Stability of Unbraced Pallet Racks

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STABILITY OF UNBRACED PALLET RACKS

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Summary

A general 'exact' procedure for determining the elastic critical load of plane frames with flexible joints is described. This is followed by an approximate method of particular application to unbraced pallet racks. A worked example is given and further typical results from the two analyses are compared.

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Introduction

The general problem of the stability analysis of a plane framework of prismatic members, any of which may have partially rigid joints at one or both ends, does not appear to have been previously solved. The pallet rack is a particular case of such a frame and, for this case, computer programs are known to be available within the trade. These generally consider partial joint rigidity only in the beams, within which the destabilising effects of axial compressive loads can be ignored. Axial compressive loads in the vertical members are considered but, here, partial rigidity of the base is included by means of an equivalent ground beam.

In this paper the problem is solved in more general terms by deriving the stiffness matrix for a member carrying an axial compressive load and having partially rigid joints at one or both ends. Thus a computer program can be written that is of more general application than just to particular pallet rack configurations. In this program, the elastic critical load is approached automatically by means of the modified Southwell Plot using a technique that also gives bending moments and deflections at the load levels of interest.

The general analysis for elastic critical loads requires the use of a computer. For pallet racks and other rectangular plane frames, where partial rigidity is confined to the ends of the beams and the bases of the stanchions, the problem also admits of a remarkably accurate approximate solution suitable for manual calculation or for a small desk top computer. This approximate method is described later and the results obtained are compared with 'exact' solutions for a range of pallet racks. They are found to be sufficiently accurate for all practical purposes.

Stiffness matrix of the general member

The general member considered is shown in Fig. 1. If a clockwise positive sign convention is assumed and the flexible joints at the two ends of the member have stiffnesses k_1 and k_2 respectively, the relationships between internal moments and rotations at the joints are

$$M_1 = -k_1\theta_{H1} \quad \text{and} \quad M_2 = -k_2\theta_{H2} \qquad \text{.....................} (1)$$

The axial strain equations follow from Hooke's Law as

$$P_{x1} = -P_{x2} = \frac{EA}{L}(\delta_{x1} - \delta_{x2}) \qquad \text{.....................} (2)$$

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and the remaining relationships that are required can be obtained from the modified slope-deflection equations given by Livesley[5], thus

$$\left.\begin{aligned} M_1 &= \frac{2EI}{L}\left[2(\theta_1 + \theta_{H1})\phi_3 + (\theta_2 + \theta_{H2})\phi_4 - \frac{3}{L}(\delta_{y2} - \delta_{y1})\phi_2\right] \\ M_2 &= \frac{2EI}{L}\left[(\theta_1 + \theta_{H1})\phi_4 + 2(\theta_2 + \theta_{H2})\phi_3 - \frac{3}{L}(\delta_{y2} - \delta_{y1})\phi_2\right] \\ P_{y1} = -P_{y2} &= \frac{6EI\phi_2}{L^2}\left[(\theta_1 + \theta_{H1}) + (\theta_2 + \theta_{H2})\right] - \frac{12EI\phi_1}{L^3}\left[\delta_{y2} - \delta_{y1}\right] \end{aligned}\right\} \quad \ldots (3)$$

In the above equations, the ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are stability functions which depend on the axial load in the member which is initially unknown. It follows that the solution at a given load level must be iterative. As the expressions for the stability functions contain trigonometrical terms which have singularities within the range of interest, they are best obtained computationally by using a series approximation[6].

Expressions for flexible joint rotations θ_{H1} and θ_{H2} can now be obtained by substituting equations (1) into the first two equations (3),

$$\left.\begin{aligned} \theta_{H1} &= \frac{k_1'}{1-k_1' k_2' \phi_4^2}\Big[\theta_1(k_2'\phi_4^2 - 2\phi_3) + \theta_2(2k_2'\phi_3\phi_4 - \phi_4) \\ &\qquad + \frac{3}{L}(\delta_{y2}-\delta_{y1})(\phi_2 - k_2'\phi_2\phi_4)\Big] \\ \theta_{H2} &= \frac{k_2'}{1-k_1' k_2' \phi_4^2}\Big[\theta_1(2k_1'\phi_3\phi_4 - \phi_4) + \theta_2(k_1'\phi_4^2 - 2\phi_3) \\ &\qquad + \frac{3}{L}(\delta_{y2} - \delta_{y1})(\phi_2 - k_1'\phi_2\phi_4)\Big] \end{aligned}\right\} \quad \ldots . (4)$$

In the above equations, k_1' and k_2' are non-dimensional joint stiffness parameters given by

$$k_1' = \frac{2EI}{4EI\phi_3 + k_1 L} \qquad k_2' = \frac{2EI}{4EI\phi_3 + k_2 L} \quad \ldots\ldots\ldots\ldots\ldots (5)$$

When equations (4) are substituted into (3) the bending stiffness equations for the typical member shown in Fig. 1 are obtained. Together with equations (2), these allow the complete stiffness equations for the member to be assembled. These are given in matrix form in Appendix A.

