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Interpolation Theorems for Self-Adjoint Operators

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ABSTRACT. We prove a complex and a real interpolation theorems on Besov spaces and Triebel-Lizorkin spaces associated with a self-adjoint operator $\mathcal{L}$, without assuming the gradient estimate for its spectral kernel. The result applies to the cases where $\mathcal{L}$ is a uniformly elliptic operator or a Schrödinger operator with electro-magnetic potential.

1. Introduction and main result

Interpolation of function spaces has played an important role in classical Fourier analysis and PDEs [1, 6, 12, 15, 18]. Let $\mathcal{L}$ be a selfadjoint operator in $L^2(\mathbb{R}^n)$. Then, for a Borel measurable function ϕ: $\mathbb{R} \to \mathbb{C}$, we define $\phi(\mathcal{L})$ using functional calculus. In [15, 11, 2, 17] several authors introduced and studied the Besov spaces and Triebel-Lizorkin spaces associated with Schrödinger operators. In this note we present an interpolation result on these spaces for $\mathcal{L}$.

Let $\{\varphi_j\}_{j=0}^\infty \subset C_0^\infty(\mathbb{R})$ be a dyadic system satisfying (i) $\operatorname{supp}\varphi_0 \subset \{x : |x| \le 1\}$, $\operatorname{supp}\varphi_j \subset \{x : 2^{j-2} \le |x| \le 2^j\}$, $j \ge 1$, (ii) $|\varphi_j^{(k)}(x)| \le c_k 2^{-kj}$ for all j, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, (iii) $\sum_{j=0}^\infty |\varphi_j(x)| \approx 1$, $\forall x$. Let $\alpha \in \mathbb{R}$, $1 \le p, q \le \infty$. The inhomogeneous *Besov space associated with* $\mathcal{L}$, denoted by $B_p^{\alpha,q}(\mathcal{L})$, is defined to be the completion of $\mathcal{S}(\mathbb{R}^n)$, the Schwartz class, with respect to the norm

$$\|f\|_{B_p^{\alpha,q}(\mathcal{L})} = \big(\sum_{j=0}^\infty 2^{j\alpha q}\|\varphi_j(\mathcal{L})f\|_{L^p}^q\big)^{1/q}.$$

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Similarly, the inhomogeneous *Triebel-Lizorkin space associated with* $\mathcal{L}$, denoted by $F_p^{\alpha,q}(\mathcal{L})$, is defined by the norm

$$\|f\|_{F_p^{\alpha,q}(\mathcal{L})} = \|\big(\sum_{j=0}^{\infty} 2^{j\alpha q}|\varphi_j(\mathcal{L})f|^q\big)^{1/q}\|_{L^p}\,.$$

The following assumption on the kernel of $\phi_j(\mathcal{L})$ is fundamental in the study of function space theory. Let $\phi(\mathcal{L})(x,y)$ denote the integral kernel of $\phi(\mathcal{L})$.

Assumption 1.1. *Let $\phi_j \in C_0^\infty(\mathbb{R})$ satisfy conditions (i), (ii) above. Assume that there exist some $\varepsilon > 0$ and a constant $c_n > 0$ such that for all j*

$$|\phi_j(\mathcal{L})(x,y)| \le c_n \frac{2^{nj/2}}{(1+2^{j/2}|x-y|)^{n+\varepsilon}}\,. \tag{1}$$

This is the same condition assumed in [28, 18] except that we drop the gradient estimate condition on the kernel. This is the case when $\mathcal{L}$ is a Schrödinger operator $-\Delta+V$, $V \ge 0$ belonging to $L^1_{loc}(\mathbb{R}^n)$ [14, 19] or $\mathcal{L}$ is a uniformly elliptic operator in $L^2(\mathbb{R}^n)$ [8, Theorem 3.4.10].

In what follows, $[A,B]_\theta$ denotes the usual complex interpolation between two Banach spaces; $(A,B)_{\theta,r}$ the real interpolation, see Section 2. The notion T: $X \to Y$ means that the linear operator T is bounded from X to Y.

