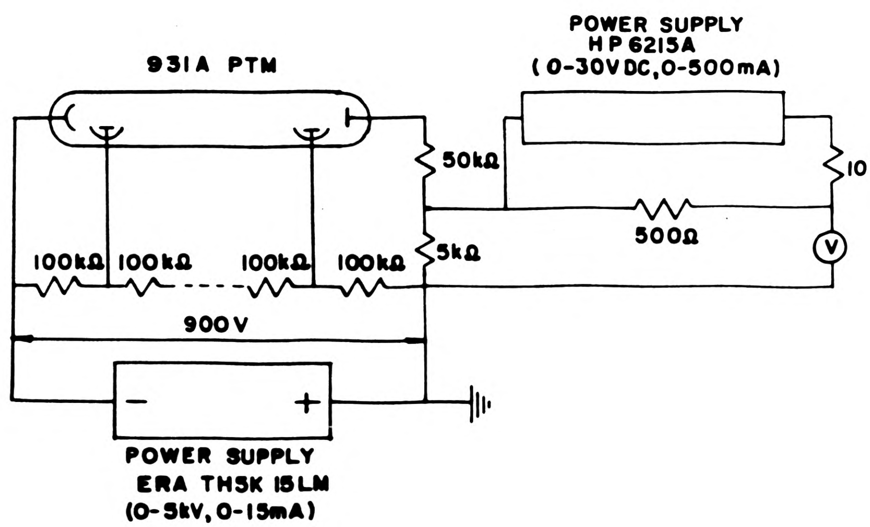

Figure 2. Photomultiplier circuit diagram.

*The slides must be in the form of a right-and-left-hand pair if they are to be in register when arranged face-to-face with zero horizontal displacement.

APPLICATION TO FREE CONVECTION IN HORIZONTAL FLUID LAYERS

When heat is first applied at the lower surface of a layer of fluid enclosed at the top and bottom by rigid, horizontal, conducting boundaries, it is transferred to the upper plate by conduction. With increasing heat input, a point is reached where motion of the fluid commences. The initiation of motion has been shown to depend on the critical value (1.70×10^3) of a non-dimensional parameter, the Rayleigh number (Ra), which is defined in the Symbols.

The motion at Rayleigh numbers slightly greater than critical usually appears as one of two regular flow patterns when viewed from above. In the one case, commonly called cellular flow, the fluid (if it is a liquid) rises in a series of vertical plumes located at the centers of a network of hexagons. At the upper surface of the layer the rising fluid flows radially outward and descends along the outer edges of the hexagons. At the lower surface the flow is inward to the center of the hexagons. If the fluid is a gas the flow is observed to be in the opposite direction. The other flow pattern has the appearance of long vortices, known as rolls, in which the fluid circulates continuously between the upper and lower surfaces of the layer (as with the cellular flow pattern the direction of motion depends on the nature of the fluid). These rolls lie side by side in a regular pattern; for example, they may have the appearance of a system of concentric circles or a system of parallel straight lines. As the Rayleigh number of the system is increased the regular flow pattern breaks up and eventually (at a Rayleigh number* of about 3×10^4) the flow becomes completely disordered. The flow pattern consists of irregularly shaped cells with shapes and dimensions which continually vary with time. This disordered fluid motion provides a very suitable means for the study of turbulence.

The flow patterns associated with the disordered fluid motion at high Rayleigh numbers have been examined qualitatively by Silveston[15], Somerscales and Dropkin[17], Elder[3] and Rossby[14]. In each case (except for the work reported in reference 15, where the Schlieren technique was used) the pattern was made visible by the addition of a very small amount of aluminum dust to the fluid. Under suitable illumination the flow appears as a pattern of light and dark areas.

Because of the irregular nature of the flow pattern and because some of the cells are crossed by faint, dark lines it is difficult to decide which regions of the photograph constitute individual cells. In these circumstances the spatial variation of the autocorrelation of the transparency provides the best means of determining the characteristic horizontal spatial scales of the flow system.

