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The Effects of Frictional Restraint on the Stability of a Simple Strut

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Synopsis

As a preliminary to an investigation of the effects of friction on the behaviour of Drive In and Drive Thru Racking systems, this paper illustrates the effect of a friction restraint to the sway of a simple strut on the collapse load, and on the stability of the post buckling path. The paper shows that for quite small frictional resistance, of the order of 1-2%, the collapse load can be increased by several times.

INTRODUCTION

The work described in this paper was designed as a preliminary to an investigation of the behaviour of Drive In and Drive Thru pallet racking systems, and the effects on the behaviour of these of the presence of the frictional forces which can be developed between the pallet and its supporting rail. It was appreciated in the early stages, however, that this study has a direct relevence to the testing of structures which sway, and to the problems encountered in designing a suitable loading system. It is therefore in the context of the latter problem that this paper should be read, In the meantime work is proceeding on an investigation into the behaviour of Drive In racking systems, which will be completed shortly.

The requirements of a loading system for structures which sway are that the loading device must not develop any significant resistance to sway as the structure deforms, that it must be controllable to a suitable degree of accuracy, and, particularly with large structures, it must be safe to operate.

A number of alternative approaches are shown in fig.(1). The dead-load system in fig.(1a) meets all the requirements for small structures, but with large structures not only do the physical problems of the size and cost of the loading meduim become very significant, but also, unless a fluid such as water which can be pumped is used, the placing of the load as the structure approaches the collapse condition can become hazardous.

Fig. (1b) shows an alternative employing a loading frame and hydraulic jacks mounted on a slip surface with a very low coefficient of friction. This meets all the requirements, except that the presence of horizontal frictional forces could affect the nature of the collapse of the structure. Lightfoot[1] and others have successfully made use of this system.

The loading mechanism in fig.(1c) used by Yarimci[2] is another attractive alternative, but again is not a perfect loading device, and as well as errors inherent in the geometry of the mechanism, is susceptible to friction in its joints. The more elaborate system in fig.(1d) employs a hydraulic jack operating in a horizontal direction and controlled by a servo-mechanism which is designed to maintain the line of application of the load in a vertical direction as the structure sways. The system has also been used successfully by Stark and Tilburgs[3], but as with the other devices is subject to small errors of alignment.

Finally fig.(1e) offers a further alternative which has many attractions, not the least of which is its simplicity. It comprises two test structures, one a mirror image of the other, which are strained against each other. It depends, of course, for its effectiveness on the degree to which the two test structures can be made identical.

All of these loading systems, with the exception of the deadload arrangement, are imperfect in some degree. As an example, if the friction coefficient in the system in fig.(1b) is too great, sway may be prevented, and collapse of the frame be delayed, giving an erroneous test result. This paper sets out to demonstrate the demands that a particular test structure makes upon such a loading system, if the results obtained in a structural test are not to be significantly distorted.

