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Establishment of Weak Conditions for Darboux-Goursat-Beudon Theorem

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Abstract

Abstract. This paper is devoted to the establishment of weak conditions for Darboux-Goursat-Beudon (DGB) theorem in order to improve analogous results in [1, 7]. By adapting a technique proposed by [7] in another setting [8] via majorant method, we obtain the generalization of DGB theorem.

Mathematics Subject Classifications: 35A10, 58A99

Keywords: Cauchy-Kovalevskaya theorem, connected open neighborhood, support of function

1 Introduction

It is well known that the classical Cauchy and Goursat theorems [1, 2, 5, 6, 7, 8] play an important role in the theory of differential equations and their solutions. As a result, the study of Darboux-Goursat-Beudon problem is attaining more prominence. In particular, during the last two decades many useful and interesting contributions have been made in the investigation of existence and uniqueness of the solution. The extended DGB problem has been initiated in [1] and [7] and the theory of analytic function of several variables [3] and [4] has been applied to ensure the existence and the uniqueness of the generalized DGB problem. In this paper, we shall continue this study and investigate the

existence and uniqueness of the solution of DGB problem with weak hypotheses in this framework.

2 Utility notions and basic results

We start by presenting some basic notations. Let \mathbb{R}^{n+1} be the $(n + 1)$ -dimensional Euclidean space, \mathbb{R}_+ the set of real numbers ≥ 0 , \mathbb{R}_+^{n+1} be the set of all $r = (r_0, r_1, \dots, r_n)$ with $r_j \in \mathbb{R}_+$ and \mathcal{C}^{n+1} be the $(n + 1)$ -dimensional complex space with variables $z = (z_0, z_1, \dots, z_n)$ and Ω be an open subset of \mathcal{C}^{n+1} containing the origin. We use the standard multi-index notation. More precisely, let \mathbb{Z} be the set of integers, > 0 or ≤ 0 , and \mathbb{Z}_+ be the set of integers ≥ 0 . Then \mathbb{Z}_+^{n+1} is the set of all $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$ with $\alpha_j \in \mathbb{Z}_+$ for each $j = 0, 1, \dots, n$. The length of $\alpha \in \mathbb{Z}_+^{n+1}$ is $|\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_n$; $\alpha \leq \beta$ means $\alpha_j \leq \beta_j$ for every $j = 0, 1, \dots, n$; and $\alpha < \beta$ means $\alpha \leq \beta$ and $\alpha \neq \beta$. If $\alpha \in \mathbb{Z}_+^{n+1}$ and $\beta \in \mathbb{Z}_+^{n+1}$, we define the operation $+$ by

$$\alpha + \beta = (\alpha_0 + \beta_0, \alpha_1 + \beta_1, \dots, \alpha_n + \beta_n) \in \mathbb{Z}_+^{n+1}.$$

Moreover, we let $\alpha! = \alpha_0! \alpha_1! \dots \alpha_n!$ and if $\alpha \leq \beta$

$$\binom{\beta}{\alpha} = \binom{\beta_0}{\alpha_0} \binom{\beta_1}{\alpha_1} \dots \binom{\beta_n}{\alpha_n} = \frac{\beta!}{\alpha! (\beta - \alpha)!}, \tag{1}$$

$$D^\alpha = \left(\frac{\partial}{\partial z_0}\right)^{\alpha_0} \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n},$$

and use the notations $D_j = \frac{\partial}{\partial z_j}$ and

$$D^\alpha = D_0^{\alpha_0} D_1^{\alpha_1} \dots D_n^{\alpha_n} \tag{2}$$

respectively. Let u be a continuous function in Ω ; by the support of u , denoted by $\text{supp } u$, we mean the closure in Ω of $\{z : z \in \Omega, u(z) \neq 0\}$. Let $\mathcal{C}^k(\Omega)$, $k \in \mathbb{Z}_+$, $0 \leq k \leq \infty$, denote the set of all functions u defined in Ω , whose derivatives $D^\alpha u(z)$ exist and continuous for $|\alpha| \leq k$. Using the multi-index notation, we may write the Leibnitz formula

$$D^\beta (uv) = \sum_{\alpha \leq \beta} \frac{\beta!}{\alpha! (\beta - \alpha)!} D^{\beta - \alpha} u D^\alpha v, \tag{3}$$

where we assume $u, v \in \mathcal{C}^{|\alpha|}(\Omega)$. If $u \in \mathcal{C}^\infty(\Omega)$, we may consider the Taylor expansion at the origin

$$u(z) = \sum_{\alpha \in \mathbb{Z}_+^{n+1}} \frac{D^\alpha u(0)}{\alpha!} z^\alpha. \tag{4}$$