Measurements were made on one of the flow pattern photographs obtained by Somerscales and Dropkin (Figure 3 in reference 17). This photograph is a plan view of the flow pattern observed at a Rayleigh number of 1.31×10^5 in a 50 centistoke silicone fluid of Prandtl number 300. The apparatus shown in Figure 3 was used to obtain the flow pattern.

Because of the conditions under which the photograph of the flow pattern was obtained it probably represents the flow in regions of the fluid immediately adjacent to the upper boundary of the fluid layer. This is a speculative conclusion and it would be worthwhile to investigate this point in more detail using a lighting system which illuminates different layers of the fluid.

A typical autocorrelation curve is shown in Figure 4. Such curves were obtained for a number of values of the angle θ and were found to have substantially the same character. Two features of the autocorrelation curve are of interest: the FWHM (as discussed in section 1) and the nature of the autocorrelation curve for large values of displacement ($\triangle s$).

The FWHM varies with angle θ between 0.25 in. and 0.27 in. with an average value of 0.26 in. On the average then the small structure observed at a Rayleigh number of 1.31×10^5 in the apparatus shown in Figure 3 is 0.26 in.

The correlation curve has a marked change in slope at points corresponding to a length of about 0.89 in. This is possibly associated with the cellular structure of the flow. An examination of the photograph shows this to be reasonable. If this conjecture is true then there should be in an area of 6 in. by 8 in. (the area of the test chamber) an average of 80 cells. Counting the cells in the original photograph (admittedly an uncertain process) gives a value of approximately 60 cells. The small difference between the two values is interesting but may only be coincidental. One difficulty with this interpretation rises from the dimensions of the cellular structure. This is of the same order of magnitude as the area of the stop in the correlator. In these circumstances the stop is probably not

* The exact value appears to depend on the Prandtl number of the fluid.

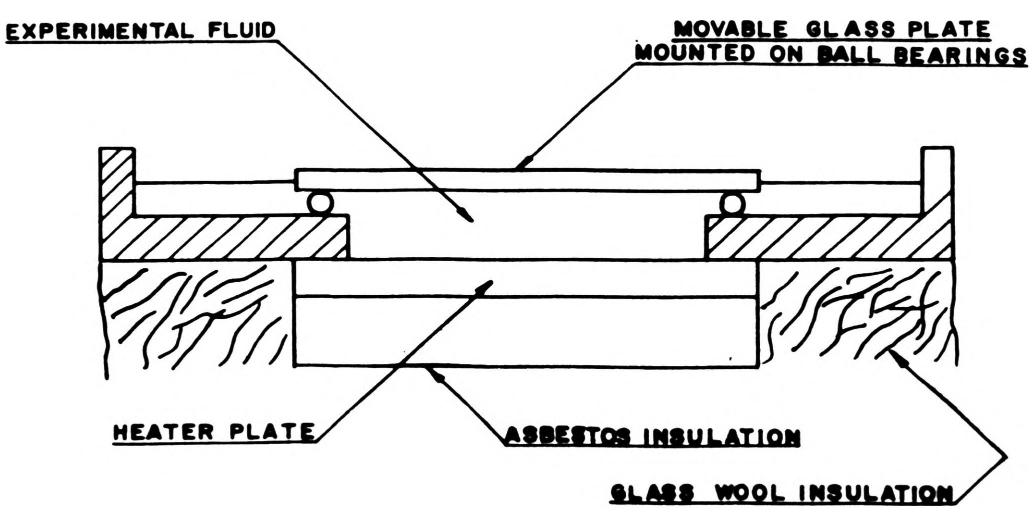

Figure 3. Schematic diagram of test chamber (from reference 17).