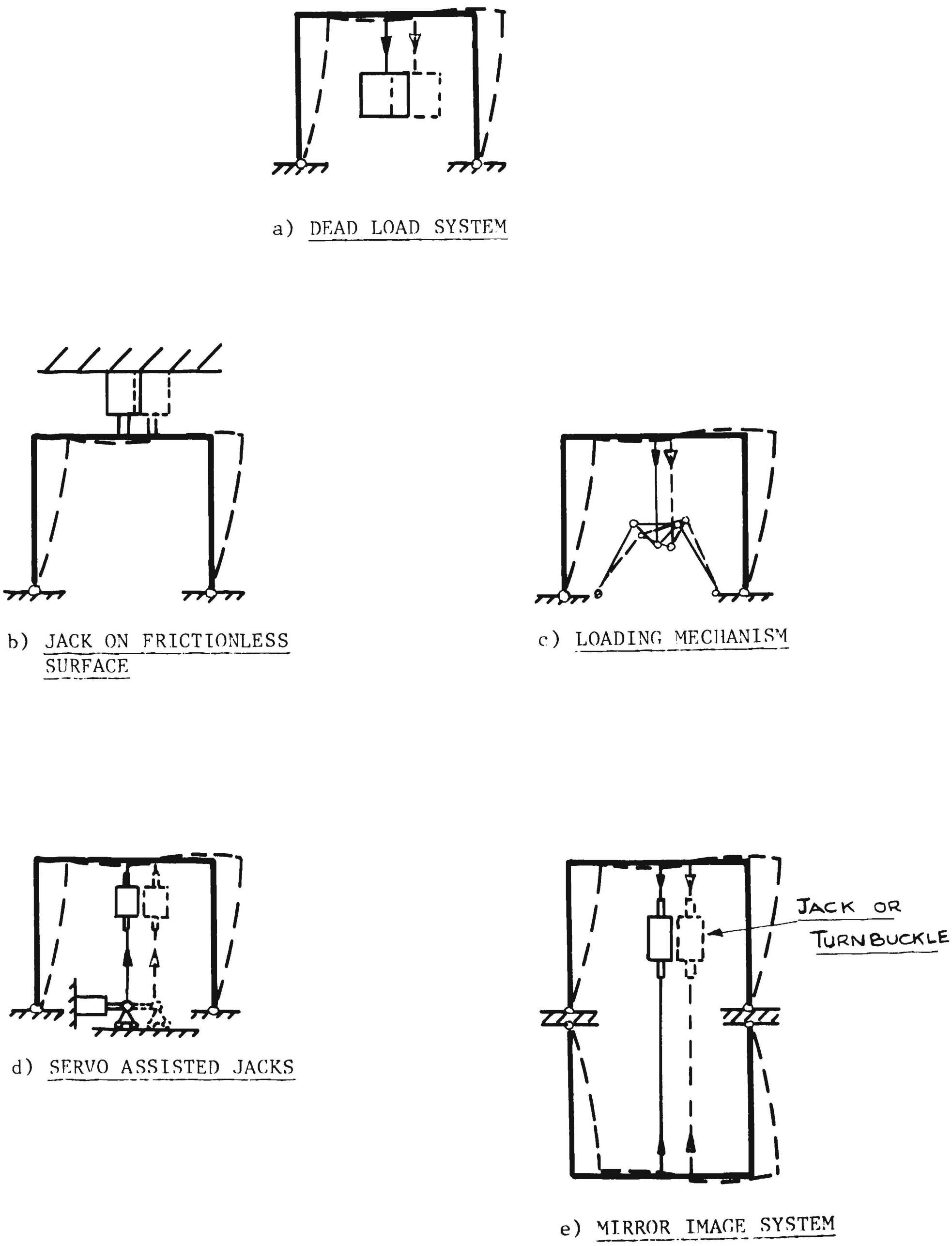


FIG.(1) ALTERNATIVE LOADING SYSTEMS FOR STRUCTURES WHICH SWAY

THEORETICAL APPROACH

In this paper the load system of fig.(1b), being perhaps the simplest to apply is treated in some detail. The structure is assumed to be wholly elastic in its behaviour and in this context it is worth noting that many storage structures are slender enough to remain elastic up to the point of collapse.

Here we treat with the simplest structures of those exhibiting sway at collapse, that of the vertical cantilever, axially loaded, and shown in fig.(2). The characteristics of the loading system we are modelling are represented by a vertical load P, which is applied experimentally as a dead load, and a horizontal reaction H which is developed by friction between a steel disc carrying a load W and a horizontal glass plate.

The strut is assumed to be imperfectly straight in its initial unloaded condition. An analysis of a perfectly straight strut, axially loaded, yields no useful information because as P, the load, is increased, H does not develop. The mere presence of an incipient horizontal reaction in the perfect system is enough to prevent a sway mode of buckling.

The magnitude of the initial imperfection at the 'free' end of the strut is denoted by Δ_o as shown in fig.(2).

The idealised behaviour of the reaction H is shown in fig.(3). Sway of the strut cannot occur until a reaction of $\pm H_o$ is developed in the system. At this value of H, slip begins, and during slip, and thus during the sway of the structure, the value of the horizontal reaction remains unchanged at $H = \pm H_o$

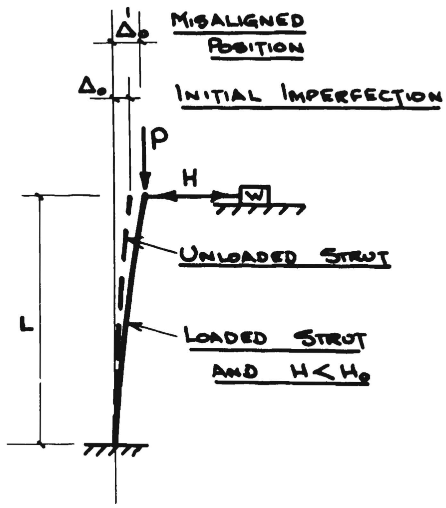


Fig.(2) STRUCTURAL SYSTEM

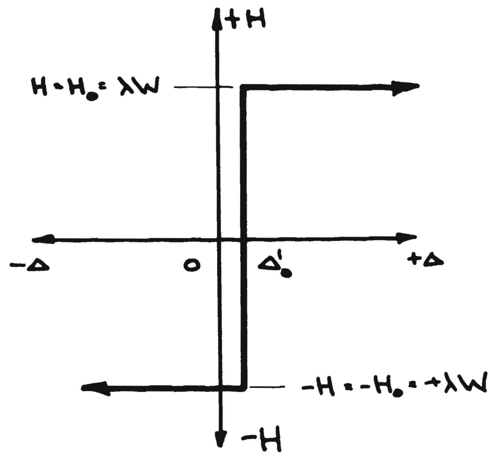


Fig.(3) LOAD-DEFLECTION RELATIONSHIP FOR H

It can be noted here, that when the structure is assembled, the value of H must necessarily be less than $|H_o|$ but will not in general be less than zero. Thus with no load P on the strut, the diplacement of the 'free' end in the horizontal direction may be typically, Δ_o' as shown in fig.(3). The value of H at this point is dependent on the lateral stiffness of the strut.

The next stage in the proceedings is to investigate the behaviour of the strut when subjected to an axial load P and a horizontal load H as shown in fig.(5a) and independent of the friction device.

If the strut were perfectly straight and H = 0, then the initial equilibrium path is stable and follows the line Δ=0 until the first critical load

$$P_{CRIT\ (1)} = \frac{\pi^2 EI}{4 L^2}$$

is reached. At this point the initial path becomes unstable, and sway of the structure occurs. This sway mode is shown in fig.(4a). At loads above the critical load, the path for which $\Delta = 0$ is a valid equilibrium path, but unstable, and cannot be followed by a practical strut.

Notwithstanding this, and still considering the perfect system, the next buckling mode occurs at a value of P such that

$$P_{CRIT\ (2)} = \tan P_{CRIT\ (2)}$$

where

$$P_{CRIT\ (2)} = \sqrt{\frac{P_{CRIT\ (2)} L^2}{EI}}$$

This mode is a non-sway mode, and is shown in fig.(4b).

