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THE INFLUENCE OF AMBIGUITY AND NOISE ON THE MEASUREMENT OF TURBULENT SPECTRA BY DOPPLER SCATTERING*

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ABSTRACT

A fundamental uncertainty in velocity measurement by Doppler scattering is caused by the finite residence time of the scattering particles in the observation volume; the arrival of scattering particles at arbitrary times gives rise to fluctuations in phase (and hence frequency) of the observed Doppler frequency. An estimate is obtained for the spectrum of these frequency fluctuations (called ambiguity noise). The frequency at which the spectral levels of a turbulent signal and the ambiguity noise are equal, provides a limit to the temporal resolution of an instantaneous velocity measurement; this limit is obtained, and shown to be quite restrictive. The influence of electronic noise is also analyzed and found to be negligible.

An experimental installation is described in which instantaneous fluctuating turbulent velocities may be measured by Doppler scattering using coherent radiation from a laser. Measurements are presented of the spectra of ambiguity noise and electronic noise. The agreement with theory is excellent.

INTRODUCTION

Recently several investigators[1,2,3,6] have measured mean velocities and mean square fluctuating velocities in flowing liquids by examining the frequency shift in monochromatic coherent radiation scattered from particles in the liquid, using a laser source for the incident radiation. The scattered and unscattered radiation is heterodyned on a photocell, producing an electrical signal at the difference frequency, and the spectrum of this difference signal is examined using conventional techniques.

Even in a steady laminar flow, however, the Doppler frequency is not steady. The signal received by the photocell is the sum of the signals scattered by all the scatterers present in the measuring volume at that instant; they may be taken all to have the same frequency, but each has a phase dependent on its location in the volume as well as an intensity dependent on its size.

As scatterers leave the scattering volume, and new ones enter, the signal gradually looses coherence; when the entire population has changed, all coherence is lost.

An individual scatterer may be taken as producing a characteristic signal like

$$f(t) = \cos \omega_o t \; \frac{1}{\sqrt{2\pi}\sigma} e^{-t^2/2\sigma^2}$$

where ω_o is the Doppler frequency, and the Gaussian form has been chosen because the intensity profile of the laser beam is Gaussian; σ is an arbitrary measure of beam width. The net signal may be represented as

$$u(t) = \int_{-\infty}^{+\infty} f(t-\tau)\,d\zeta(\tau) \;, \quad \overline{d\zeta(\tau)\,d\zeta(\tau')} = \begin{cases} \sigma, & \tau \neq \tau' \\ d\tau, & \tau = \tau' \end{cases} \tag{2}$$

where $d\zeta(\tau)$ is, in fact, statistically independent at two different times. This is similar to the shot effect (c.f. Rice, in Wax[4]).

σ may be related to more meaningful parameters in the following way: - roughly speaking, the shifts are averaged over the beam. This gives the beam an effective width in time of

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-t^2/2\sigma^2} \, dt = T/\sqrt{2\pi}\sigma = 1 \tag{3}$$

At the mean velocity this corresponds to a distance of

$$\sqrt{2\pi}\sigma U \tag{4}$$

as an effective beam width. A spatial variation of wave number K must have $\pi/K \geqslant \sqrt{2\pi}\sigma\bar{U}$ in order not to be seriously smoothed by averaging. Then the wave number limit of resolution of the beam is

$$K_* = \sqrt{\pi/2}\,(1/\sigma U) \tag{5}$$

so that (1) may be written as

$$f(t) = \cos\omega_o t \; (K_*\bar{U}/\pi) e^{-t^2 K_*^2 U^2/\pi} \tag{6}$$

By Campbell's theorem[4] the spectrum of (2) may easily be obtained as FF^*, where F is the Fourier transform of f:

$$FF^* = \frac{1}{4}\left[e^{-\pi(\omega+\omega_o)^2/2K_*^2U^2} + e^{-\pi(\omega-\omega_o)^2/2K_*^2U^2} \right] \tag{7}$$

The half width of the spectral line may be taken as the root-mean-square deviation from the mean Doppler frequency, or

$$\Delta\omega = UK_*/\sqrt{\pi} \tag{8}$$

so that the better the resolution, the faster the population of scatterers changes, and the wider is the line.

THEORETICAL ANALYSIS

Measuring the Instantaneous Velocity

The foregoing analysis is well known, and clearly must be taken into consideration in measuring mean square fluctuating velocities, since when the deviation due to the fluctuating velocity becomes of the order of (8), the fluctuating velocity can no longer be distinguished from the fluctuations due to incoherence. The center frequency* is

$$\omega_o = (2\pi U/\lambda)\; 2\sin(\theta/2) \tag{9}$$

where θ is the scattering angle, and λ is the wavelength of the incident radiation, and as the limit of resolution we have

$$(\Delta\omega/\omega_o) = K_*\lambda/4\pi^{3/2} \sin(\theta/2) \leqslant u'/U \tag{10}$$

The matter of the measurement of instantaneous fluctuating velocity however, is somewhat more serious, and involves the question of the spectrum of the fluctuations in phase. For example, if the fluctuations in phase occurred at a frequency corresponding to the spacing between particles this could be a high frequency relative to that of the fluctuation in velocity and could be averaged out during a half period of the latter, reducing the line width and improving the resolution. Unfortunately, this attractive possibility does not correspond to reality; to see this we must obtain estimates for the spectrum of the phase. This question is particularly dif-

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*Throughout we are considering only the component of velocity in the direction of the difference of the incident and scattered wavenumber vectors.

ficult, since it is essentially that of the joint distribution at two times of the time between successive zeros of a random function, a problem that has never been successfully solved.[4]

The Fluctuating Frequency

Expression (2) may be written, using (6) as

$$\begin{aligned} u(t) &= \int \cos \omega_o(t-\tau)(K_*U/\pi)e^{-(t-\tau)^2K_*^2U^2/\pi}\, d\zeta(\tau) = \\ &= \cos \omega_o t \int \cos \omega_o\tau(K_*U/\pi)e^{-(t-\tau)^2K_*^2U^2/\pi}\, d\zeta(\tau) + \\ &\quad \sin \omega_o t \int \sin \omega_o\tau\,(K_*U/\pi)e^{-(t-\tau)^2K_*^2U^2/\pi}\, d\zeta(\tau) \end{aligned} \tag{11}$$