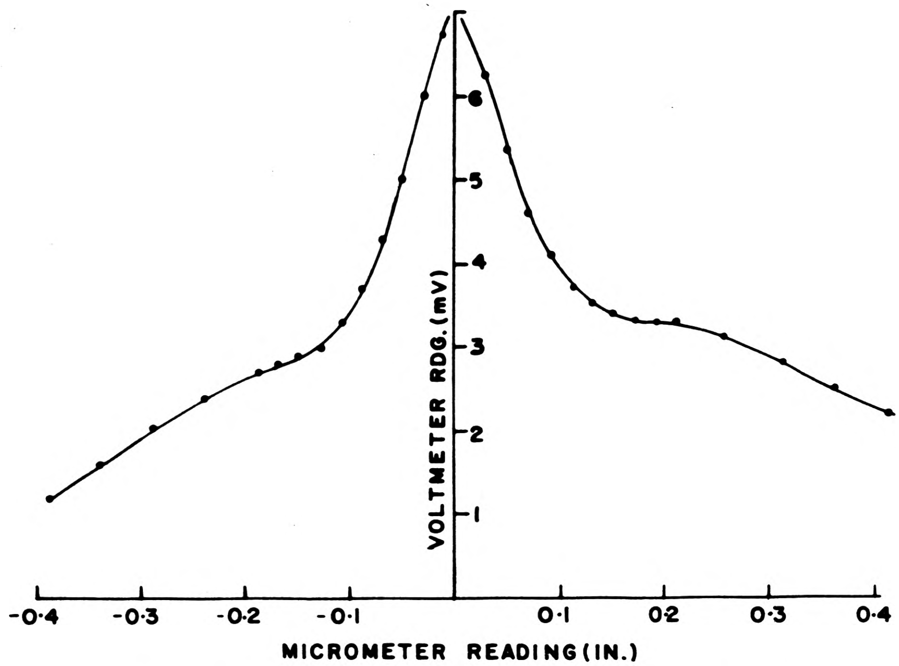

Figure 4. Typical autocovariance curve obtained from Figure 3 of reference 17 (reduction scale of photograph 2.21:1).

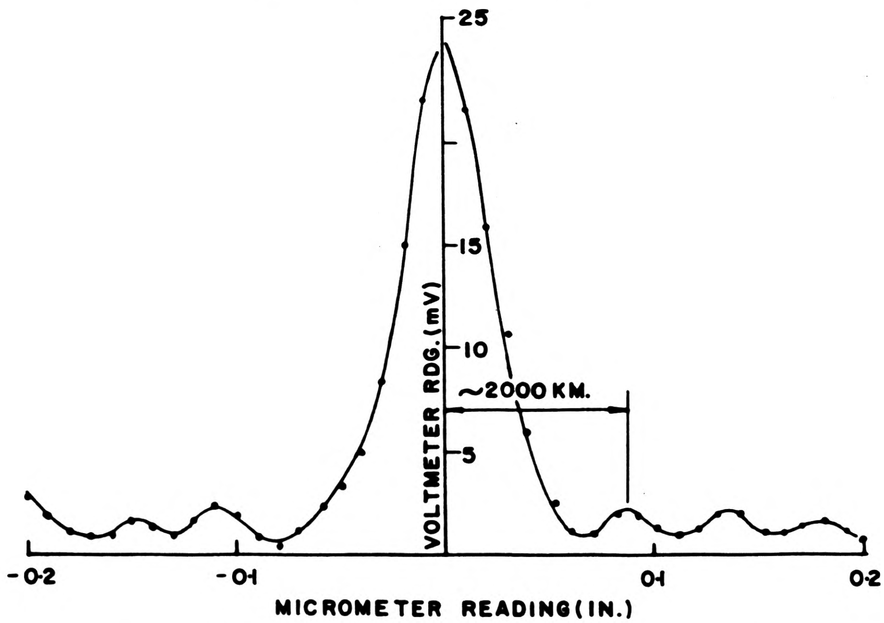

Figure 5. Typical autocovariance curve obtained from Figure 1 of reference 11.

sufficiently large to provide suitable area averaging. Furthermore the relative dimensions of the cells and the test chamber are such that a statistically significant sample of cells may not be present. This could also be the cause of the slight assymetry in the correlation curve (Figure 4).

APPLICATION TO ASTROPHYSICAL AND METEOROLOGICAL SITUATIONS AND COMPARISON WITH LABORATORY OBSERVATIONS

The autocorrelation measurements were made on a photograph of the solar granulation (which has a complicated "cellular" structure not unlike the flow patterns seen in heated horizontal fluid layers) and on a photograph of a probably cellular cloud pattern. In the former case such measurements had already been made by Leighton[11,12] and the primary intention of repeating them was to check those measurements. The results however are of intrinsic interest in that they demonstrate very clearly the type of information that can be obtained using the optical autocorrelation technique.