The unstable equilibrium path continues above this load to a further point of bifurcation of equilibrium paths, when a second sway mode becomes possible at

$$P_{CRIT\ (3)} = \frac{9\pi^2 EI}{4 L^2}$$

This mode is shown in fig.(4c)

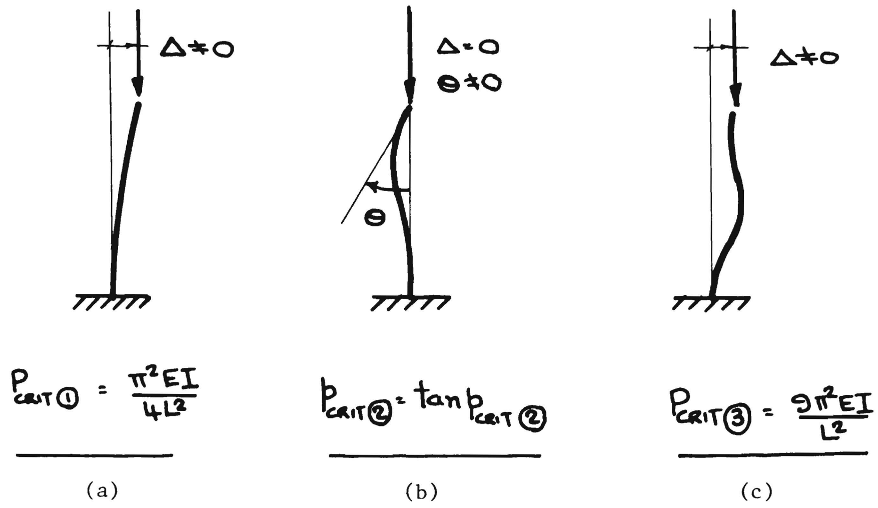
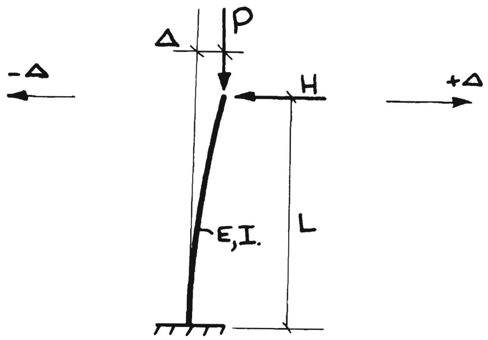
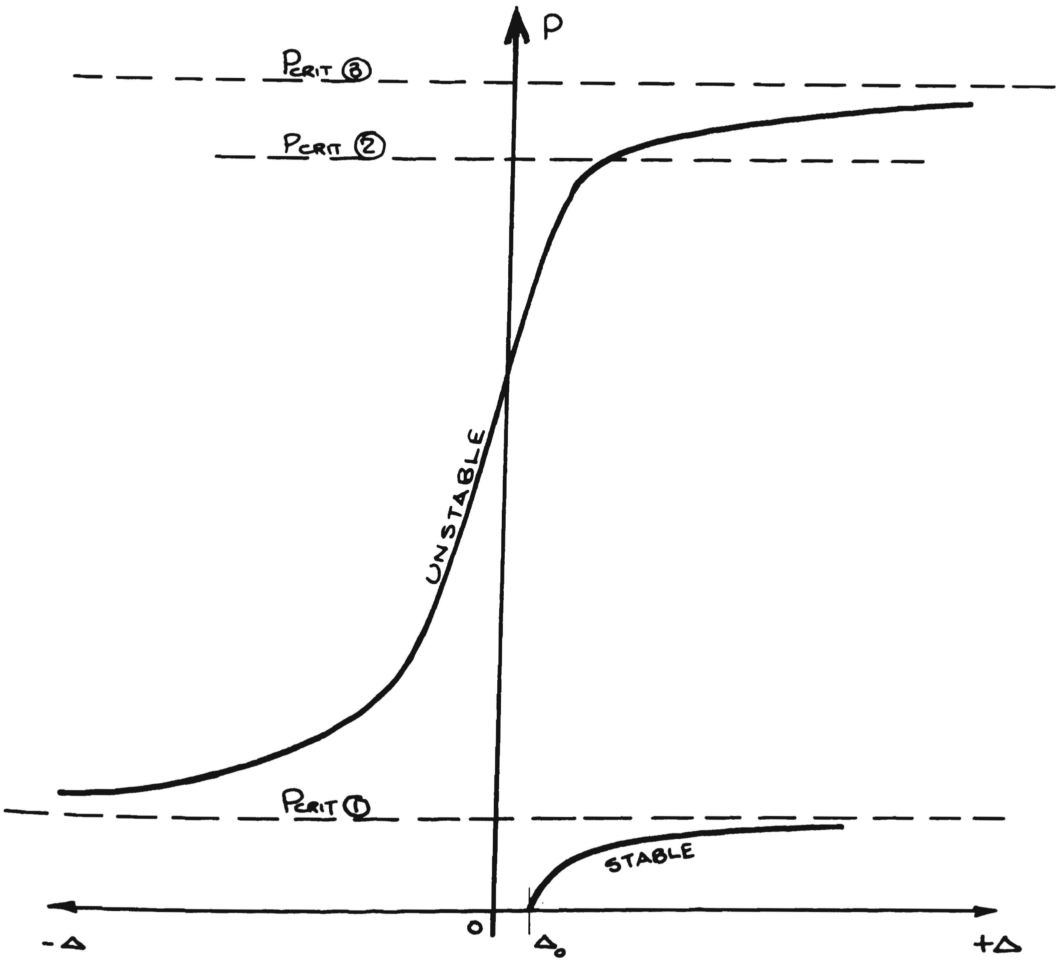


Fig.(4) BUCKLING MODES



(a) TEST STRUCTURE



(b) EQUILIBRIUM PATHS FOR H = 0

Fig.(5)

We now turn to the imperfect system, loaded with an axial load P and a horizontal force H as shown in fig.(5a)

When H=0 the initial equilibrium path approaches the line $P=P_{CRIT(1)}$ from below as P is increased, as in fig.(5b). This path is stable, in contrast to the complementary path which at one extreme approaches the line $P=P_{CRIT(1)}$ from above, and at the other, the line $P=P_{CRIT\ (3)}$ from below. These paths are illustrated in fig.(5b), for a typical value of the initial imperfection.