Let us define

$$f(t) = \int \cos \omega_o\tau\,(K_*U/\pi)e^{-(t-\tau)^2K_*^2U^2/\pi}\, d\zeta(\tau) \tag{12}$$

$$g(t) = \int \sin \omega_o\tau(K_*U/\pi)e^{-(t-\tau)^2K_*^2U^2/\pi}\, d\zeta(\tau)$$

Then we may write

$$u(t) = \left(f^2 + g^2\right)^{1/2} \cos(\omega_o t-\Phi) \tag{13}$$

$$\Phi(t) = \tan^{-1}(g/f)$$

Now the usual sort of measuring circuit would remove the amplitude information from the signal u(t) by amplification and clipping, keeping only the information on zero-crossing, and would then produce a signal proportional to frequency

$$\omega = \omega_o - \frac{d\Phi}{dt} \tag{14}$$

Suppose we write

$$\omega_o \int e^{i\alpha t} dZ(\alpha),\quad \overline{dZ(\alpha)dZ^*(\alpha')} = \begin{cases} 0, & \alpha\neq\alpha' \\ \dfrac{\omega_o^2}{U^2}\,\Phi\left(\dfrac{\alpha}{U}\right)\dfrac{d\alpha}{U}, & \alpha = \alpha' \end{cases}$$

$$\dot\Phi = \int e^{i\alpha t}\, dN(\alpha),\quad \overline{dN(\alpha)dN^*(\alpha')} = \begin{cases} 0, & \alpha\neq\alpha' \\ \eta(\alpha)\,d\alpha & \alpha=\alpha' \end{cases}$$

where $\overline{\omega_o}$ is the mean center frequency; and Φ is the one dimensional velocity spectrum. Then

$$\omega = \int e^{i\alpha t}\, dZ(\alpha) - \int e^{i\alpha t}\, dN(\alpha) \tag{16}$$

and the spectrum of the composite signal is given by

$$\frac{\overline{\omega_o}^2}{U^3}\,\Phi\left(\frac{\alpha}{U}\right) + n(\alpha) \tag{17}$$

The Spectrum of the Frequency

The problem is thus reduced to what is $n(\alpha)$? Certain things can be said immediately; if $d\zeta(\tau)$ is statistically independent at different times, then f and g are Gaussian random variables. The correlations are given by

$$\begin{aligned} \overline{f(t)f(t')} &= \frac{1}{2}\int (K_*^2U^2/\pi^2)e^{-(t-t'+\eta)^2K_*U^2/\pi-\eta^2K_*^2U^2/\pi}\,d\eta \\ &= \frac{K_*U}{2\pi\sqrt{2}}\,e^{-K_*^2U^2(t'-t)^2/2\pi} = \overline{g(t)g(t')} \end{aligned} \tag{18}$$

$$\overline{f(t)g(t')} \cong 0$$

An approximation is involved in writing (18), namely that

$$\exp(-\pi\overline{\omega_o}^2/8\,K_*^2U^2) \ll 1 \tag{19}$$

or small bandwidth; otherwise each component of (18) would contain an additional small term periodic in time, of period $2\overline{\omega_o}$.

Hence, if we take f and g to be the x and y coordinates of a point in the plane, the point moves in such a way that is has a circulatory symmetric Gaussian distribution. Φ is the angle subtended by the radius vector to the point (with the x-axis, say).

It is evident that Φ is not stationary. If Φ begins from zero, say, then for times short compared to to $2\pi^{3/2}/UK_*$ the probability of finding $|\Phi|>2\pi$ will be small; as t increases, it is more and more likely that Φ will have made one or more revolutions of the origin. Hence, the value of the $\dot\Phi$ spectrum at the origin will be given by

$$\int_{-\infty}^{+\infty} \dot\Phi(t)\,\dot\Phi(t+\eta)\,d\eta = \frac{d}{dt}\overline{\Phi^2}\;; \tag{20}$$

Since Φ is non-stationary, (20) will be non-zero. Now, the joint characteristic functional of f and g is determined entirely by one parameter, $\Delta\omega$, and consequently, that of Φ will also. Hence, we must have

$$\frac{d}{dt}\overline{\Phi^2} \propto \Delta\omega \tag{21}$$

with an unknown coefficient (hopefully of order unity).

To obtain the exact form of the spectrum, consider the relation between r, the radius vector, and $\dot\Phi$, the angular velocity. It is easy to show that $V_\Phi = \dot\Phi r$ and r are statistically independent. Since the distribution has circular symmetry, we may consider the joint distribution of $\dot\Phi$ and r where $\Phi = 0$. Then $V_\Phi = \dot g$ and $r = f$ and they are independent. Hence V_Φ and r are independent for all Φ. We can write, differentiating (13),

$$\dot\Phi = \frac{f\dot g-g\dot f}{f^2+g^2} = \frac{V_\Phi}{r} \tag{22}$$

Taking the correlation of rV_Φ at two different times, using the fact that f and g are independent, stationary Gaussian variables with the same correlation, we have

$$\overline{\dot\Phi\dot\Phi'} = \frac{2\sigma^4(\rho'^2-\rho\rho'')}{\overline{rr'}}\;\overline{\frac{1}{r}\frac{1}{r'}} \tag{23}$$

where ρ is the correlation coefficient of f:

$$\rho = e^{-K_*^2U^2\tau^2/2\pi} = e^{-\Delta\omega^2\tau^2/2} \tag{24}$$

Now we must evaluate $\overline{rr'}$ and $\overline{\frac{1}{r}\frac{1}{r'}}$. Using the fact that f and g are independent Gaussian variables, it is straight-forward to obtain the density for r:

$$\frac{r}{\sigma^2}\,e^{-r^2/2\sigma^2} \tag{25}$$

from which we obtain

$$\overline{r^2} = 2\sigma^2 \tag{26}$$

and

$$\overline{\frac{1}{r^2}} = \int_0^\infty \frac{1}{r\sigma^2}\,e^{-r^2/2\sigma^2}\,dr = \infty \tag{27}$$

Thus, we have immediately that $\overline{\dot\Phi^2} = \infty$. This is not surprising: since V_Φ and r are independent, the same values of V_Φ may occur for small and large r: hence, when r is small, very large values of $\dot\Phi$ may occur. Thus we may expect the spectrum of $\dot\Phi$ to be very broad - the behavior of infinity will depend on the type of singularity at the origin.