Critical load prediction by the modified Southwell Plot

Typical pallet racks, because of their flexible joints, fall into the class of structure which fails primarily by elastic instability. Plasticity in the members only becomes significant at a late stage of loading when the structure is close to failure. For this reason the theoretical elastic critical load is of considerable importance in pallet rack design. Strictly speaking, the elastic critical load is defined by purely axial loads which cause no deflection and is the load level at which bifurcation of equilibrium becomes possible with an infinitesimally small disturbing force. However, there is a closely related load, the elastic failure load, at which deflections become infinite under a more general applied loading and it is this load that is of more practical interest to the designers of pallet racks. Numerically, the two loads are almost identical as both are associated with a stiffness matrix that becomes singular due to the destabilising effect of axial compressive loads but, in practical terms, a method that approaches the elastic failure load through successive analyses is going to give the designer much more information about the behaviour of his structure. Such a method is the modified Southwell Plot.

If a structure is subject to proportionately increasing loads measured by a load factor λ and resulting in a deflection Δ at some critical point, the modified Southwell Plot is a graph of λ/Δ versus λ as shown in Fig. 2. It can be shown theoretically[3] that, if the deflection pattern caused by the applied loads has a dominant component similar to the first critical mode of elastic buckling, the plot is a straight line cutting the axis $\lambda/\Delta = 0$ at the elastic failure load λ_{crit}. In practice, this implied assumption is not completely satisfied so that the plot of λ/Δ versus λ is slightly curved and it is necessary to approach the critical load in a series of steps, as shown in Fig. 2. As each step involves an analysis of the structure, it is convenient to start the process at levels of load which will give useful information and the author has found it convenient to base an initial prediction on analyses at the working loads and 1.5 times the working loads. Two such analysis allow a prediction of the critical load to be made but this is likely to be an overestimate as a consequence of curvature of the plot. It is therefore necessary to carry out the next analysis at an arbitrarily chosen value of the load factor which is less than that predicted and then to make a further prediction based on the two most recent analyses. In the author's program, it has proved successful to make a prediction λ_{pred} based on two successive analyses λ_{i-1} and λ_i and then to carry out the next analysis at a load factor of $\lambda_{i+1} = 0.75\,\lambda_{pred} + 0.25\,\lambda_i$. The analysis is terminated when two successive values of λ_{pred} are sufficiently close. Fig. 2 shows a typical plot obtained by using the above procedure. The remaining data for the rack analysed is the same as that shown in Fig. 8.

At any stage in the analysis, it is possible for the critical load λ_{crit} to be exceeded. When this is the case, the determinant of the stiffness matrix will become negative and it is necessary to recognise this case, reduce the applied load, and start the analysis again.

The assembly of the complete stiffness matrix of the structure and the solution for displacements and internal forces follows the conventional procedure which can be found in any text book concerned with the matrix methods of structural analysis. The only difficulties that arise are those occasioned by the fact that the stability functions are dependent on the member axial loads which are unknown at the commencement of the analysis. At a given load level, an analysis with $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 1$ (member axial loads zero) will give a linear elastic solution which provides initial values of the required axial loads. These can then be used in a second analysis to obtain a better set of axial loads and the iterative process continued until suitable accuracy has been obtained. As might be anticipated, this process converges rapidly and two of three cycles are usually sufficient.

However, if the structure is being analysed at a load level very close to the failure load, the deflections may be very large and numerical ill-conditioning can occur in the calculation of the member axial loads thus rendering the calculation procedure unstable. For this reason it is usually better to base the entire calculation on a set of axial loads calculated on the basis of a linear elastic analysis at unit load factor, increasing these in proportion as the load level increases. This completely avoids any necessary for iteration and will usually give more than adequate accuracy as the elastic failure load of most rectangular structures, including pallet racks, is not particularly sensitive to the precise distribution of axial load between vertical columns.

Finally, it may be observed that the modified Southwell Plot can always be used to obtain a true elastic critical load if all of the loading is applied axially except for a small disturbing force at the critical joint. When this technique is used, the device of using axial loads calculated once and for all on the basis of a linear elastic analysis at unit load factor is, of course, exact.

Approximate calculation of the critical loads of pallet racks

The previous sections of this paper are concerned with an analysis that requires a computer. Furthermore, the necessity for repeated analysis of the complete structure means that the calculation of the elastic failure load of a large rack is by no means a small problem. There is a clear need for a rapid approximate method of analysis suitable for hand calculation or a small desk-top computer. The accuracy of the method described in this section is such that, unless information is required concerning bending moments and displacements prior to failure, no more complex analysis is warranted.