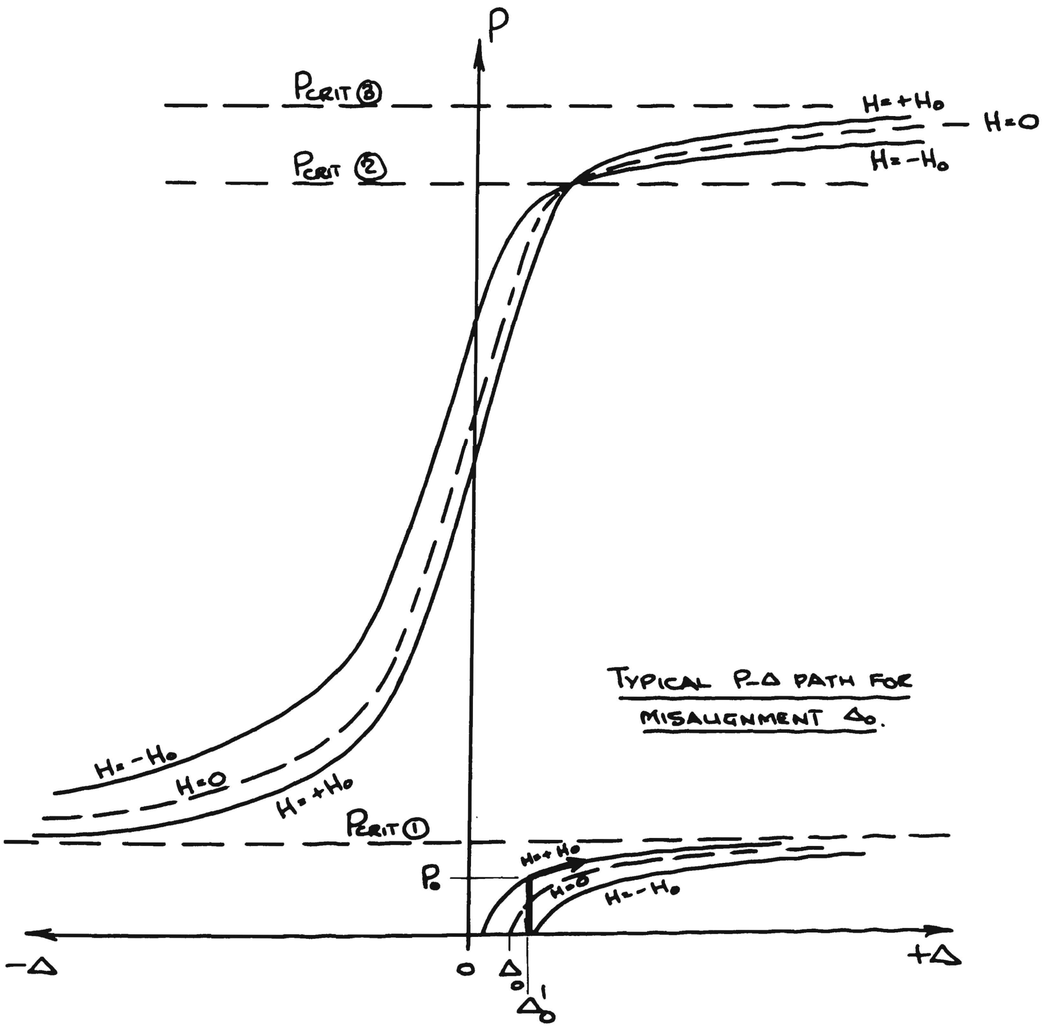
When $H \neq 0$ and is less than some critical value $H = H_{CRIT}$ the equilibrium paths are not greatly changed as shown in fig.(6). When $H > H_{CRIT}$ the effect is to reverse the direction of sway of the equilibrium paths in the region of the lowest critical load $P = P_{CRIT\ (1)}$ as shown in fig.(7)

It will be shown that the critical value of H, at which this reversal takes place is given by

$$H_{CRIT} = \frac{\pi^4}{32}\ \frac{\Delta_1}{L}$$

where Δ_1 is the magnitude of the initial imperfection parameter for the lower of the two sway modes considered.

In the model structure, where the force H is a reaction developed by friction between the weight W and the plate on which it rests, the initial equilbrium path is defined by $\Delta = \Delta'_o$ while $H < H_o$ because before the load P is applied, but while the friction device is in place, the structure may be misaligned (or prestressed) into a position at which $\Delta = \Delta'_o = \Delta_o$. The limits on Δ'_o are determined by H_o and the stiffness of the strut.



EQUILIBRIUM PATHS FOR $H_o < H_{crit}$ Fig.(6)

When the load P is subsequently applied, the initial equilibrium path is defined by $\Delta = \Delta'_o$ until $H = \pm H_0$, that is until the line $\Delta = \Delta'_o$ intersects an equilibrium path for which $H = +H_0$ or $H = -H_0$. The load at which this happens, the breakaway load, is denoted by P_0. Typical such paths and intersections are shown in figs. (6) and (7).