The joint density for r and r' may be obtained as

$$\frac{I_o\left(\dfrac{\rho}{1-\rho^2}\dfrac{rr'}{\sigma^2}\right)}{\sigma^4(1-\rho^2)}\,e^{-\left(\frac{1}{2\sigma^2}\frac{(r^2+r'^2)}{1-\rho^2}\right)}\,rr' \tag{28}$$

where I_o is the modified Bessel function of the first kind of order zero. Using the asymptotic behavior of this

$$\begin{aligned} &I_o(x)\sim 1, \quad x\to 0 \\ &I_o(x)\sim\frac{e^x}{\sqrt{2\pi x}}, \quad x\to\infty \end{aligned} \tag{29}$$

we may obtain asymptotic expressions for $\overline{rr'}$ and $\overline{\frac{1}{r}\frac{1}{r'}}$, for $\rho\to 0$ and $\rho\to 1$. $\overline{rr'}$ is straight forward.

$$\overline{rr'}\sim 2\sigma^2\rho,\quad \rho\to 1$$

$$\overline{rr'} \sim \sigma^2 \frac{\pi}{2} (1-\rho^2)^2 \sim \sigma^2 \frac{\pi}{2}, \ \rho \to 0 \tag{30}$$

$\overline{\frac{1}{r}\frac{1}{r'}}$, is well behaved as $\rho \to 0$, giving

$$\overline{\frac{1}{r}\frac{1}{r'}} \sim \frac{1}{\sigma^2} \frac{\pi}{2} \tag{31}$$

As $\rho \to 1$, however, the behavior is quite complicated, and we obtain

$$\overline{\frac{1}{r}\frac{1}{r'}} \sim \frac{1}{2\sigma^2} \int_{\frac{1-\rho^2}{2\rho^2}}^{\infty} \frac{dt}{t} e^{-t} \sim \frac{-\ln \tau \Delta\omega}{\sigma^2} \tag{32}$$

Since the singularity is logarithmic, the spectrum evidently drops off as inverse frequency. Since it is determined by a single parameter, we may expect the break to come near $\alpha \sim \Delta\omega$.

In order to obtain n(0)

$$n(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\sigma^4 \Delta\omega^2 \rho^2}{\overline{rr'}} \ \overline{\frac{1}{r}\frac{1}{r'}} \, d\tau \tag{33}$$

we will approximate the integrand in (33) by the small- ρ asymptotic forms for $\tau\Delta\omega \geqslant 1$. The small-$\rho$ contribution is

$$\frac{2\Delta\omega}{\pi} \int_1^{\infty} e^{-x^2} dx = 0.28 \ \frac{\Delta\omega}{\pi} \tag{34}$$

and as $\rho \to 1$ we have

$$\frac{\Delta\omega}{\pi\sqrt{2}} \int_0 dt \ e^{-t} t^{-\frac{1}{2}} + \frac{\Delta\omega}{2\pi} \int_{\frac{1}{2}}^{\infty} dt \left(\frac{1}{t} - \sqrt{2}\frac{)}{t}\right) e^{-t} \sim \frac{\Delta\omega}{\sqrt{2}\pi} \tag{35}$$

Hence, altogether we have roughly

$$n(0) = 1.5 \frac{\Delta\omega}{\pi} \tag{36}$$

We expect $n(\alpha)$ to be flat at this value at least to $\alpha \sim \Delta\omega$, and to fall slowly thereafter. From a practical point of view, this point is usually beyond the range of measurement, so that we may expect $n(\alpha)$ to be experimentally white.

The Limit of Measurement

The limit of measurement of the spectrum will now be (somewhat arbitrarily) determined by the point α_o where the signal/noise ratio becomes unity, or where

$$\frac{\overline{\omega}_o^2}{U^3} \Phi \left(\frac{\alpha_o}{U}\right) = n(\alpha_o) \tag{37}$$

If we change to Kolmogorov variables, $\alpha_o \eta / U = \tilde{K}_o$, $K_* \eta = \tilde{K}_*$, $\Phi(\alpha_o/U) = \epsilon^{2/3} \eta^{5/3} \Phi(K_i)$, where η is the Kolmogorov microscale, $\eta = (\nu^3/\epsilon)^{\frac{1}{4}}$, we have

$$\Phi(K_o) = K_* \ 6.83 \times 10^{-3} \ R^2 \tag{38}$$

where

$$R = \frac{U}{\nu 2 \sin(\theta/2)} \tag{39}$$

The crests of the light waves make an angle of $\theta/2$ with the flow direction; if $\lambda/2$ is taken as the smallest length that can be resolved, then $\lambda/2 \sin(\theta/2)$ is the smallest length that can be resolved in the mean flow direction.

It seems reasonable to require that $\tilde{K}_o = K_*$; that is, that the resolution due to ambiguity be the same as that due to finite beam width. Hence

$$R = \left(\frac{\Phi(\tilde{K}_o)}{6.83 \times 10^{-3} \tilde{K}_o}\right)^{\frac{1}{2}} \tag{40}$$

Pao[5] gave numerical values for $\tilde{\Phi}(\tilde{K})$ (it must be born in mind that Pao's values are normalized on the half-line, whereas we have normalized on the whole-line: thus Pao's values must be divided by two for use in (40)); Pao's spectrum is plotted in figure 1, together with the spectra $\tilde{K}_* \ 6.83 \times 10^{-3} R^2$ for various values of $\tilde{K}_o$ and corresponding R from (40). The values of R obtained from (40) are plotted in figure 2.