Theorem 1.2 (complex interpolation)**.** *Suppose that $\mathcal{L}$ is a selfadjoint operator satisfying Assumption 1.1. Let $0<\theta<1$, $s=(1-\theta)s_0+\theta s_1$, $s_0, s_1 \in \mathbb{R}$ and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

(a) If $1<p_i<\infty$, $1<q_i<\infty$, $i=0,1$, then

$$[F_{p_0}^{s_0,q_0}(\mathcal{L}), F_{p_1}^{s_1,q_1}(\mathcal{L})]_\theta = F_p^{s,q}(\mathcal{L}).$$

(b) If $1 \le p_i \le \infty$, $1 \le q_i \le \infty$, $i=0,1$, then

$$[B_{p_0}^{s_0,q_0}(\mathcal{L}), B_{p_1}^{s_1,q_1}(\mathcal{L})]_\theta = B_p^{s,q}(\mathcal{L}).$$

c) If $T: F_{p_0}^{s_0,q_0}(\mathcal{L}) \to F_{\bar{p}_0}^{\bar{s}_0,\bar{q}_0}(\mathcal{L})$ and $T: F_{p_1}^{s_1,q_1}(\mathcal{L}) \to F_{\bar{p}_1}^{\bar{s}_1,\bar{q}_1}(\mathcal{L})$, then $T: F_p^{s,q}(\mathcal{L}) \to F_{\bar{p}}^{\bar{s},\bar{q}}(\mathcal{L})$, where $\bar{s},\bar{p},\bar{q}$ and $\bar{s}_i,\bar{p}_i,\bar{q}_i$, satisfy the same relations as those for s,p,q and s_i,p_i,q_i, $1<p_i,q_i<\infty$. Similar statement holds for $B_p^{s,q}(\mathcal{L})$.

Complex interpolation method originally was due to Calderón [4] and Lions and Peetre [16]; see also [13, 24]. The classical interpolation theory for Besov and Triebel-Lizorkin spaces on $\mathbb{R}^n$ has been given systematic treatments in [20], [3], and [25, 26]. There are interesting discussions on interpolation theory in [20] and [26, 25, 22] for generalized Besov spaces associated with differential operators, which requires certain Riesz summability for $\mathcal{L}$ that seems a nontrivial condition to verify. Nevertheless, we would like to mention that the Riesz summability, the spectral multiplier theorem and the decay estimate in (1) are actually intimately related [10, 18].

The real interpolation result for $B_p^{\alpha,q}(\mathbb{R}^n)$, $F_p^{\alpha,q}(\mathbb{R}^n)$ can be found in [20] and [26, 25]. Following the proof as in the classical case, but applying the estimate in (1) in stead of spectral multiplier result, we obtain

Theorem 1.3 (real interpolation)**.** *Suppose that $\mathcal{L}$ satisfies Assumption 1.1. Let $0 < \theta < 1$, $1 \le r \le \infty$, $s = (1-\theta)s_0 + \theta s_1$, $s_0 \neq s_1$.*
(a) If $1 \le p < \infty, 1 \le q_1, q_2 \le \infty$, then

$$(F_p^{s_0,q_0}(\mathcal{L}), F_p^{s_1,q_1}(\mathcal{L}))_{\theta,r} = B_p^{s,r}(\mathcal{L}).$$

(b) If $1 \le p, q_1, q_2 \le \infty$, then

$$(B_p^{s_0,q_0}(\mathcal{L}), B_p^{s_1,q_1}(\mathcal{L}))_{\theta,r} = B_p^{s,r}(\mathcal{L}).$$

The homogeneous spaces $\dot{B}_p^{\alpha,q}(\mathcal{L})$ and $\dot{F}_p^{\alpha,q}(\mathcal{L})$ can be defined using $\{\varphi_j\}_{j=-\infty}^{\infty}$ in (i) to (iii), instead of $\{\varphi_j\}_{j=0}^{\infty}$. Then the analogous results of Theorem 1.2 and Theorem 1.3 hold.

2. INTERPOLATION FOR $\mathcal{L}$

Theorem 1.2 and Theorem 1.3 are part of the abstract interpolation theory for $\mathcal{L}$. In this section we present the outline of their proofs. It was mentioned in [22] that the interpolation associated with $\mathcal{L}$ is a "subtle and difficult" subject, which normally relies on the very property of $\mathcal{L}$.

2.1. **Complex interpolation.**
The proof of Theorem 1.2 is similar to that given in [25] in the Fourier case. The insight is that the three line theorem (involved in Riesz-Thorin or Calderón's constructive proof for L^p spaces) reflects the fact that the value of an analytic function in the interior of a domain is determined by its boundary values.