Let $\mathcal{H}(\Omega)$ denote the set of all holomorphic functions in Ω , that is, functions $u(z) \in \mathcal{C}^\infty(\Omega)$ given by their Taylor expansion in some neighborhood of the origin in Ω . A linear partial differential operator $P(z; D)$ is defined by

$$P(z; D) = \sum_{|\alpha| \leq m} a_\alpha(z) D^\alpha, \tag{5}$$

where the coefficients $a_\alpha(z)$ are in $\mathcal{H}(\Omega)$. If for some α of length m the coefficient $a_\alpha(z)$ does not vanish identically in Ω , m is called the order of $P(z; D)$. We set $\mathcal{I}_\beta = \{(j, k) : j = 0, 1, \dots, n, \text{ and } k = 0, 1, \dots, \beta_j - 1\}$.

3 Weak conditions and main results

Consider Darboux-Goursat-Beudon problem:

$$(P) \begin{cases} Pu = \sum_{|\alpha| \leq m} a_\alpha D^\alpha u = 0 \\ D_0^k (u - \varphi)|_{z_0=0} = 0, 0 \leq k < m - 1 \\ (u - \varphi)|_{z_0=0} = 0. \end{cases}$$

In fact, $(m, 0, \dots, 0) = \beta, a_\beta(0) = 0$, with $\varphi(z) = z_0^N, N \in \mathbb{Z}_+$,

$$a_{m,0,\dots,0}(0, z') = 0, \tag{6}$$

$$D_0 a_{m,0,\dots,0}(0, z') = 0, \tag{7}$$

and

$$a_{m-1,1,0,\dots,0}(0, z') \neq 0. \tag{8}$$

It is easy to observe that these conditions are weaker than those given in [1, 7]. Moreover, the problem mentioned above admits a unique analytic solution $u_N = u$. We shall prove that $(\varphi \neq 0)$,

$$D_0^k u(0, z') = 0, \forall k = 0, 1, \dots, N - 1. \tag{9}$$

We have three cases to consider: $N < m, N = m$ and $N > m$.

First case $N < m$: By hypothesis $u = u_m$ satisfies the equality

$$D_0^k (u - \varphi)|_{z_0=0} = 0, \quad 0 \leq k < m - 1. \tag{10}$$

As $N < m$, we have

$$D_0^k (u - \varphi)|_{z_0=0} = 0, \quad 0 \leq k \leq N - 1, \tag{11}$$

and

$$D_0^k u(0, z') = 0, \quad 0 \leq k \leq N - 1. \quad (12)$$

Second case $N = m$: Now, $u = u_m$ satisfies the equation $Pu = 0$, that is, for every $z \in \Omega$

$$\begin{aligned} & \sum_{|\alpha| \leq m} a_\alpha(z) D^\alpha u(z) \\ &= a_{m,0,\dots,0}(z) D_0^m u(z) + \sum_{i=1}^n a_{m-1,0,\dots,0, \underset{i^{th} \text{ place}}{1}, 0,\dots,0}(z) D_i D_0^{m-1} u(z) \\ &+ a_{m-1,0,\dots,0}(z) D_0^{m-1} u(z) + \sum_{\substack{|\alpha| \leq m \\ \alpha_0 < m-1}} a_\alpha(z) D^\alpha u(z) \\ &= 0. \end{aligned} \quad (13)$$

Hence for $z = (0, z')$, we have

$$\begin{aligned} & \sum_{i=1}^n a_{m-1,0,\dots,0, \underset{i^{th} \text{ place}}{1}, 0,\dots,0}(0, z') D_i D_0^{m-1} u(0, z') \\ &+ a_{m-1,0,\dots,0}(0, z') D_0^{m-1} u(0, z') \\ &= 0. \end{aligned} \quad (14)$$