From figure 2, it can be seen that, for $\tilde{K}_o = 1$, $R = 0.62$. With typical values of $\lambda = 6 \times 10^{-5}$ cm, $\nu = 10^{-2}$ cm^2/sec and $2 \sin(\phi/2) = 0.2$ (corresponding to measurements in water), we have

$$U \sim 20 \text{ cm/sec} \tag{41}$$

Relaxation of the resolution requirements to about $\tilde{K}_o = 0.5$ increases the permissible U by nearly an order of magnitude; it should be remembered, however, that (40) is actually quite conservative. A signal/noise ratio of unity does not allow accurate determination of spectral values; it can be seen from figure 1 that for $\tilde{K}_o \sim 1$, the spectrum begins to deviate perceptably at about $\tilde{K}_o = 0.5$.

It might be thought that the stringency of the requirement could be reduced somewhat by increasing the value of θ. Unfortunately, however, forward scattering is much more efficient; it is difficult to obtain a Doppler signal when $2 \sin(\theta/2)$ is much above 0.7. Another practical limitation is the value of the Doppler frequency; the higher the Doppler frequency, the more difficult the electronic circuitry becomes. We can write

$$R = \frac{U^2}{\nu f_o} \tag{42}$$

where f_o is the Doppler center frequency in cycles/sec. For work in water, for $\tilde{K}_o = 1$, if it is desired to keep $f_o \not> 5 \times 10^5$, we have

$$U \sim 56 \text{ cm/sec} \tag{43}$$

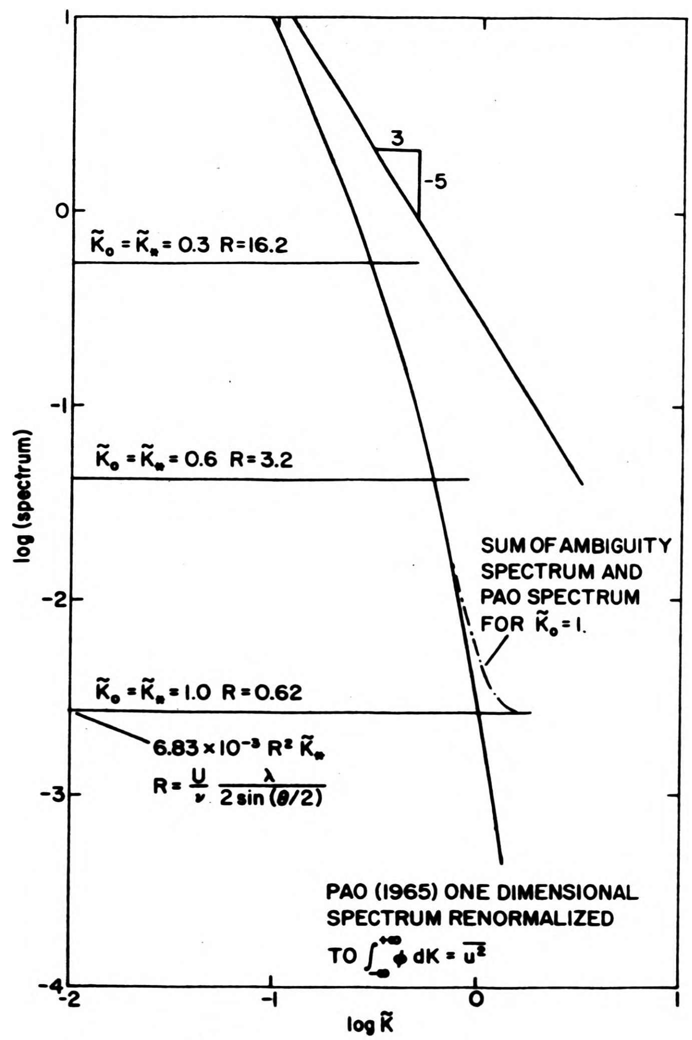

Figure 1. Pao one-dimensional spectrum renormalized to $\int_{-\infty}^{\infty} \phi dK = \overline{(u')^2}$

$$\overline{rr'} \sim \sigma^2 \frac{\pi}{2}(1-\rho^2)^2 \sim \sigma^2 \frac{\pi}{2}, \ \rho \to 0 \qquad (30)$$

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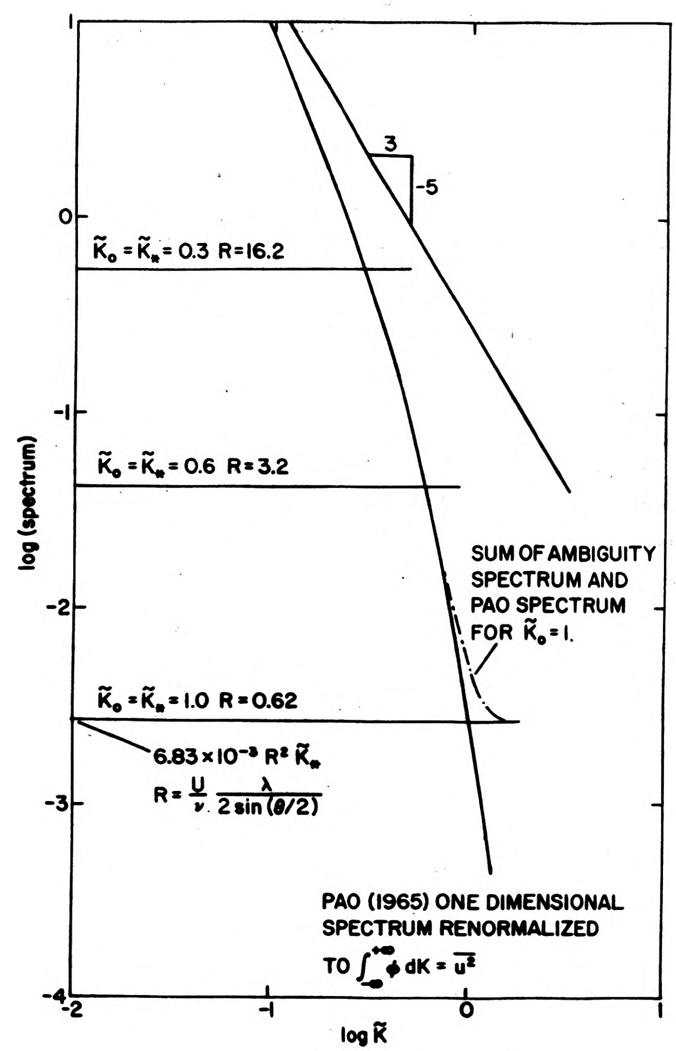