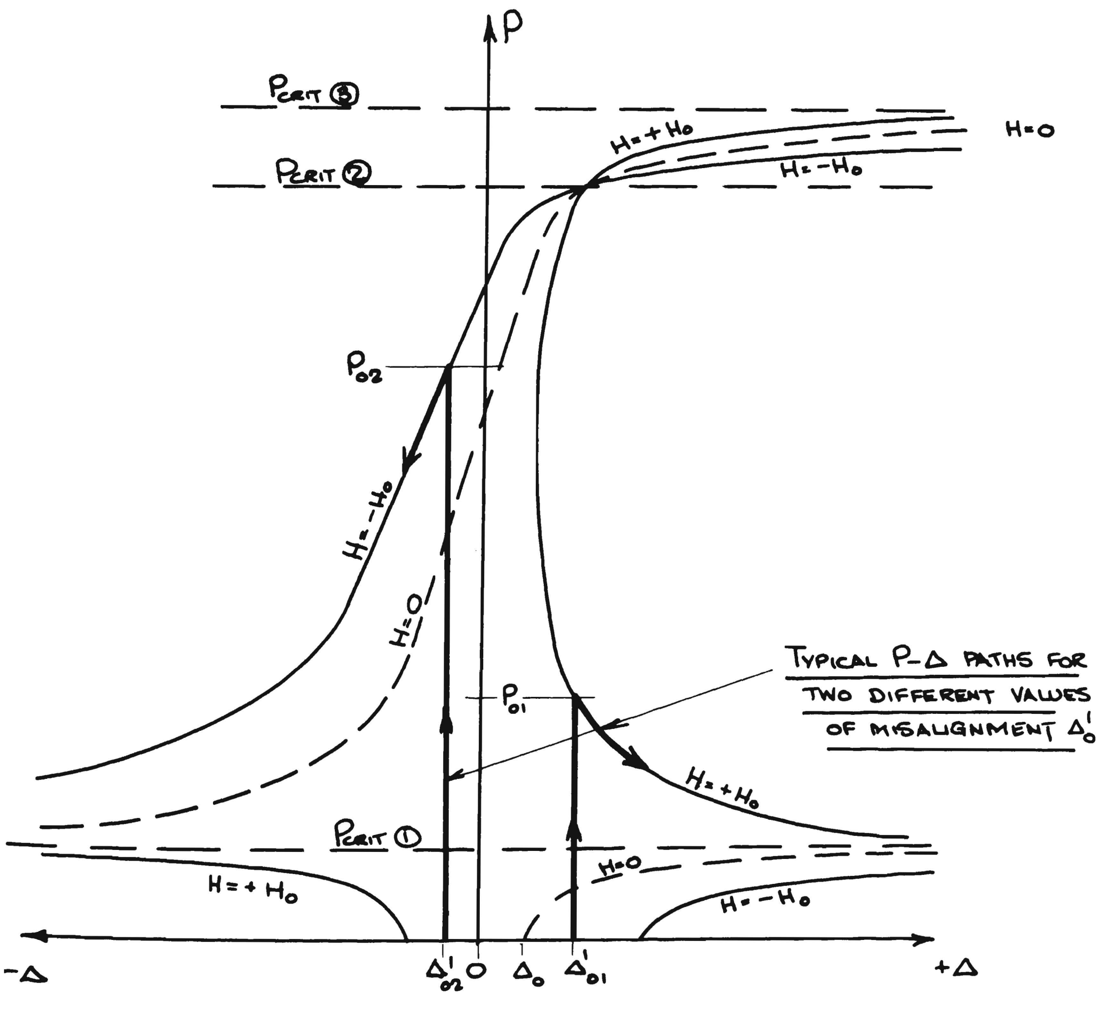
In fig.(6), $H_0 < H_{CRIT}$ and the breakaway load P_0 is therefore always less than $P_{CRIT(1)}$. In fig.(7), $H > H_{CRIT}$ and the breakaway load always in excess of $P_{CRIT\ (1)}$

In fig.(6) when $H_0 < H_{CRIT}$ the only consequence on the observed behaviour of the system of the presence of a frictional resistance to sway, is a discontinuity in the equilibrium path when slip occurs. The observed buckling load will in all cases be less than the theoretical value.

This is the case whatever initial conditions are met before loading starts, that is, although the strut may be displaced from its at rest position by misalignment, the magnitude of this misalignment is limited to that corresponding to $H = \pm H_0$. The only effect of the misalignment is to change the load P_0 at which the slip occurs. It does not affect the final buckling load.

When $H_0 > H_{CRIT}$ however, the range of possible misalignments is increased, and allows collapse to occur in a direction opposite to the bias induced by the nature of the initial imperfection.

Thus, in fig.(7) two typical load-deflection paths are shown denoted by $\Delta = \Delta'_{o1}$ and $\Delta = \Delta'_{o2}$ each corresponding to a different value of the initial misalignment. In both cases the breakaway load is defined by the intersection of the initial path $\Delta = \Delta'_o$ and the complementary equilibrium path for which $H = +H_0$ or $H = -H_0$



EQUILIBRIUM PATHS FOR $H_0 > H_{CRIT}$ Fig.(7)

In both cases the breakaway load P_{01} or P_{02} is in excess of the lowest buckling load, and will lead to a false impression of the collapse load of the structure for the unwary experimenter. Because the breakaway load now lies on the unstable complementary equilibrium path, the ensuing behaviour of the structure is catastrophic.

The maximum value of the breakaway load that can be achieved for a given system depends on the initial imperfections, that is on their size and nature, as well as on the value of H_0 and Δ_o' . It can be seen from fig.(7) however that the highest value of P_0 is always less than $P_{CRIT\ (2)}$ the critical load for the non-sway mode. Substantial buckling of the strut may occur in this mode before sway of the structure occurs.

THE ANALYSIS

The analysis of the structure is made on conventional lines, from a consideration of the system shown in fig.(8). The strut is assumed to be perfectly elastic, and loaded by an axial force P and a horizontal load H which is positive in a direction opposite to that of the imperfection parameter Δ .