The method arises initially out of the work of Horne[2] who demonstrated a surprisingly simple procedure for obtaining a good estimate of the elastic critical loads of conventional rectangular plane frames of the type shown in Fig. 3a. In his procedure, an elastic analysis is carried out in which the total vertical load on each storey is applied as side load at that level, as shown in Fig. 3b. If, in a given storey of height h_i the relative sway of the floors above and below is u_i, the sway index ϕ_i is defined by $\phi_i = u_i/h_i$. Horne showed that a remarkably accurate value of the elastic critical load was given by

$$\lambda_{crit} = 1/\phi_i \qquad \text{(6)}$$

and a value that was always safe by $\lambda_{crit} = 0.9/\phi_i$.

Obviously, Horne's method is directly applicable to storage racks provided that a linear elastic analysis of plane frames is available which takes account of the relevant joint flexibilities. Such an analysis will usually require a computer. However, the method is capable of further simplification without significant loss of accuracy whereupon it becomes a perfectly reasonable manual method.

For an analysis under side loads alone, the full frame can be advantageously reduced to the simple substitute frame shown in Fig. 3c which is often known as the "Grinter" frame. This equivalent frame has, in each storey, one column of stiffness equal to the total column stiffness of all the columns in that storey of the complete structure($K=\sum K_c = \sum I_c/h$) and one beam of stiffness equal to three times the total beam stiffness ($K_B=3\sum K_b = 3\sum I_b/L$) at that level. The multiplier of three occurs because each beam restrains two columns and has approximately 50% increase of stiffness if the end rotations are approximately equal.

It was the author's proposal[1] that applying Horne's method to the Grinter frame would allow a simple manual analysis for elastic critical loads giving good results at a minimal cost in computational effort.

The practical justification for this suggestion is that the complete substitute frame can be solved manually for this load case in a single moment distribution process using Naylor's no-shear method[4]. The bending moments in the frame shown in Fig. 3c are thus obtained by a moment distribution process which usually converges rapidly because the beams are stiffer than the no-shear stanchions. The sway indices ϕ_i then follow rapidly using the slope-deflection equations

$$\left.\begin{aligned} \theta_u &= \frac{M_{Bu}}{4EK_{Bu}} \\ \phi_i &= \frac{u_i}{h_i} = \theta_u - \frac{2M_u - M_l}{6EK} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots (7)$$

where the various quantities are defined in Fig. 4.

The extension of the above procedure to pallet racks requires that modified stiffness factors are derived to take account of the flexible joints at the ends of the beams and the bases of the columns. The required modifications now follow. It is interesting to observe that the same substitute frame can also be applied to predict the sidesway stiffness of diaphragm-braced frames[1] when a similar set of stiffness factors are required.

(a) Factor for beam stiffness

For analysis using the substitute frame, all beams of the complete structure are assumed to deform anti-symmetrically as shown in Fig. 5. The slope deflection equation for a single beam reduces to

$$M = -k\theta_H = \frac{6EI_b}{L}(\theta + \theta_H) \qquad \text{......(8)}$$

Thus $\theta_H = \dfrac{-6EI_b\theta}{6EI_b + kL}$ and $M = \dfrac{6EI_b k\theta}{6EI_b + kL}$

It follows that a beam which has at its ends flexible joints with rotational stiffness k behaves as though its bending stiffness $K_b = I_b/L$ is modified by a factor

$$\frac{kL}{6EI_b + kL} \qquad \text{......(9)}$$

(b) Factor for column stiffness

Typical columns in the substitute frame do not require any modification for flexible joints and their stiffness is that for conventional no-shear analysis, namely $^1/4$ K. The bottom storey column may require further modification for base flexibility and for this case the required conditions are shown in Fig. 6. The relevant slope-deflection equations for a single column are

$$\left.\begin{aligned} M &= \frac{2EI_c}{h}\left(2\theta + \theta_H - 3\frac{u}{h}\right) \\ M_1 &= -k_1\theta_H = \frac{2EI_c}{h}\left(2\theta_H + \theta - 3\frac{u}{h}\right) \end{aligned}\right\} \qquad \text{......(10)}$$

and these, together with the no-shear condition $M + M_1 = 0$, give

$\theta_H = \dfrac{EI_c\theta}{EI_c + k_1 h}$ and $M = \dfrac{EI_c k_1\theta}{EI_c + k_1 h}$

It follows that a column in a no-shear moment distribution which has a flexible joint of rotational stiffness k_1 at its base behaves as though its bending stiffness $K_c = I_c/h$ is modified by a factor

$$\frac{k_1 h}{EI_c + k_1 h} \qquad \text{......(11)}$$

in addition to the usual no-shear factor of $^1/4$.