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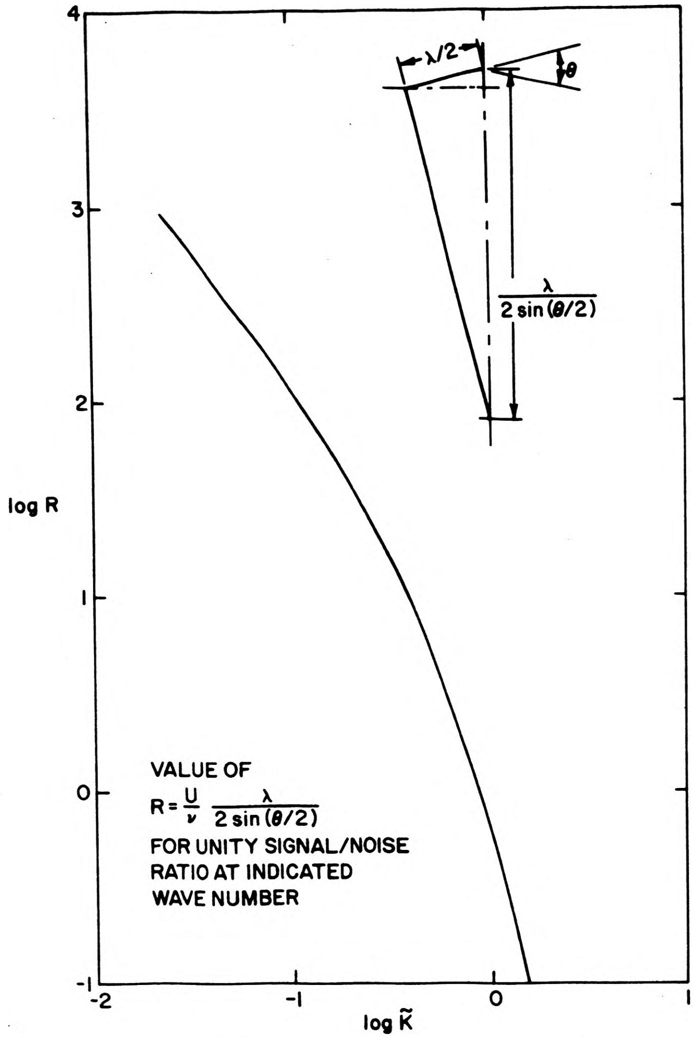

Figure 2. Value of $R = (U/\nu)(\lambda/2\sin(\theta/2))$ for unity signal/noise ratio at indicated wave number.

Hence, practically speaking, in water, it is difficult to measure the entire spectrum above about 1 m/sec (presuming that a certain amount of correction can be done by subtracting the ambiguity spectrum).

Electronic Noise

If white noise is filtered around a frequency ω_o by a narrow band filter of half width $\Delta\omega_f$, it is not difficult to show that the resulting signal can also be represented as

$$f(t)\cos\omega_o t + g(t)\sin\omega_o t \tag{44}$$

where f and g are uncorrelated, identically distributed Gaussian processes with identical spectra centered on the origin, having the shape of the filter function; thus the spectra have a half power point of $\Delta\omega_f$ (this is, in fact, a general property of narrow-band signals).

If the filtered white noise is combined with a sinusoidal signal

$$A\cos\omega_o t \tag{45}$$

the result is

$$(A+f)\cos\omega_o t + g\sin\omega_o t = [(A+f)^2+g^2]^{\frac{1}{2}}\cos\left(\omega_o t-\tan^{-1}\left[\frac{g}{f+A}\right]\right) \tag{46}$$

The r.m.s. value of the signal is $A/\sqrt{2}$; the r.m.s. value of the noise $\sqrt{\overline{f^2}} = \sqrt{\overline{g^2}}$. Thus the signal/noise ratio is

$$A/\sqrt{2\overline{f^2}} = a, \text{ say.} \tag{47}$$

Again the frequency is given by

$$\omega = \omega_o - \frac{d}{dt}\tan^{-1}\left(\frac{g}{A+f}\right) = \omega_o - \frac{(A+f)\dot{g}-g\dot{f}}{(A+f)^2+g^2} \cong \omega_o - \dot{g}/A \tag{48}$$

to first order. Allowing ω_o to fluctuate in proportion to the velocity, the spectrum of this signal is given by

$$\frac{\omega_o^2}{U^3}\Phi(\alpha/U) + \frac{\alpha^2}{A^2}\psi(\alpha) \tag{49}$$

where $\Phi(K)$ is the one-dimensional turbulent spectrum, and $\psi(\alpha)$ is the filter function, which must satisfy

$$\overline{f^2} = 2\Delta\omega_f\psi(0) \tag{50}$$

Note that, because of the presence of A, so long as a is large enough, the likelihood of the point with coordinates A+f, g going around the origin is much reduced, so that ϕ is stationary; hence the value of the spectrum at the origin is zero. Also, since the magnitude of r is much less likely to be small, $\dot{\phi}$ does not attain such large values, so that $\overline{\dot{\phi}^2}$ is finite.