Definition 2.2. *Let (A_0, A_1) be an interpolation couple , i.e., A_0, A_1 are (complex) Banach spaces, linearly and continuously embedded in a Hausdorff space $\mathcal{H}$. The space $A_0 \cap A_1$ is endowed with the norm $\|a\|_{A_0\cap A_1} = \max\{\|a\|_{A_j}, j = 0, 1\}$. The space $A := A_0 + A_1$ is endowed with the norm*

$$\|a\|_A = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a_0 \in A_0, a_1 \in A_1\}.$$

Let $S = \{z \in \mathbb{C} : 0 \le \Re z \le 1\}$ and $\bar{S}$ its closure. Denote F the class of all A-valued functions $f(z)$ on $\bar{S}$ such that $z \mapsto f(z) \in A$ is analytic in S and continuous on $\bar{S}$, satisfying

(i)
$$\sup_{z\in\bar{S}} \|f(z)\|_A \textit{ is finite.}$$

(ii) *The mapping $t \mapsto f(j+it) \in A_j$ are continuous from $\mathbb{R}$ to A_j, $j = 0, 1$.*

Then F is a Banach space with the norm

$$\|f\|_F = \max_j\{\sup_t \|f(j+it)\|_{A_j}\}.$$

For $0 < \theta < 1$ we define the interpolation space $[A_0, A_1]_\theta$ as

$$[A_0, A_1]_\theta := \{a \in A : \exists f \in F \textit{ with } f(\theta) = a\}.$$

Then $[A_0, A_1]_\theta$ is a Banach space equipped with the norm

$$\|a\|_\theta := \inf\{\|f\|_F : f \in F \textit{ and } f(\theta) = a\}.$$

2.3. **Outline of the proof of Theorem 1.2.** Let $\{\phi_j\}, \{\psi_j\}$ satisfy the conditions in (i)-(iii) and $\sum_j \psi_j(x)\phi_j(x) = 1$. Define the operators $S : f \mapsto \{\phi_j(\mathcal{L})f\}$, and $R : g \mapsto \sum_j \psi_j(\mathcal{L})g$. The proof for part (a) follows from the commutative diagram

$$\begin{array}{ccc} F_p^{s,q}(\mathcal{L}) & \xrightarrow{S} & L^p(\ell^q) \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle Id} \\ F_p^{s,q}(\mathcal{L}) & \xleftarrow{R} & L^p(\ell^q) \end{array}$$

and Lemma 2.4 and Lemma 2.5, which are interpolation results for Banach space valued L^p and ℓ^q spaces [25].

Lemma 2.4. *Let $0 < \theta < 1$, $1 \le p_0, p_1 < \infty$ and $p^{-1} = (1-\theta)p_0^{-1} + \theta p_1^{-1}$. Let A_0, A_1 be Banach spaces. Then*

$$[L^{p_0}(A_0), L^{p_1}(A_1)]_\theta = L^p([A_0, A_1]_\theta). \tag{2}$$

If $p_1 = \infty$, then (2) holds with $L^{p_1}(A_1)$ replaced by $L_0^\infty(A_1)$, the completion of simple A_1-valued functions with the esssup norm.

As in [25], denote $\ell^q(A_j)$ the space of functions consisting of $a = \{a_j\}$, $a_j \in A_j$ (A_j being Banach spaces) equipped with the norm

$$\|a\|_{\ell^q(A_j)} = \left(\sum_j \|a_j\|_{A_j}^q\right)^{1/q}.$$

Lemma 2.5. *Let $0 < \theta < 1$, $1 \le q_0, q_1 < \infty$ and $q^{-1} = (1-\theta)q_0^{-1} + \theta q_1^{-1}$. Let A_j be Banach spaces, $j \in \mathbb{N}$. Then*

$$[\ell^{q_0}(A_j), \ell^{q_1}(B_j)]_\theta = \ell^q([A_j, B_j]_\theta). \tag{3}$$

If $q_1 = \infty$, then

$$[\ell^{q_0}(A_j), \ell^{\infty}(B_j)]_\theta = \ell^q([A_j, B_j]_\theta) = [\ell^{q_0}(A_j), \ell_0^{\infty}(B_j)]_\theta, \tag{4}$$

where $\ell_0^\infty(B_j) := \{\{c_j\} \in \ell^\infty(B_j) : \|c_j\|_{B_j} \to 0 \text{ as } j \to \infty\}$.

If $1 \le q_0, q_1 < \infty$, (3) also follows from Lemma 2.4 as a special case where the underlying measure space can be taken as $(X, \mu) = \mathbb{Z}$. If $q_1 = \infty$, then the remark in [25, Subsection 1.18.1] shows that the second statement in Lemma 2.5 is also true.

In the diagram above in order to show S, R are continuous mappings, we need the following well-known lemma.