(c) Initial fixed end moments for bottom column

The initial fixed end moments for a typical storey i of height h_i in the substitute frame are unaffected by the flexible beams. If the storey shear is S_i, they are simply $-S_i h_i/2$ at each end of the column concerned. However, the bottom storey column requires special consideration as shown in Fig. 7. The relevant slope-deflection and shear equilibrium equations for the column of the substitute frame are

$$\left.\begin{aligned} M_u &= \frac{2EI}{h}\left(\theta_H - 3\frac{u}{L}\right) = 2EK\left(\theta_H - 3\frac{u}{L}\right) \\ M_l &= -k_l'\,\theta_H = \frac{2EI}{h}\left(2\theta_H - 3\frac{u}{L}\right) = 2EK\left(2\theta_H - 3\frac{u}{L}\right) \\ M_u &+ M_l + SL = 0 \end{aligned}\right\} \quad \ldots (12)$$

where k_l' is the total rotational stiffness of the lower joints of all of the columns. Rearranging these equations gives

$$\theta_H = \frac{Sh}{2} \cdot \frac{1}{EK + k_l'}$$

and

$$\left.\begin{aligned} M_u &= -\frac{Sh}{2}\left[\frac{2EK + k_l'}{EK + k_l'}\right] \\ M_l &= -\frac{Sh}{2}\left[\frac{k_l'}{EK + k_l'}\right] \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (13)$$

(d) Carry over factors

The carry-over factors are in all cases equal to -1 as is usual in no-shear moment distribution and are unaffected by joint flexibility in either beams or columns.

(e) Evaluation of storey sway indices

For a typical storey, equations (6), given previously for a conventional frame still apply provided that the modified beam stiffness according to equation (9) is used for K_{Bu}.

Example of manual calculation

As the above procedure is so simple to apply, an example can be given in full for the 3 storey, 3 bay pallet rack shown in Fig. 8. The addition of further bays or storeys adds very little extra work. The calculations proceed as follows:-

$$\text{Basic beam stiffness } K_B = 3\sum K_b = \frac{3 \times 3 \times 1.3372}{106.84} = 0.1126 \text{ in}^3$$

$$\text{Modification factor for flexible joints} = \frac{kL}{6EI_b + kL}$$

$$= \frac{638 \times 106.84}{6 \times 29500 \times 1.3372 + 638 \times 106.84} = 0.2236$$

$\therefore$ Beam stiffness for Naylor analysis $= 0.1126 \times 0.2236 = 0.02519\ in^3$

Basic column stiffness $K = \sum K_c = \dfrac{4 \times 1.6700}{60} = 0.1113\ in^3$

$\therefore$ Column stiffness for Naylor analysis $= \frac{1}{4} \times 0.1113 = .02783\ in^3$

Modification factor for stiffness of lower columns $= \dfrac{k_1 L}{EI + k_1 L}$

$$= \frac{800 \times 60}{29560 \times 1.67 + 800 \times 60} = 0.4935$$

$\therefore$ Stiffness of lower column for Naylor analysis $= 0.4935 \times .02783 = .01374\ in^3$

Distribution factor at lower storey to beam $= \dfrac{0.02519}{0.02519 + 0.02783 + 0.01374}$

$$= 0.3773$$

——— " ——— upper column $= \dfrac{.02783}{.02519 + .02783 + .01374}$

$$= 0.4169$$

——— " ——— lower column $= \dfrac{.01374}{.02519 + .02783 + .01374}$

$$= 0.2058$$

The remaining distribution factors follow similarly and are given in Table 1.

Total vertical load on rack per storey $= 0.0209 \times 106.84 \times 3 = 6.699$ kip

$\therefore$ Fixed end moments for upper storey $= \dfrac{-Sh}{2} = \dfrac{-6.699 \times 60}{2}$

$$= -201.0 \text{ kip in}$$

Fixed end moments for middle storey $= \dfrac{-13.398 \times 60}{2}$

$$= -401.9 \text{ kip in}$$

Fixed end moments for bottom storey:-

at upper end, $M_u = \dfrac{-Sh}{2}\left[\dfrac{2EK + k'_1}{EK + k'_1}\right]$

$$= \frac{20.097 \times 60}{2}\left[\frac{2 \times 29500 \times 0.1113 + 800 \times 4}{29500 \times 0.1113 + 800 \times 4}\right] = -908.3 \text{ kip in}$$

at lower end, $M_1 = \frac{-Sh}{2}\left[\frac{k_1'}{EK + k_1'}\right]$

$$= \frac{-20.097 \times 60}{2}\left[\frac{800 \times 4}{29500 \times 0.1113 + 800 \times 4}\right] = 297.5 \text{ kip in}$$

This completes the information necessary to carry out the no-shear moment distribution procedure which is given in Table 1. It should be noted that, at each level, the carryover factor for column moments is -1.