Again, placing (48) in Kolmogorov variables, we have (ignoring the roll-off of the filter function)

$$\tilde{\Phi}(\tilde{K}) + \frac{R^2(U/f_o\eta)}{32\pi^3 a^2 \frac{\Delta\omega_f}{\omega_o}} \tag{51}$$

This is a somewhat artificial situation, since we do not ordinarily have a signal of constant amplitude A, but a signal of exactly the form (44), so that the sum should be analyzed by the same method as that used for the ambiguity alone ((50) really corresponds to $\Delta\omega/\Delta\omega_f \to 0$). Our result is suitable only for evaluating the electronics, with an artificial sinusoidal input; hence, we really have no $\tilde{\Phi}$, and no η. Thus, in our presentation of data, we will use the U and η corresponding to a possible flow.

In a real flow, with mixed noise and turbulence, expression (23) is valid, where ρ is the composite correlation of signal plus noise. In our measurements, ordinarily the signal/noise ratio is of the order of 30 or better, while $\Delta\omega_f$ is not much more than 3 $\Delta\omega$. Under these circumstances, the composite ρ and its first and second derivative do not differ from ρ for the ambiguity signal by more than 1%. Hence, the presence of noise will only increase the ambiguity spectrum by an amount of the order of 1%. This is a predicated on the filter function falling off faster than ω^{-2}, so that second derivatives of the noise correlation are finite at the origin. Our filter falls off as ω^{-4}.

EXPERIMENTAL APPARATUS

In figure 3 is shown our optical set-up. Our scattering angle is about 7°, and our scattering agent is a volume solution of milk of concentration 2×10^{-4}. The measurements were made at the center of a uniform flow about two diameters downstream of the entrance in a 2" diameter glass tube. The flow enters the tube through a 36:1 contraction, after passing through two screens and a fine-mesh honeycomb. The polarization rotator on the laser is adjusted for optimum signal strength (it controls the relative energies in the scattered and reference beams). The polarization rotator in the incident beam is adjusted for optimum strength (so that the reference beams have the same polarity). The polaroid filter is used as a fine adjustment on relative beam intensity. The entire optical system rests on a 700 lb. concrete table supported by four viscoelastic sandwich pads.

Beam diameters at the focal point were measured <u>in situ</u> using an 80x B&L binocular microscope with a calibrated reticle in the right eye piece. Apparent beam widths were interpreted by assuming that the human eye perceives the beam edge at an intensity 2^{-9} times the maximum intensity at the center[7].

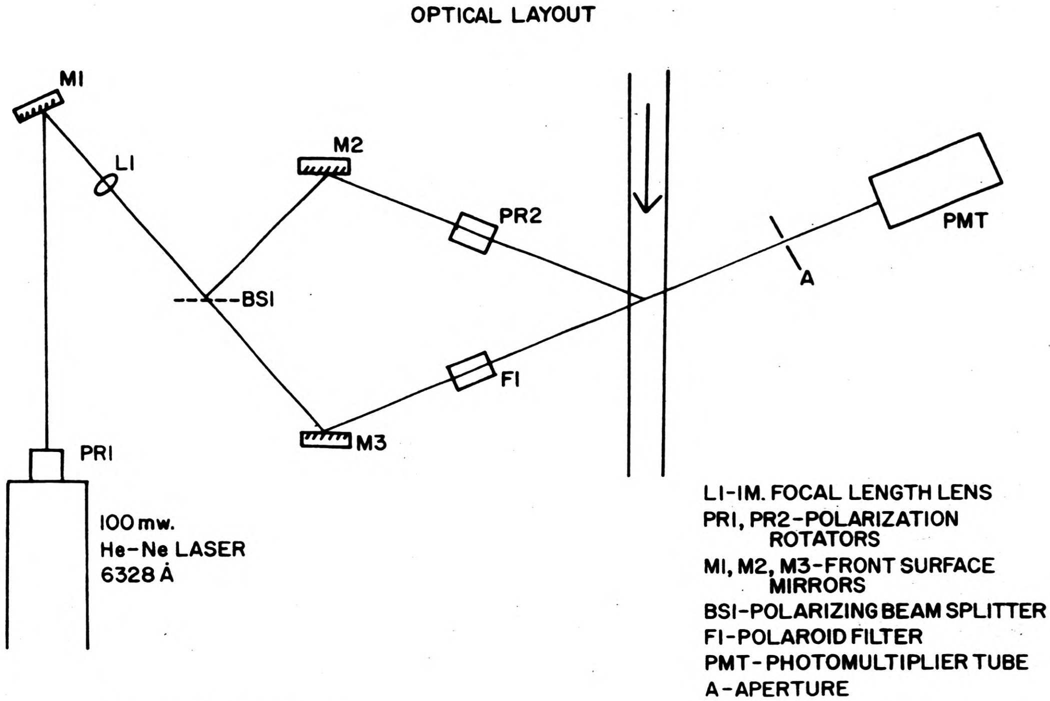

Figure 3. Optical Layout.

The signal was processed by the circuitry shown in block diagram in figure 4. This is a straight-forward blocking amplifier, tunable band-pass filter, and frequency-voltage converter. Spectra were measured using a Hewlett-Packard 302-A wave analyzer: the internal 100 kc carrier was fed to a Ballentine true-RMS voltmeter, and the mean square output integrated for thirty seconds by an analog integrator.

Data were reduced to universal form by using the measured value of K_* (since, without turbulence, there is no η), and a nominal value of $\nu = 10^{-2}$ cm^2/sec.

DISCUSSION

The measured ambiguity spectra are shown in figure 5. Signal/noise ratios were observed to be 30/1. These were measured after the filter, by blocking the incident beam. Since, for optimum signal strength, reference and scattered intensities should be equal, the observed s/n ratio may be too high by as much as a factor of two. The fact that the ambiguity spectra measured for three quite different beam diameters collapse indicates, however, that our conclusion regarding the negligibility of electronic noise was correct. Since there are no adjustable constants in either the theory or the experiment, the excellent quantitative agreement must be regarded as gratifying.