Lemma 2.6. *Let $h(x)$ be a monotonely nonincreasing, radial function in $L^1(\mathbb{R}^n)$. Let $h_j(x) = 2^{jn/2}h(2^{j/2}x)$ be its scaling. Then for all f in $L^1_{loc}(\mathbb{R}^n)$*

$$|\int h_j(x-y)f(y)dy| \le c_n \|h\|_1 Mf(x),$$

where Mf denotes the usual Hardy-Littlewood maximal function.

Evidently the decay estimate in (1) and Lemma 2.6 imply the continuity of S and R, in light of the $L^p(\ell^q)$-valued maximal inequality.

The proof for $B_p^{s,q}(\mathcal{L})$ in part (b) proceeds in a similar way.

2.7. **Real interpolation.** Peetre's K-functional [21] is defined as

$$K(t, a) := K(t, a; A_0, A_1) = \inf(\|a_0\|_{A_0} + t\|a_1\|_{A_1}),$$

where the infimum is taken over all representations of $a = a_0 + a_1$, $a_i \in A_i$. Let $0 < q \le \infty, 0 < \theta < 1$. For a given interpolation couple (A_0, A_1), the real interpolation space $(A_0, A_1)_{\theta,q}$ is given by

$$(A_0, A_1)_{\theta,q} = \{a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty t^{-\theta q} K(t,a)^q \frac{dt}{t}\right)^{1/q} < \infty\}$$

with usual modifications if $q = \infty$.

Proof of Theorem 1.3 is similar to [26, Subsection 2.4.2] and [3, Theorem 6.4.5]. Define $\ell^{s,q}(A) = \{a = \{a_j\} : \|a\|_{\ell^{s,q}(A)} = \|\{2^{js}\|a_j\|_A\}\|_{\ell^q} < \infty\}$. For Besov spaces it follows from

$$(\ell^{s_0,q_0}(A_0), \ell^{s_1,q_1}(A_1))_{\theta,q} = \ell^{s,q}((A_0, A_1)_{\theta,q}),$$

$s = (1-\theta)s_0 + \theta s_1$, $q^{-1} = (1-\theta)q_0^{-1} + \theta q_1^{-1}$ and the commutative diagram for $B_p^{s,q}(\mathcal{L})$. Consult [26, 25] or [20, Chapter 5,Theorem 6]; both of their proofs rely on retraction method. Also see [3] for a different proof in the special case involving Sobolev spaces. In the general case [3] suggests using a more concrete characterization of the K-functional for the Lorentz space L^{pq}.

For the F-space the proof follows from the commutative diagram for $F_p^{s,q}(\mathcal{L})$ and

$$(L^{p_0}(A_0, w_0), L^{p_1}(A_1, w_1))_{\theta,p} = L^p((A_0, A_1)_{\theta,p}, w),$$

where $p^{-1} = (1-\theta)p_0^{-1} + \theta p_1^{-1}$, $w = w_0^{1-\theta} w_1^{\theta}$, w_0, w_1 being two weight functions [20, Chapter 5].

2.8. **Schrödinger operators with magnetic potential.** From [14], [28] or [18] we know that if the heat kernel of $\mathcal{L}$ satisfies the upper Gaussian bound

$$|e^{-t\mathcal{L}}(x,y)| \le c_n t^{-n/2} e^{-c|x-y|^2/t} \tag{5}$$

then the kernel decay in Assumption 1.1 holds. Let

$$H = -\sum_{j=1}^{n} (\partial_{x_j} + i a_j)^2 + V,$$

where $a_j(x) \in L^2_{loc}(\mathbb{R}^n)$ is real-valued, $V = V_+ - V_-$ with $V_+ \in L^1_{loc}(\mathbb{R}^n)$, $V_- \in K_n$, the Kato class [23]. Proposition 5.1 in [7] showed that (5) is valid for $-\Delta + V$ if $V_+ \in K_n$ and $\|V_-\|_{K_n} < \gamma_n := \pi^{n/2}/\Gamma(\frac{n}{2} - 1)$, $n \ge 3$, whose proof evidently works for $V_+ \in L^1_{loc}$. By the diamagnetic inequality [23, Theorem B.13.2], we see that (5) also holds for H provided $\|V_-\|_{K_n} < \gamma_n$, $n \ge 3$.

As another example, a *uniformly elliptic operator* is given by

$$\mathcal{L} = -\sum_{j,k=1}^{n} \partial_{x_j}(a_{jk} \partial_{x_k}),$$

where $a_{jk}(x) = a_{kj}(x) \in L^\infty(\mathbb{R}^n)$ are real-valued and satisfy the ellipticity condition $(a_{jk}) \approx I_n$. Then [19, Theorem 1] tells that (5) is true provided that the infimum of its spectrum $\inf \sigma(\mathcal{L}) = 0$.