The autocorrelation curve made from Figure 1 of reference 11 is shown in Figure 5. The characteristic scales measured from this figure are in agreement with Leighton's values. It can be seen that the autocorrelation is periodic at large values of the displacement*. This would appear to correspond to a preferred dimension in the flow field of about 2000 km. So the photograph of the solar granulation can be interpreted as being made up of a fine scale structure arranged into "cells" of about 2000 km. diameter.

It is worthwhile noting that the characteristics of the solar granulation photograph are such that to obtain the dimensions of the cells would have been extremely tedious by any other method.

* The persistent periodicity does not appear in Leighton's correlation curve[11]. The reason for this difference is not clear.

Some cloud photographs taken from artificial earth satellites[5,8] show patterns which correspond very closely to the cellular flow patterns seen in the laboratory. The physical mechanism operating to produce these cloud formations is unknown. However, Hubert[5] has proposed that this flow situation is related to the flow of a fluid in a heated horizontal layer except that turbulent rather than molecular transport processes are dominant.

One of the cloud photographs taken from a Nimbus earth satellite was subjected to correlation analysis (see result in Figure 6). This photograph was chosen because it provided a large field of possible cellular cloud formations with the cloud structure being substantially smaller than the area of the photograph and the area of the stop in the correlator. Furthermore, of the available photographs, this had a minimum of distortion due to the curvature of the earth. The characteristic size of the small structure in the photograph is approximately 100 statute miles as determined from the FWHM of the correlation curve. In addition there appears to be a cellular flow with dimensions of about 280 statute miles.

The nature of the cloud pattern photograph was such that the large scale periodic structure would have been extremely difficult to measure by simple visual study of the photograph.

ERROR CONSIDERATIONS

The major source of error in the correlation measurements is considered to be in the initial alignment of the plates. This was minimized by the procedure for determining the point of zero displacement described in section 3. The residual uncertainty is estimated to be $\pm$ 0.005 in. The uncertainty in the measurement of the voltage output of the photomultiplier tube is $\pm$ 2 percent.

An attempt was made to estimate the effect of plate grain "noise" on the correlation measurements. The conclusions were uncertain due to difficulties in uniformly developing the plates, however this probably adds not more than 2 percent to the measured correlation.

DISCUSSION AND CONCLUSIONS

A technique has been demonstrated for the measurement of spatial scales in free convection flows. By employing photographs of the flow pattern taken at different times it would be possible to extend the method to the measurement of temporal correlations, as has been done by Bahng and Schwarzschild[1]. These two types of statistical measurements would be very useful in the study of Howard's thermal layer model of free convection at high Rayleigh numbers[4,14,18]. Optical correlation would be used in this case to measure the spatial scales and temporal periodicities in the flow field.

A preliminary investigation of these scales at a Rayleigh number of 1.40 x 10^7 has been conducted by Lazzara[10] using temperature measurements. One of the results of this study was the detection of a large scale periodic phenomenon which seemed to be confined to the regions of the fluid close to the upper and lower surfaces. As noted above, it is not certain that a flow of this type was observed in the experiments reported in this paper. Some statistical arguments were advanced for this and it is also possible that the substantially lower Rayleigh number (1.31 x 10^5) may have some effect. A program for the measurement of the scales of fluid motion in free convection for a wide range of Rayleigh numbers and with fluids of different Prandtl numbers would provide valuable information on the problem of free convection at high Rayleigh numbers.

The autocorrelation measurements on the solar granulation and terrestrial cloud photographs revealed large scale ordered flows which would have been difficult to detect by other means. Because of the different physical phenomena involved one would hesitate to apply conclusions obtained from laboratory flows to the solar granulation. However, for the cloud photographs, regardless of the valid objections (based on the physics of the atmospheric flow) raised by Hubert[5] to laboratory investigation of this atmospheric phenomonon, such a comparison could be very fruitful. In particular, a wide range of flow conditions can be examined in the laboratory under carefully controlled conditions. Such experiments might reveal significant conditions that could be searched for in a program of field measurements.