The initial imperfection is assumed to be of the form

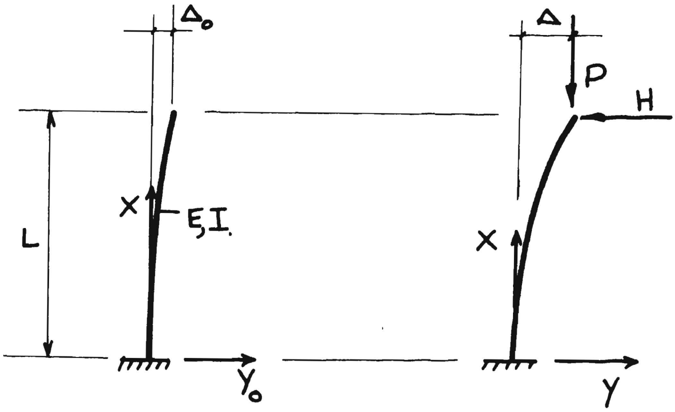
$$y = y_o = \Delta_1(1-\cos\frac{\pi x}{2L}) + \Delta_2\ (1 - \cos\frac{3\pi x}{2L}) \qquad (1)$$

so that

$$y_o = \Delta_o = \Delta_1 + \Delta_2 \text{ at } x=L$$

For the loaded condition, we can write

$$EI\ \frac{d^2(y-y_o)}{dx^2} = p(\Delta-y) - H(L-x) \qquad (2)$$



(a) UNLOADED STRUT (b) LOADED STRUT

Fig.(8) IMPERFECT ELASTIC FIXED ENDED STRUT

To reduce the complexity of the algebraic expressions, we employ a non dimensional format by writing

$$p^2 = \frac{PL^2}{EI}\ ,\quad h^2 = \frac{HL^2}{EI}\ ,\quad x = \frac{X}{L}\ ,\quad y = \frac{Y}{L}\ ,\quad \delta = \frac{\Delta}{L}\ ,\ \text{etc}$$

so that equation (2) becomes

$$\frac{d^2(y-y_o)}{dx^2} = p^2\,(\delta - y) - h^2(1-x) \qquad (3)$$

The solution to this equation, having substituted for Y_0 using equation (1) is

$$y-y_o = A\cos px + B\sin px + \delta - \delta_o + \frac{p^2\delta_1\cos\frac{\pi x}{2}}{(p^2-\pi^2/4)} + \frac{p^2\delta_2\cos\frac{3\pi x}{2}}{(p^2-9\pi^2/4)} + \frac{h^2(1-x)}{p^2} \qquad (4)$$

and the boundary conditions are

$$\text{at } x=0\ ,\ y-y_o = 0 \text{ and } \frac{d(y-y_o)}{dx} = 0 \qquad (5)$$

so that

$$A = \frac{h^2}{p^2} - \delta + \frac{\delta_1}{\left(1-\frac{4p^2}{\pi^2}\right)} + \frac{\delta_2}{\left(1-\frac{4p^2}{9\pi^2}\right)} \qquad (6)$$

$$B = \frac{-h^2}{p^3} \qquad (7)$$

substituting for A and B in equation (4) yields

$$y-y_o = \frac{h^2}{p^2}\{\cos px - \frac{\sin px}{p} - (1-x)\} + (\delta - \delta_o)(1-\cos px)$$

$$+ \frac{4p^2}{\pi^2}\frac{\delta_1}{(1-\frac{4p^2}{\pi^2})}(\cos px - \cos\frac{\pi x}{2}) + \frac{4p^2}{9\pi^2}\frac{\delta_2}{(1-\frac{4p^2}{9\pi^2})}(\cos px - \cos\frac{3\pi x}{2})$$

(8)

when $x = 1$, $y_o = \delta_o = \delta_1 + \delta_2$ and after some re-arrangement of terms,

$$\delta = \frac{\delta_1}{(1-\frac{4p^2}{\pi^2})} + \frac{\delta_2}{(1-\frac{4p^2}{9\pi^2})} + \frac{h^2(\cos p - \frac{\sin p}{p})}{p^2 \cos p} \quad (9)$$

Equation (9) yields the equilibrium paths shown in figs.(5) and (6) for different values of $H = H_0$.

The breakaway loads of fig.(7) are obtained by the simultaneous solution of equation (9) and $\Delta = \Delta_o'$ (10)

It can be shown by expanding the trigonametical functions of p close to $p^2 = \pi^2/4$ that the value of $H = H_0$ at which the breakaway load is $P_0 = \frac{\pi^2 EI}{4L^2}$ is given by

$$h_o^2 = \frac{\pi^4}{32}\delta_1$$

that is $H_o = \frac{\pi^4}{32}\frac{EI\,\Delta_1}{L^3} = H_{CRIT}$

If in a test the observed buckling load is not to be significantly affected by frictional resistance, then

$$H_{CRIT} < \frac{\pi^4}{32}\frac{EI\,\Delta_1}{L^3}$$

The theoretical buckling load is $P_{CRIT(1)}$ and the maximum permitted coefficient of friction in the system shown in fig.(1b) is therefore given by

$$\lambda_{CRIT} = \frac{H_{CRIT}}{P_{CRIT\ (1)}} = \frac{\pi^2}{8}\frac{\Delta_1}{L}$$

$$\doteqdot 1.2\ \frac{\Delta_1}{L}$$

EXPERIMENTAL WORK

The purpose of the experimental work was to establish the relationship between collapse or breakaway load, and the limiting horizontal reaction, for a particular value of the initial misalignment.

A high tensile steel strip was chosen for a model strut, a sketch of which is shown in fig.(9). The frictional restraint comprised a weighted steel disc resting on a horizontal glass plate. The tests were composed of two parts. The first to determine the family of equilibrium paths for different values of the horizontal force H and the second to determine the breakaway, or collapse loads, P_0, for a range of values of the weight on the steel disc. A Southwell plot was also made to give an indication of the value of initial imperfection , and slip tests were made to determine the coefficient of friction between the steel disc and the plate.

In all cases, the vertical was a dead load.