The storey sways, and from these the load factors against elastic buckling in each storey, can now be found as follows

Rotation at lower storey beam $\theta_{u1} = \frac{M_{BU1}}{4EK_{Bu}}$

$$= \frac{735.5}{4 \times 29500 \times .02519} = 0.2474 \text{ radians}$$

$\therefore$ Lower storey sway index $\phi_1 = \theta_{u1} - \frac{2M_{u1} - M_{l1}}{6EK}$

$$= 0.2474 - \frac{-2 \times 507.2 + 698.6}{6 \times 29500 \times 0.1113} = 0.2635$$

$\therefore$ Load factor against failure in bottom storey $= {}^1/\phi_1 = 3.795$

Rotation at middle beam $\theta_{u2} = \frac{578.4}{4 \times 29500 \times .02519} = 0.1946$ radians

$\therefore$ Middle storey sway index $\phi_2 = 0.1946 - \frac{-2 \times 575.5 + 228.3}{6 \times 29500 \times 0.1113} = 0.2414$

$\therefore$ Load factor against failure in middle storey $= {}^1/\phi_2 = 4.142$

similarly, load factor against failure in upper storey = 5.726

Failure in the lower storey is clearly critical and the load factor against elastic buckling for the rack is therefore 3.795.

In general, the critical storey is often the lowest storey and is always near the bottom of the structure. It is therefore never necessary to calculate the sway index of more than the lowest few storeys in order to establish the critical load of the rack. Indeed, in a tall rack structure of many levels, there is little loss of accuracy if the moment distribution process is confined to (say) the lowest third of the structure and carry-over to and from the less critical regions is ignored.

Comparison of exact and approximate methods

The approximate method described above is best justified by systematic comparison with the 'exact' analysis for a range of racks. The simple rack described in Fig. 8 provides a reasonable basis for such a comparison and a number of racks were investigated having the same basic design but varying the number of storeys, bays, loads and joint stiffnesses. Some typical results obtained during this investigation are shown in Table 2. The results presented include not only those for the 'exact' and 'approximate' methods described above but also the result of applying Horne's method in full. All three compare remarkably well so that, if the critical load is all that is required, the approximate method is perfectly adequate. It may be noted that neither the worked example nor the analyses reported in Table 2 include Horne's factor 0.9 mentioned in connection with equation (6). This is because the author has found that, for the racks that he has investigated, both Horne's method and the approximate method described above have given consistently safe results without this factor.

Conclusions

Two alternative practical approaches to the calculation of the elastic critical loads of pallet racks have been described and compared. The 'exact' method requires the use of a computer but gives useful information regarding bending moments and deflections at the working loads. It also provides a yardstick whereby the alternative approximate method may be evaluated. The approximate method requires only a simple manual calculation yet gives remarkably accurate results. Both methods are offered as being useful tools to the designers of pallet racks.

References

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APPENDIX A

Stiffness equations for a typical member with flexible joints (Fig. 1)

$$\begin{bmatrix} P_{x1} \\ P_{y1} \\ M_1 \\ \hline P_{x2} \\ P_{y2} \\ M_2 \end{bmatrix} = \left[\begin{array}{ccc|ccc} A & O & O & & & \\ O & B & C & & \text{Symmetrical} & \\ O & C & D & & & \\ \hline -A & O & O & A & O & O \\ O & -B & -E & O & B & E \\ O & -C & G & O & E & F \end{array}\right] \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \theta_1 \\ \hline \delta_{x2} \\ \delta_{y2} \\ \theta_2 \end{bmatrix}$$

$$k_1' = \frac{2EI}{4EI\,\phi_3 + k_1 L} \qquad k_2' = \frac{2EI}{4EI\,\phi_3 + k_2 L}$$

$$\psi = 1 - k_1'\,k_2'\,\phi_4^2$$

$$A = \frac{EA}{L}$$

$$B = \frac{6EI}{\psi L^3}\left[2\phi_1 - 3(k_1' + k_2')\,\phi_2^2 + 2k_1'\,k_2'\,(3\phi_2^2\,\phi_4 - \phi_1\,\phi_4^2)\right]$$

$$C = \frac{6EI\phi_2}{\psi L^2}\left[1 - 2k_1'\,\phi_3 - k_2'\,\phi_4 + k_1'\,k_2'\,\phi_3\,\phi_4\right]$$