In conclusion it should be emphasized that the results reported here are preliminary to a more extensive investigation of free convection in a heated fluid layer.

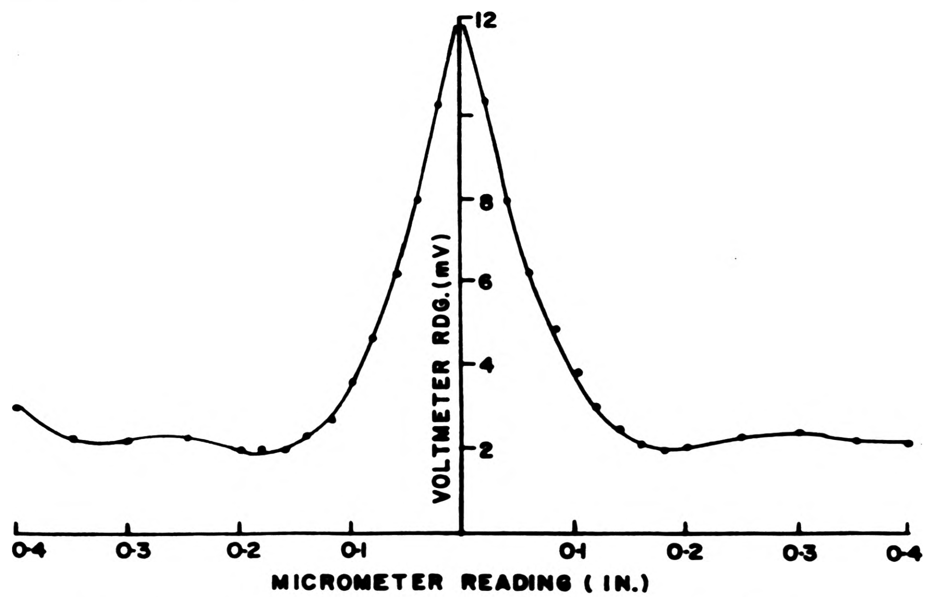

Figure 6. Autocovariance curve obtained from NIMBUS satellite cloud photographs between 120°W - 134°W, 22°S - 30°S (approximate scale: 0.10in. = 40 statute miles).

ACKNOWLEDGMENTS

Drawings for the construction of the optical correlator were provided by the Bell Telephone Laboratories with the invaluable assistance of Dr. E. R. Kretzmer. The correlator was constructed by J. Marshall and the success of the work is directly related to his skillful workmanship.

High quality prints of certain cloud photographs were supplied by the Environmental Science Services Administration through the good offices of Dr. L. F. Hubert, Chief, Synoptic Meteorology Branch.

SYMBOLS

A	area of averaging (in.^{2});
C	autocovariance (lumens^2); specific heat of experimental fluid (see definition of Rayleigh number) (BTU/lbm°F);
g	acceleration of gravity (4.17×10^8 ft/h^2);
k	thermal conductivity of experimental fluid (BTU/h ft°F);
L	distance between the upper and lower surfaces of the fluid (ft.);
R	coefficient of autocorrelation (dimensionless);
Ra	Rayleigh number ($g \beta \rho C L^3 \Delta t/\nu k$) (dimensionless);
s	radial polar coordinate (inches);
T	transparency at point s, θ on the photographic slide (lumens);
$\bar{T}$	spatial mean transparency (lumens);
T'	transparency fluctuation at point s, θ on the photographic slide ($=T-\bar{T}$) (lumens);
t_H	temperature of the lower (hot) surface of the fluid layer (°F);
t_c	temperature of the upper (cold) surface of the fluid layer (°F);
β	coefficient of volume expansion of the experimental fluid (ft^2/h);
Δs	displacement along a radius (in.);
Δt	temperature difference between upper and lower surfaces of the fluid layer ($t_H - t_c$) (°F);
θ	angular polar coordinate (degrees);
λ	spatial scale (in.);
ν	kinematic viscosity of the experimental fluid (ft^2/h);
ρ	density of the experimental fluid (lbm/ft^3);
σ_T^2	variance of T (lumens^2).