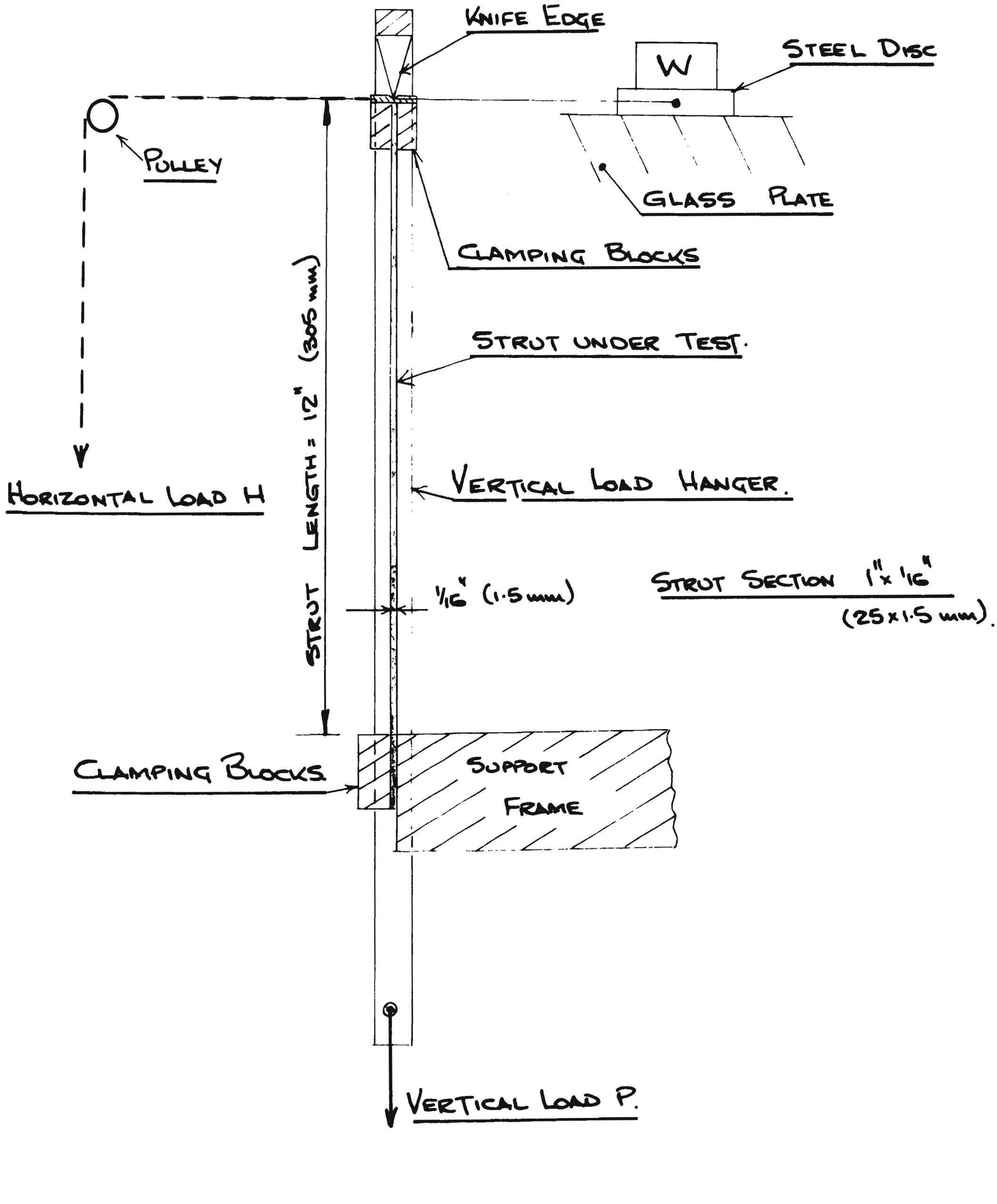


Fig. (9) SECTION THROUGH THE EXPERIMENTAL RIG

The first series of test to determine the family of equilibrium paths, were made as follows:

The initial path, being stable, was simply obtained by observing the lateral sway of the strut as the load was increased to a value close to $P_{CRIT\ (1)}$. The complementary path presented a little difficulty because of its unstable nature. A number of methods are available for dealing with this, and in this experiment, the strut was first loaded with the horizontal load, but prevented from swaying by a stop against which it was allowed to rest. The axial load P was then applied in appropriate increments until the strut just left the stop and tried to follow the complementary path. The load at which the strut left the stop was taken to be a point of statical equilibrium.

This process was repeated for a number of values of the horizontal force, and the results are shown in fig.(10).

The second set of tests were made with a range of weights W placed on the disc. It was convenient, however, to pre-load the strut thus inducing an initial misalignment of $\Delta = \Delta_o'$, prior to connecting the disc. The misalignment thus obtained, Δ_o' was $1.5\ x\Delta_o$, the measured initial imperfection. Once this was done, the load on the strut was increased until collapse occurred. Within the range tested, this collapse load increased as the frictional resistance to sway increased.

Fig.(10) shows the results of this series of tests also. In theory the collapse of the structure should occur without any initial sway movement. In practice there was some small movement of the structure as the collapse load was approached. The arrow heads on the diagram mark the extremities of extrapolations of the observed load