References

[1] M. Beals, W. Strauss, L^p estimates for the wave equation with a potential. *Comm. P.D.E.* **18** (1993), no. 7-8, 1365–1397.

[2] J. Benedetto, S. Zheng, Besov spaces for the Schrödinger operator with barrier potential, to appear in *Complex Analysis and Operator Theory*, Birkhäuser.

[3] J. Bergh, J. Löfström, *Interpolation Spaces*. Springer-Verlag, 1976.

[4] A. Calderón, Intermediate spaces and interpolation. *Stud. Math.*, Seria Specjalna, **1** (1963), 31–34.

[5] ———, Intermediate spaces and interpolation, the complex method. *Studia Math.* **24** (1964), 113–190.

[6] S. Cuccagna, P. Schirmer, On the wave equation with a magnetic potential. *Comm. Pure Appl. Math.* **54** (2001) no. 2, 135–152.

[7] P. D'Ancona, V. Pierfelice, On the wave equation with a large rough potential. *J. Funct. Anal.* **227** (2005), no. 1, 30–77.

[8] E. Davies, *Heat Kernels and Spectral Theory.* Cambridge University Press, Cambridge, 1989.

[9] ———, Uniformly elliptic operators with measurable coefficients. *J. Funct. Anal.* **132** (1995), no.1, 141–169.

[10] X. Duong, E. Ouhabaz, A. Sikora, Plancherel type estimates and sharp spectral multipliers. *J. Funct. Anal.* **196** (2002), 443–485.

[11] J. Dziubańsk, Triebel-Lizorkin spaces associated with Laguerre and Hermite expansions. *Proc. Amer. Math. Soc.* **125** (1997), 3547–3554.

[12] J. Ginibre, G. Velo, Generalized Strichartz inequalities for the wave equation. *J. Funct. Anal.* **133** (1995), 50-68.

[13] P. Grisvard, Commutativité de deux foncteurs d'interpolation et applications. *J. Math. Pures Appl.* **45** (1966), 143–290.

[14] W. Hebisch: A multiplier theorem for Schrödinger operators. *Colloq. Math*, **60/61** (1990), no. 2, 659–664.

[15] A. Jensen, S. Nakamura, Mapping properties of functions of Schrödinger operators between L^p spaces and Besov spaces. *Advanced Studies in Pure Math.* **23** (1994), 187–209.

[16] J. Lions, J. Peetre, Sur une classe d'espaces d'interpolation. *Inst. Hautes Études Sci. Publ. Math.* **19** (1964), 5–68.

[17] G. Ólafsson, S. Zheng, Function spaces associated with Schrödinger operators: the Pöschl-Teller potential. *J. Fourier Anal. Appl.* **12** (2006), no.6, 653–674.

[18] G. Ólafsson, S. Zheng, Harmonic analysis related to Schrödinger operators. *Contemporary Mathematics* **464** (2008), 213–230.

[19] E. Ouhabaz, Sharp Gaussian bounds and L^p-growth of semigroups associated with elliptic and Schrödinger operators. *Proc. A.M.S.* **134** (2006), no. 12, 3567–3575.

[20] J. Peetre, *New Thoughts on Besov Spaces.* Duke Univ. Math. Series I. Durham: Duke Univ. 1976.

[21] ———, A theory of interpolation of normed spaces. Notes, Brasilia, 1963 (Notas de Matematica, no.39, 1968).

[22] H.-J. Schmeisser, H. Triebel, *Topics in Fourier Analysis and Function spaces.* Wiley-Interscience, 1987.

[23] B. Simon, Schrödinger semigroups. *Bull. Amer. Math. Soc.* **7** (1982), no.3, 447–526.

[24] M. Taibleson, On the theory of Lipschitz spaces of distributions on Euclidean n-spaces, I,II. *J. Math. Mech.* **13** (1964), 407–479; **14** (1965), 821–839.

[25] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators.* Amsterdam, North-Holland, 1978.

[26] H. Triebel, *Theory of Function Spaces.* Birkhäuser, 1983.

[27] K. Yajima, The $W^{k,p}$-continuity of wave operators for Schrödinger operators. *J. Math. Soc. Japan* **47** (1995), 551–581.

[28] S. Zheng, Littlewood-Paley theorem for Schrödinger operators. *Anal. Theory. Appl.* **22** (2006), no.4, 353–361.

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