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APPENDIX

Suppose the transparency of a photographic slide at some point distance Δs/2 from an appropriate origin is given by

$$T(\Delta s/2\theta) = \bar{T} + T'(\Delta s/2,\theta)$$

where

$$\bar{T} = \frac{1}{A}\int_A T(\Delta s/2,\theta)dA$$

Then the autocovariance which is defined as

$$C(\Delta s,\theta) = \frac{1}{A}\int_A T'(+\Delta s/2,\theta)T'(-\Delta s/2,\theta)dA$$

can be written

$$C(\Delta s,\theta) = \frac{1}{A}\int_A \left[T(+\Delta s/2,\theta)-\bar{T}(\theta)\right]\left[T(-\Delta s/2,\theta)-\bar{T}(\theta)\right]dA$$

Or, employing a simpler notation

$$C(\Delta s,\theta) = \overline{(T_+ - \bar{T}_+)(T_- - \bar{T}_-)}$$

Then it can be shown that

$$C(\Delta s,\theta) = \overline{T_+T_-} - \bar{T}_+\bar{T}_-$$

or

$$C(\Delta s,\theta) = \frac{1}{A}\int_A T(-\Delta s/2,\theta)T(+\Delta s/2,\theta)dA - \frac{1}{A^2}\int_A T(-\Delta s/2,\theta)dA\int_A T(+\Delta s/2,\theta)dA \qquad (3)$$

The coefficient of autocorrelation is defined as

$$R(\Delta s,\theta) = \frac{C(\Delta s,\theta)}{\sigma_T^2}$$

or

$$\sigma_T^2 = \frac{1}{A}\int_A T'^2(0,\theta)dA$$

$$= \frac{1}{A}\int_A \left[T(0,\theta) - \bar{T}(\theta)\right]^2 dA$$

$$= \frac{1}{A}\int_A T^2(0,\theta)dA - \left[\frac{1}{A}\int_A T(0,\theta)dA\right]^2$$

Hence we obtain equation (4) in the text.

It can be shown (21) all values of the coefficient of autocorrelation lie between zero and one.

To obtain a measure of the rate at which R varies about the point $\Delta s = 0$ we can expand R in a Taylor series about $\Delta s = 0$ (21)

$$R(\Delta s,\theta) = R(0,\theta) + \Delta s R'(0,\theta) + \frac{(\Delta s)^2}{2!} R''(0,\theta) + ----$$

Since R is a maximum at $\Delta s = 0$ it follows that

$$R'(0,\theta) = 0$$

also

$$R(0,\theta) = 1$$

Hence

$$R(\Delta s,\theta) \approx 1 + \frac{(\Delta s)^2}{2} R''(0,\theta)$$

If we write

$$\lambda = \left[-\frac{1}{2} R''(0,\theta) \right]^{-\frac{1}{2}}$$

where the minus sign is introduced because $R''(0,\theta)$ is negative because R is a maximum at $\Delta s = 0$, then

$$R(\Delta s,\theta) \approx 1 - \left(\frac{\Delta s}{\lambda}\right)^2$$

It can be seen that λ is a direct measure of the rate at which R changes in the vicinity of $\Delta s = 0$. Now the rate at which R changes will depend on the mean width of the curves representing $T'(\Delta s,\theta)$, as can be seen from Figure 7. Hence λ is a measure of the mean width of the transparency curves.

According to the preceding discussion the most logical procedure for the determination of λ would be to fit a parabola through the autocorrelation curve in the vicinity of its maximum value. However, in view of the tediousness of such a procedure and because in general only an order of magnitude figure is required, it is conventional to adopt a simpler, approximate technique. The half-width of the correlation curve at its half-maximum is used to represent the scale λ. Conventionally the full width at the half-maximum (FWHM) (approximating 2λ) is generally employed.