$$D = \frac{2EI}{\psi L}\left[2\phi_3 - 4k_1'\,\phi_3^2 - k_2'\,\phi_4^2 + 2k_1'\,k_2'\,\phi_3\,\phi_4^2\right]$$

$$E = \frac{-6EI\phi_2}{\psi L^2}\left[1 - k_1'\,\phi_4 - 2k_2'\,\phi_3 + 2k_1'\,k_2'\,\phi_3\,\phi_4\right]$$

$$F = \frac{2EI}{\psi L}\left[2\phi_3 - k_1'\,\phi_4^2 - 4k_2'\,\phi_3^2 + 2k_1'\,k_2'\,\phi_3\,\phi_4^2\right]$$

$$G = \frac{2EI\phi_4}{\psi L}\left[1 - 2(k_1' + k_2')\phi_3 + 4k_1'\,k_2'\,\phi_3^2\right]$$

APPENDIX B

Notation

A	=	cross-sectional area of member
E	=	Young's modulus
I	=	second moment of area of member
h	=	storey height
K	=	stiffness of member = I/L
k	=	rotational stiffness of joint = M/θ_H
k'	=	dimensionless parameter in stiffness calculation (equation 5)
L	=	length of member
M	=	bending moment
P	=	joint load in stiffness calculation
u	=	storey sway
Δ	=	critical deflection in modified Southwell plot
δ	=	deflection
λ	=	load factor
θ	=	joint rotation
θ_H	=	rotation associated with joint flexibility
ϕ	=	stability function or sway index u/h

		Lower beam			Middle beam			Upper beam	
	.2058	.3773	.4169	.3442	.3115	.3442	.5250	.4750	Distribution factors
-297.5	-908.3		-401.9	-401.9		-201.0	-201.0		Fixed end moments
-269.6	269.6	494.3	546.2	-546.2					
			-395.5	395.5	358.0	395.5	-395.5		
-81.4	81.4	149.2	164.9	-164.9		-313.1	313.1	283.3	
			-164.9	164.5	148.9	164.5	-164.5		
-33.9	33.9	62.1	68.6	-68.6		-86.4	86.4	78.1	
			-53.4	53.4	48.3	53.4	- 53.4		
-11.0	11.0	20.1	22.3	-22.3		-28.0	28.0	25.4	
			-17.3	17.3	15.7	17.3	- 17.3		
						-9.1	9.1	8.2	
					- etc. -				
-698.6	-507.2	735.5	-228.3	-575.5	578.4	-2.9	-399.1	399.1	Solution

Table 1 No-shear moment distribution in 3-storey substitute frame

No. of storeys	No. of bays	Column base k (kip in/radian)	Beam k (kip in/radian)	Critical load factor		
				'Exact'	'Approximate'	Horne
1	1	0	638	6.116	6.16	6.16
1	1	0	2000	10.75	11.0	11.0
1	1	800	638	16.77	16.9	16.9
2	2	0	638	3.377	3.18	3.16
2	2	0	2000	5.282	5.06	5.00
2	2	800	638	6.702	6.54	6.50
3	3	0	638	2.178	1.95	1.94
3	3	0	2000	3.285	3.04	3.00
3	3	800	638	4.000	3.80	3.78