Set

$$D_0^{m-1} u(0, z') = U(z'), \quad a_{m-1,0,\dots,0, \underset{i^{th} \text{ place}}{1}, 0,\dots,0}(0, z') = A_i(z'),$$

$1 \leq i \leq n$ and $a_{m-1,0,\dots,0}(0, z') = A_0(z')$. By the hypothesis, we have

$$u(z_0, 0, z'') = \varphi(z_0, 0, z'') = z_0^N, \quad (15)$$

for $0 \leq k < m - 1 = N - 1$, hence $m = N > 1$, consequently,

$$u(0, 0, z'') = \varphi(0, 0, z'') = 0, \quad (16)$$

therefore $U(0, z'') = 0$.

Now, consider the following problem:

$$(P') \begin{cases} \sum_{i=1}^n A_i(z') D_i U(z') + A_0(z') U(z') = 0, \\ U|_{z_1=0} = 0, \end{cases}$$

Note that $A_1(0) \neq 0$, and $U(z') = 0$ is solution of problem (P') . It is unique by Cauchy-Kovalevskaya Theorem, that is, $D_0^{m-1} u(0, z') = 0$. Hence

$$D_0^k u(0, z') = 0, \quad (17)$$

for every $k \in \mathbb{Z}_+$ such that $0 \leq k \leq m - 1 = N - 1$.

Third case $N > m$:

$$D_0^{m-1}u(0, z') = 0, \quad \text{for } 0 \leq j \leq k - 1 < N - 1 \text{ and } m \leq k. \quad (18)$$

Differentiate $\sum_{|\alpha| \leq m} a_\alpha(z)D^\alpha u(z) = 0$, $k - (m - 1)$ times with respect to z_0 by using the Leibnitz formula, then $\forall z \in \Omega$,

$$\begin{aligned} 0 &= D_0^{k-(m-1)} \left(\sum_{|\alpha| \leq m} a_\alpha(z)D^\alpha u(z) \right) \\ &= D_0^{k-(m-1)} [a_{m,0,\dots,0}(z)D_0^m u(z) \\ &\quad + \sum_{i=1}^n a_{m-1,0,\dots,0, \underset{i^{th\ place}}{1}, 0,\dots,0}(z)D_i D_0^{m-1} u(z) \\ &\quad + a_{m-1,0,\dots,0}(z)D_0^{m-1} u(z) + \sum_{\substack{|\alpha| \leq m \\ \alpha_0 < m-1}} a_\alpha(z) D^\alpha u(z)] \\ &= \sum_{l=1}^{k-(m-1)} \binom{k-(m-1)}{l} D_0^l a_{m,0,\dots,0}(z) D_0^{k-l+1} u(z) \\ &\quad + \sum_{i=1}^n \sum_{l=1}^{k-(m-1)} \binom{k-(m-1)}{l} D_0^l a_{m-1,0,\dots,0, \underset{i^{th\ place}}{1}, 0,\dots,0}(z) D_i D_0^{k-l} u(z) \\ &\quad + D_0^{k-(m-1)} \left(\sum_{\substack{|\alpha| \leq m \\ \alpha_0 < m-1}} a_\alpha(z) D^\alpha u(z) \right) \\ &\quad + \sum_{l=0}^{k-(m-1)} \binom{k-(m-1)}{l} D_0^l a_{m-1,0,\dots,0}(z) D_0^{k-l} u(z). \end{aligned} \quad (19)$$

We have

$$\begin{aligned} &\sum_{l=0}^{k-(m-1)} \binom{k-(m-1)}{l} D_0^l a_{m,0,\dots,0}(z) D_0^{k-l+1} u(z) \\ &= a_{m,0,\dots,0}(z) D_0^{k+1} u(z) + \binom{k-(m-1)}{1} D_0 a_{m,0,\dots,0}(z) D_0^k u(z) \\ &\quad + \sum_{l=2}^{k-(m-1)} \binom{k-(m-1)}{l} D_0^l a_{m,0,\dots,0}(z) D_0^{k-l+1} u(z). \end{aligned} \quad (20)$$

For $z = (0, z')$, the expression (20) vanishes according to (6) – (7) and (18).