The measured spectra corresponding to noise plus sine wave are shown in figure 6. Since this artificial situation does not correspond to a real flow, the arbitrary values shown were used to reduce the data, to show it relative to a realistic turbulence spectrum. Again, the agreement is excellent.

The roll-off in the measured results for both ambiguity and noise are due to the low pass filter in the frequency-voltage converter, limiting system

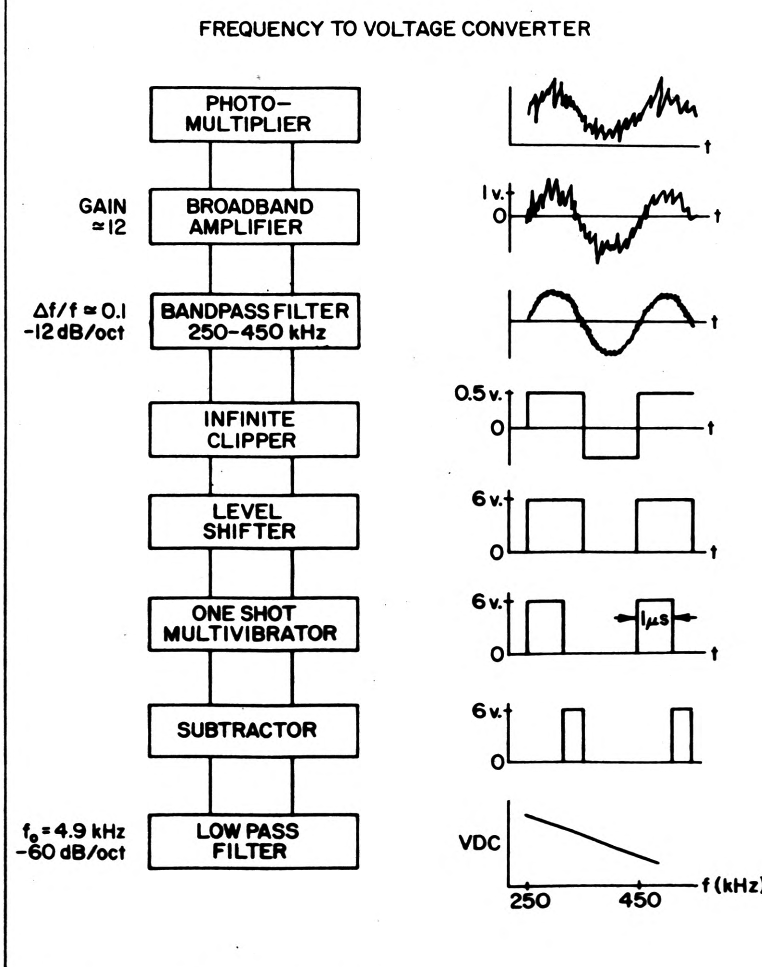

Figure 4. Frequency-to-voltage converter.

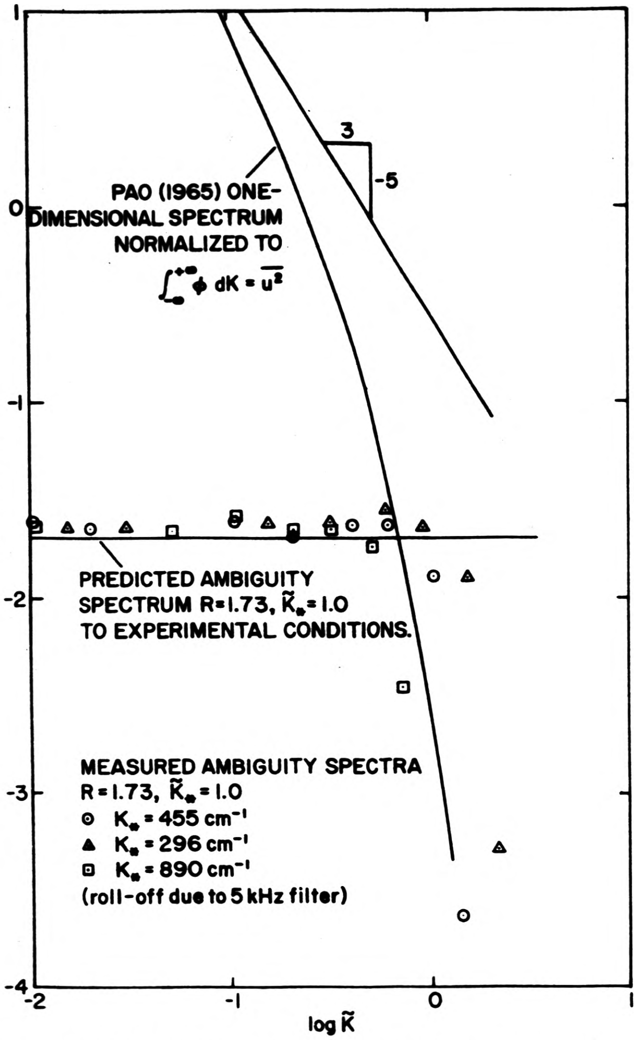

Figure 5. Comparison of predicted and measured ambiguity spectra.