Table 2 Typical comparisons of elastic critical loads

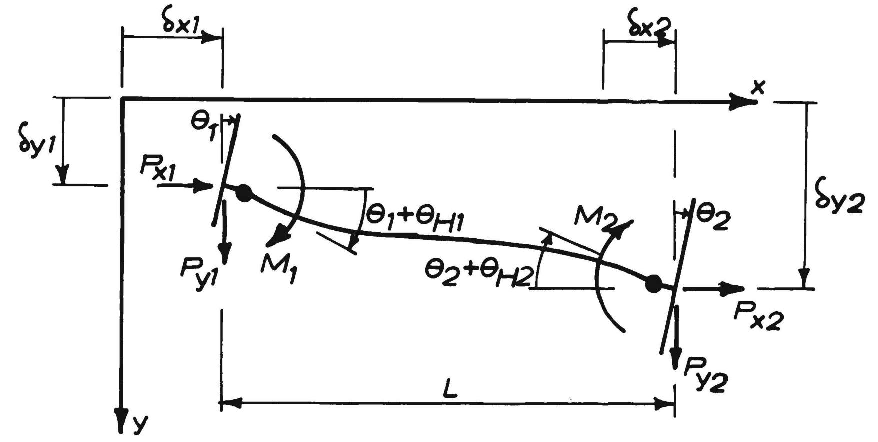

Fig 1 Typical member with flexible joints

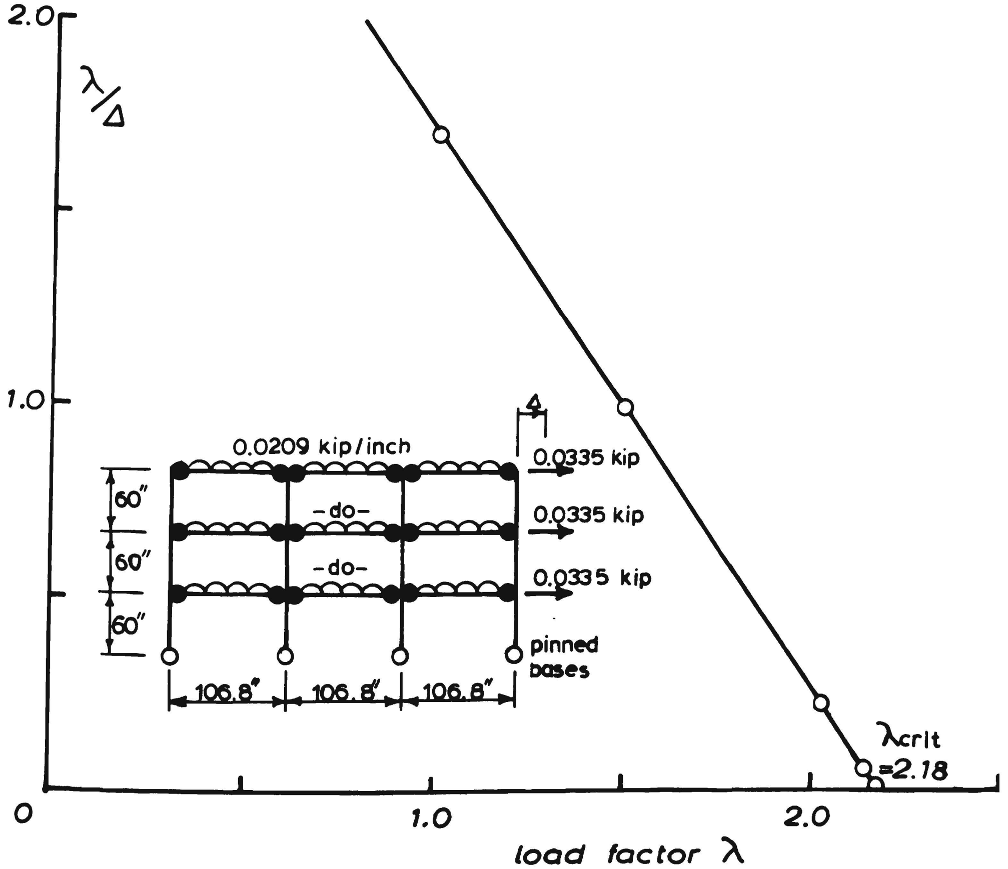

Fig 2 Modified Southwell plot for 3-storey 3-bay rack.