The expression

$$\sum_{i=1}^n \sum_{l=1}^{k-(m-1)} \binom{k-(m-1)}{l} D_0^l a_{m-1,0,\dots,0, \underset{i^{th} place}{1}, 0,\dots,0}(z) D_i D_0^{k-l} u(z) \quad (21)$$

equals

$$\sum_{i=1}^n \binom{k-(m-1)}{l} a_{m-1,0,\dots,0, \underset{i^{th} place}{1}, 0,\dots,0}(z) D_i D_0^{k-l} u(z) \quad (22)$$

for $z = (0, z')$ and for the same reason as before. The expression

$$\begin{aligned} & D_0^{k-(m-1)} \left(\sum_{\substack{|\alpha| \leq m \\ \alpha_0 < m-1}} a_\alpha(z) D^\alpha u(z) \right) \\ &= \sum_{\substack{|\alpha| \leq m \\ \alpha_0 < m-1}} \sum_{l=1}^{k-(m-1)} \binom{k-(m-1)}{l} D_0^l a_{\alpha_0, \alpha'}(z_0, z') \\ &\times D_{z'}^{\alpha'} D_0^{k-m+1-l+\alpha_0} u(z_0, z'), \end{aligned} \quad (23)$$

for $z = (0, z')$. This expression vanishes because $\alpha_0 < m - 1$ and $0 \leq l \leq k - (m - 1)$, hence $k - m + 1 - l + \alpha_0 \leq k - 1$ and the result follows from (18).

The expression

$$\begin{aligned} & \sum_{l=0}^{k-(m-1)} \binom{k-(m-1)}{l} D_0^l a_{m-1,0,\dots,0}(z) D_0^{k-l} u(z) \\ &= a_{m-1,0,\dots,0}(z) D_0^k u(z) \\ &+ \sum_{l=1}^{k-(m-1)} \binom{k-(m-1)}{l} D_0^l a_{m-1,0,\dots,0}(z) D_0^{k-l} u(z), \end{aligned} \quad (24)$$

for $z = (0, z')$, this expression becomes $a_{m-1,0,\dots,0}(0, z') D_0^k u(0, z')$ (due to (18)). Finally, we have

$$\sum_{i=1}^n a_{m-1,0,\dots,0, \underset{i^{th} place}{1}, 0,\dots,0}(0, z') D_i D_0^k u(0, z') + a_{m-1,0,\dots,0}(0, z') D_0^k u(0, z') = 0.$$

Set

$$D_0^k u(0, z') = V(z'), \quad (25)$$

$$a_{m-1,0,\dots,0, \underset{i^{th} place}{1}, 0,\dots,0}(0, z') = B_i(z'), \quad (26)$$

for $1 \leq i \leq n$ and

$$a_{m-1,0,\dots,0}(0, z') = B_0(z'). \tag{27}$$

We have $B_1(0) = a_{m-1,1,0,\dots,0}(0, 0) \neq 0$, and

$$\begin{aligned} V(0, z'') &= D_0^k u(0, 0, z'') \\ &= D_0^k \varphi(0, 0, z'') \\ &= 0, \end{aligned} \tag{28}$$

due to the fact that $\varphi(z) = z_0^N$ and $k < N$, therefore the following problem

$$\begin{cases} \sum_{i=1}^n B_i(z') D_i V(z') + B_0(z') V(z') = 0 \\ V|_{z_1=0} = 0 \end{cases}$$

admits the unique solution $V = 0$ (by Cauchy-Kovalevskaya Theorem for an operator of order 1), in other words

$$D_0^k u(0, z') = 0. \tag{29}$$

Hence

$$D_0^j u(0, z') = 0, \tag{30}$$

for $0 \leq j \leq k < N - 1$ and $m \leq k$. Now use the iterative method on k . Suppose