Suppose a periodic random process is mixed with a random noise process with respective magnitudes P(s) and N(s). These processes are assumed to be statistically independent and hence unrelated. The autocovariance of the process F(s) formed by the addition of P and N is given by

$$C_F(\Delta s) = \overline{\left[P(s)+N(s)\right]\left[P(s+\Delta s)+N(s+\Delta s)\right]}$$

$$= \overline{P(s)P(s+\Delta s)} + \overline{P(s)N(s+\Delta s)}$$

$$+ \overline{N(s)P(s+\Delta s)} + \overline{N(s)N(s+\Delta s)}$$

Since P and N are uncorrelated in this case

$$\overline{P(s)N(s+\Delta s)} = \overline{N(s)P(s+\Delta s)} = \overline{N(s)}\,\overline{P(s)}$$

It can also be shown that the autocovariance of a periodic funtion is also periodic hence $\overline{P(s)P(s+\Delta s)} = C_P(\Delta s)$ is periodic. Also for large values of Δs, $\overline{N(s)N(s+\Delta s)}$ approaches $\overline{N(s)}^2$ because N tends to become uncorrelated as Δs increases.

Hence for large Δs

$$C_F(\infty) = C_P(\infty) + 2\overline{N(s)}\,\overline{P(s)} + \overline{N(s)}^2$$

The behavior of C_F is shown in Figure 8. Hence if the autocovariance of a process F shows a periodic behavior then F must contain a periodic component.

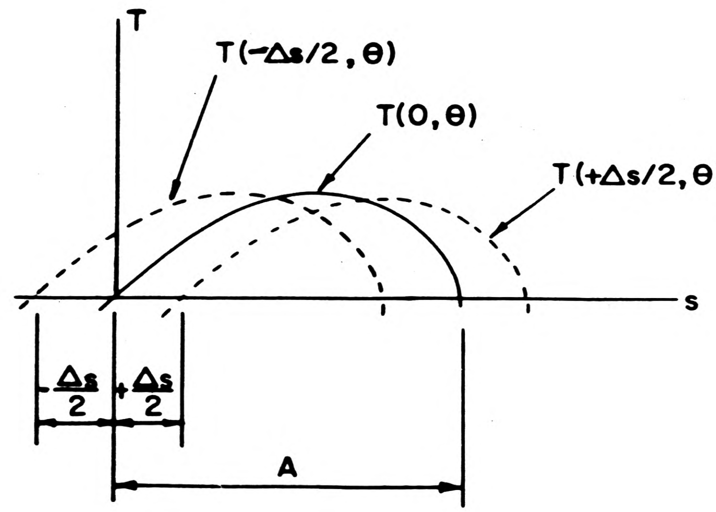

Figure 7. Illustration of autocorrelation (based on Figure 19 of reference 21).

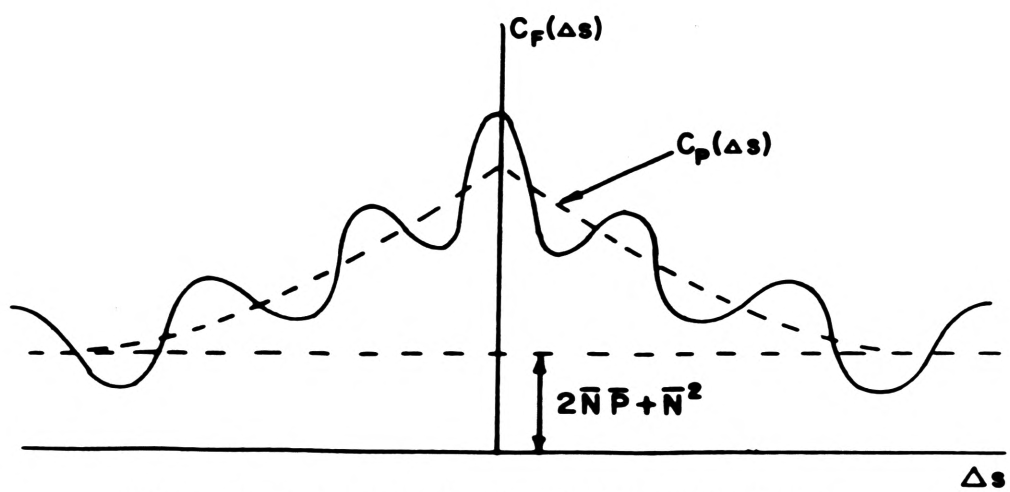

Figure 8. Autocovariance of a periodic random process mixed with a random noise process.