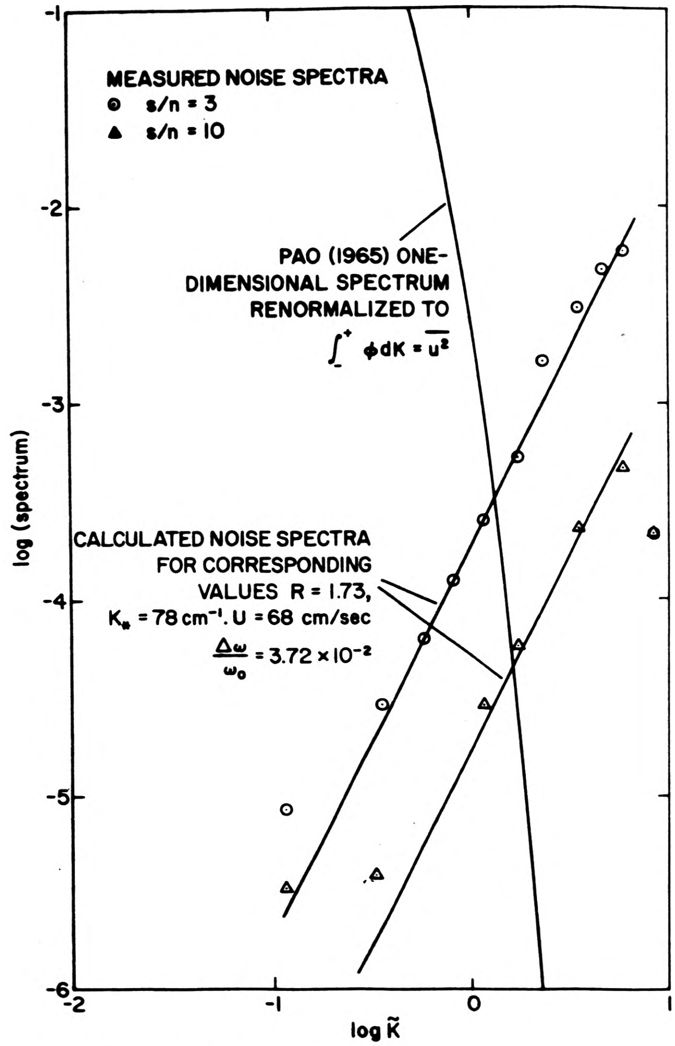

Figure 6. Comparison of predicted and calculated noise spectra.

response to 4.8 KHz. The fact that this roll-off occurs before the ambiguity spectrum meets the turbulence spectrum in one case is due to the impractically large value of K_* used; the 4.8 KHz cut-off was designed for smaller values of K_*.

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SYMBOLS

$f(t)$	(in introduction) characteristic signal produced by scattering particle passing through laser beam.
ω_o	ideal Doppler frequency (i.e.-frequency shift) of scattered radiation (rad/sec).
σ	(in introduction) transit time across beam (width defined as two radii of gyration of intensity distribution).
$u(t)$	net signal produced by stream of scattering particles.
$d\zeta(t)$	random factor accounting for presence, size, co-ordinates of beam-crossing, of scattering particles.
T	transit time across beam (width defined by energy outside equal to energy deficit - from peak value - inside).
U	mean velocity along bisector of angle between negative-incident and scattered radiation.
K_*	highest wavenumber resolved by beam (width based on UT).
F	Fourier transform of f.
$\Delta\omega$	root-mean-square deviation of Doppler signal from mean center frequency (rad/sec).
λ	wavelength of incident radiation.
θ	angle between incident and scattered radiation.
u'	root-mean-square fluctuating velocity.
$f(t),g(t)$	fluctuating amplitudes of the cosine and sine components of $u(t)$.
ϕ	fluctuating phase angle of $u(t)$.
ω	actual Doppler frequency of scattered radiation. (rad/sec).
α	frequency as an independent spectral variable (rad/sec).

$dZ(\alpha)$ fluctuating Fourier amplitude of ideal Doppler signal.

Φ one-dimensional wavenumber spectrum of (turbulent) fluctuating velocity.

$\overline{\omega_o}$ mean Doppler frequency.

$dN(\alpha)$ fluctuating Fourier amplitude of contribution to instantaneous frequency arising from ambiguity.

$n(\alpha)$ frequency spectrum of ambiguity noise.

V_ϕ circumferential velocity of moving point with coordinates f,g.

ρ correlation coefficient of f or g.

r radius vector of moving point with coordinates f,g.

σ standard deviation of distribution of f or g.

I_o modified Bessel function of the first kind of order zero.

α_o frequency (rad/sec) at which the spectral level of the ambiguity noise equals that of the turbulent signal.

η Kolmogorov microscale.

$\tilde{K}_o$ non-dimensionalized (by η) cut-off wavenumber corresponding to α_o.

ϵ turbulent dissipation of fluctuating kinetic energy per unit mass.

$\tilde{\Phi}$ non-dimensionalized (by η, ϵ) one-dimensional turbulent energy spectrum.

ν kinematic viscosity.

$\tilde{K}_*$ non-dimensionalized (by η) highest wavenumber resolved by the beam.

R Reynolds number based on mean velocity and streamwise distance subtended by ½-laser wavelength.

f_o mean Doppler center frequency in cycles/sec.

$\Delta\omega_f$ half-band width (rad/sec) of narrow-band filtered white noise.

A amplitude of arbitrary sinusoidal signal.

a signal/noise ratio.

$\psi(\alpha)$ filter function of narrow-band filtered white noise.

REFERENCES

1. Yeh, Y., and H. Z. Cummins, "Localized Fluid Flow Measurements with He-Ne Laser Spectrometer" Appl. Phys. Letters, 4, 176 (1964).

2. Foreman, J. W., R. D. Lewis, and Thornton, "Laser-Doppler Velocimeter for Measurement of Localized Fluid Velocities in Liquids" Proc. IEEE, 424 (1966).

3. Welch, N. E., and W. J. Tomme, "Analysis of Turbulence from Data Obtained with a Laser Velocimeter", AIAA paper no. 67-179 (1967).

4. Wax, N. (ed.), Noise and Stochastic Processes, Dover, New York (1954).

5. Pao, Y. H. "Structure of Turbulent Velocity and Scalar Field at Large Wave Numbers" Phys. Fluids, 8, 1063 (1965).

6. Pike, E. R., D. F. Jackson, P. J. Bourke, D. I. Page, "Measurement of Turbulent Velocities from the Doppler Shift in Scattered Laser Light", Presentation at Division of Fluid Dynamics, Am. Phys. Soc. Lehigh, November 1967.

7. Adams, Ansel, The Negative; Exposure and Development, Morgan and Morgan, Inc., Hastings on Hudson, N.Y. (1968).