$$D_0^j u(0, z') = 0, \tag{31}$$

for $0 \leq j \leq k \leq N - 1$, and $m \leq k$, one can show this property remains true for the rank $k = N - 1$. We use the same process as before instead of differentiation $k - (m - 1)$ times with respect to z_0 with $m > 1$,

$$D_0^{N-m} \left(\sum_{|\alpha| \leq m} a_\alpha(z) D^\alpha u(z) \right) = 0 \tag{32}$$

$$\begin{aligned} &= \sum_{l=0}^{N-m} \binom{N-m}{l} D_0^l a_{m,0,\dots,0}(z) D_0^{N-l} u(z) \\ &+ \sum_{i=1}^n \sum_{l=0}^{N-l} \binom{N-l}{l} D_0^l a_{m-1,0,\dots,0, \underset{i^{th\ place}}{1}, 0,\dots,0}(z) D_i D_0^{N-l-1} u(z) \\ &+ D_0^{N-l} \left(\sum_{\substack{|\alpha| \leq m \\ \alpha_0 < m-1}} a_\alpha(z) D^\alpha u(z) \right) \\ &+ \sum_{l=1}^{N-l} \binom{N-l}{l} D_0^l a_{m-1,0,\dots,0}(z) D_0^{N-l-1} u(z). \end{aligned} \tag{33}$$

The conditions (6) – (7), (30) and the previous computations allow us to write:

$$\begin{aligned} & \sum_{i=1}^n a_{m-1,0,\dots,0, \underset{i^{th} place}{1}, 0,\dots,0}(0, z') D_i D_0^{N-1} u(0, z') \\ & + a_{m-1,0,\dots,0}(0, z') D_0^{N-1} u(0, z') \\ & = 0. \end{aligned} \quad (34)$$

Set

$$D_0^{N-1} u(0, z') = W(z'), \quad (35)$$

$$a_{m-1,0,\dots,0, \underset{i^{th} place}{1}, 0,\dots,0}(0, z') = C_i(z'), \quad (36)$$

for $1 \leq i \leq n$ and

$$a_{m-1,0,\dots,0}(0, z') = C_0(z'). \quad (37)$$

We have

$$C_1(0) = a_{m-1,1,0,\dots,0}(0, z') \neq 0, \quad (38)$$

$$\begin{aligned} W(0, z'') &= D_0^{N-1} u(0, 0, z'') \\ &= D_0^{N-1} \varphi(0, 0, z'') \\ &= 0, \end{aligned} \quad (39)$$

since $D_0^{N-1}(z_0^N)|_{z_0=0} = 0$. This leads to the solution of the following problem

$$(P''') \begin{cases} \sum_{i=1}^n C_i(z') D_i W(z') + C_0(z') W(z') = 0 \\ W|_{z_1=0} = 0. \end{cases}$$

Note that $W(z') = 0$ is the solution of the problem P''' and by Cauchy-Kovalevskaya Theorem, it is unique, hence $D_0^{N-1} u(0, z') = 0$, consequently we have showed that

$$D_0^j u(0, z') = 0, \quad (40)$$

for $0 \leq j \leq k \leq N$ and $m \leq k$. We have $0 \in \text{supp } p u$ because $u(z_0, 0, z'') = z_0^N$. Now we state the main result which improves the results in [2] and [7].

Theorem 3.1 *Let Ω be an open set of \mathcal{C}^{n+1} containing the origin and $(a_\alpha)_{|\alpha| \leq m} \in C^\infty(\Omega)$. If $a_{m,0,\dots,0}(0, z') = 0$, $D_0 a_{m,0,\dots,0}(0, z') = 0$ and also $a_{m-1,1,0,\dots,0}(0, z') \neq 0$, then $\forall N > 0$, $N \in \mathbb{N}$, $\exists \Omega' \subset \Omega$ connected open neighborhood of the origin, and $u_N = u$, $u \in C^\infty(\Omega)$ such that $\sum_{|\alpha| \leq m} a_\alpha D^\alpha u = 0$ and*

$D_0^k u(0, z') = 0$ for every k , $0 \leq k \leq N - 1$, $0 \in \text{supp } p u$.

Remark: We can extend the coefficients to complex domain, solve the problem by the method discussed in this paper and obtain the solution by restriction.

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