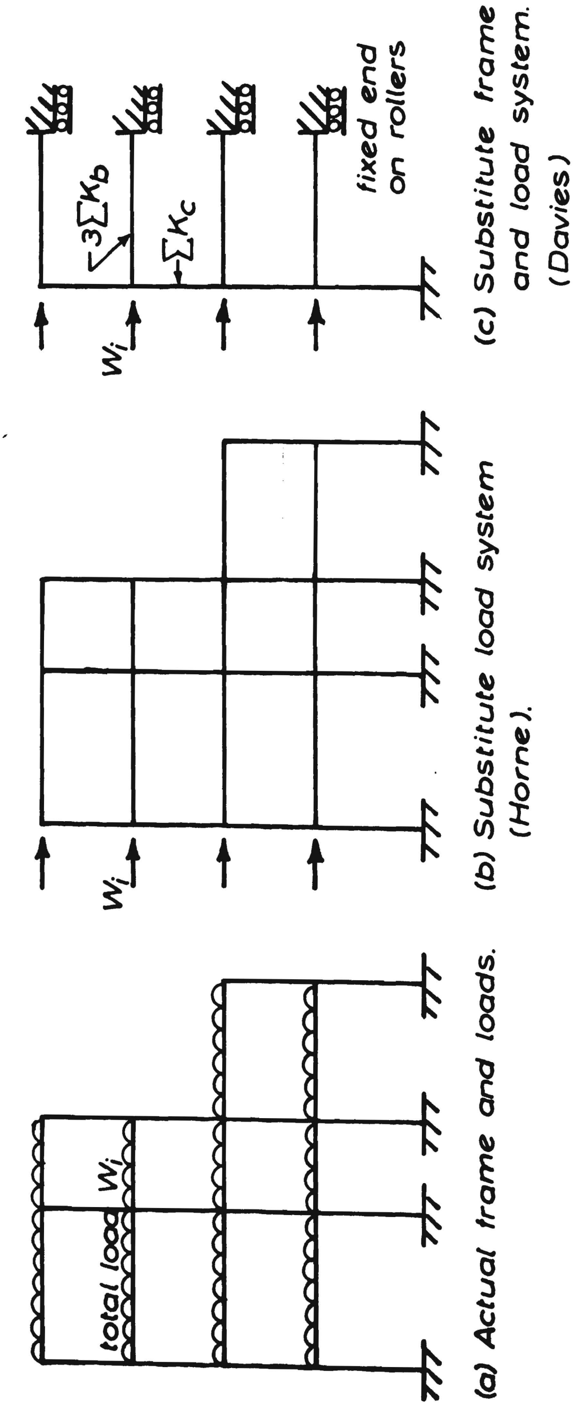

Fig 3 Approximate analyses for the elastic critical loads of rectangular frames

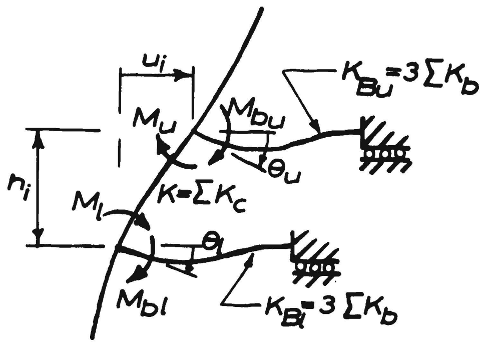

Fig 4 Typical storey of substitute frame

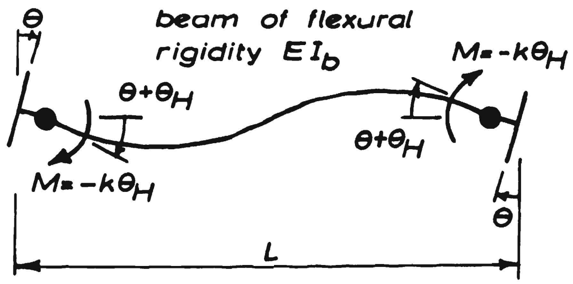

Fig 5 Typical beam

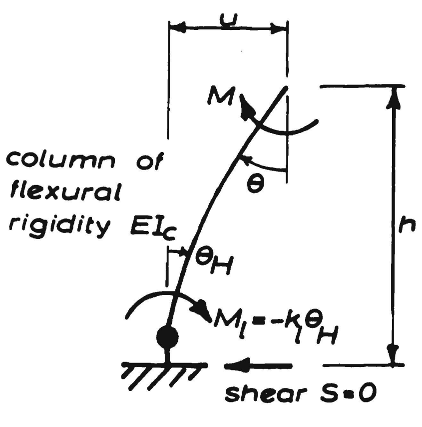

Fig 6 No shear case for bottom storey column

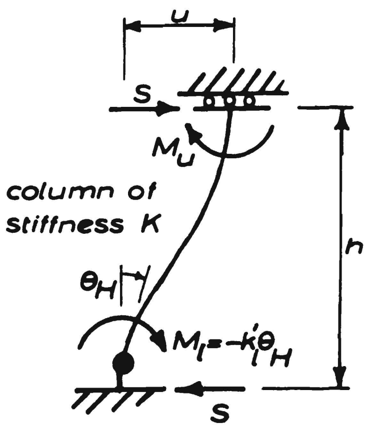

Fig 7 Fixed end moments for bottom storey column

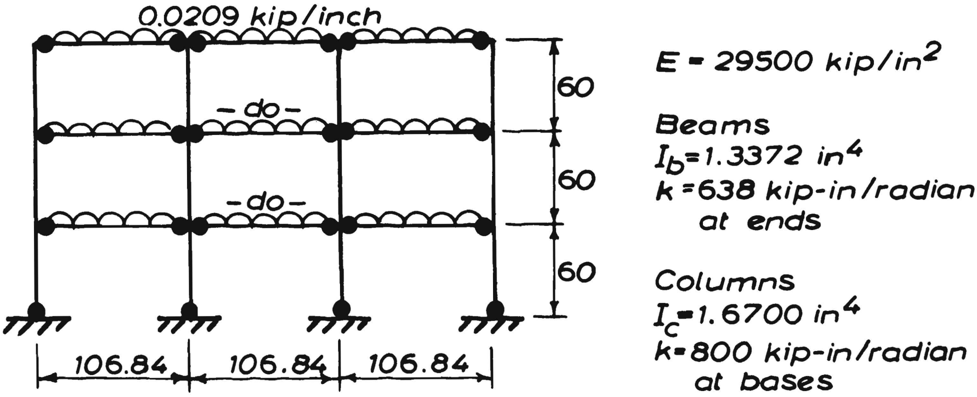

Fig 8 Rack for worked example