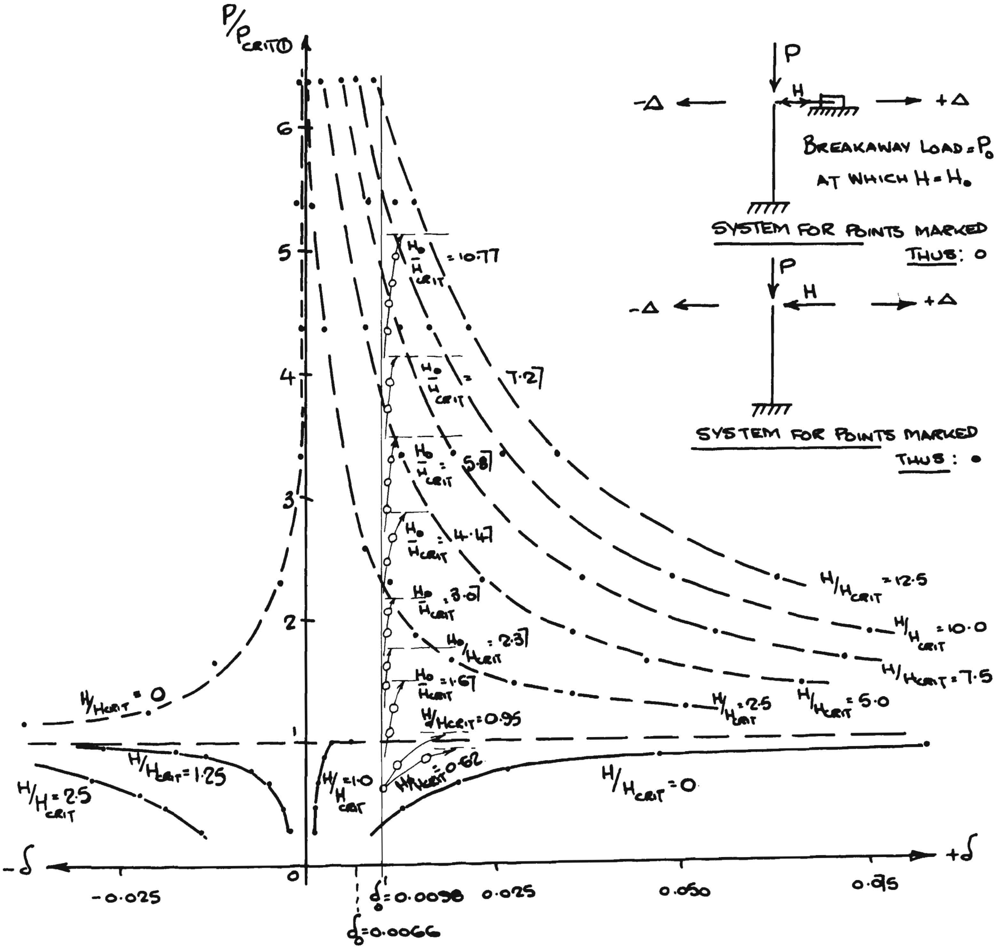


Fig.(10) EXPERIMENTAL RESULTS

deflection curves, and are placed at the maximum observed load. Collapse at this load was sudden, and deflection measurements were not possible. The match between the experimental equilibrium paths and the point at which collapse occurred is fair.

The variation of collapse load obtained theoretically from equations (9) and (10) for a range of values of the maximum available horizontal reaction H_0 is plotted in fig.(11) along with the corresponding experimental results, using the data obtained for the strut from the Southwell Plot and the tests for friction coefficient. In calculating the theoretical curves, it has been assumed that the initial imperfections were such that $\Delta_o = \Delta_1$ = 2 mm (0.08") observed from the Southwell Plot, and $\Delta_2 = 0$. $\Delta'_o = 1.5 \times \Delta_o$.

The Southwell plot is shown in fig.(12), from which an experimental value of the initial imperfection Δ_o was obtained, as well as for the effective section and material properties, $\frac{EI}{L^2}$ The value Δ_o is only an approximation, the degree of accuracy being dependent on the relative magnitudes of imperfections in the first and second sway modes corresponding to $P_{CRIT_{(1)}}$ and $P_{CRIT_{(3)}}$

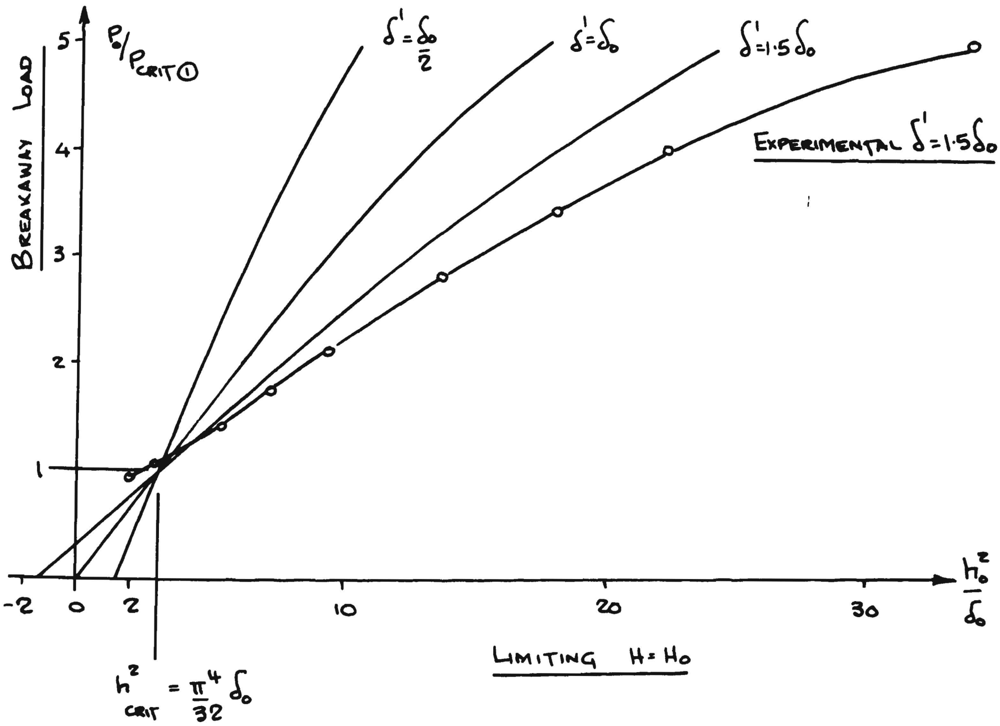


Fig.(11) RELATIONSHIP BETWEEN $P_o/P_{CRIT①}$ AND h_o^2/δ_o. THEORY AND EXPERIMENT

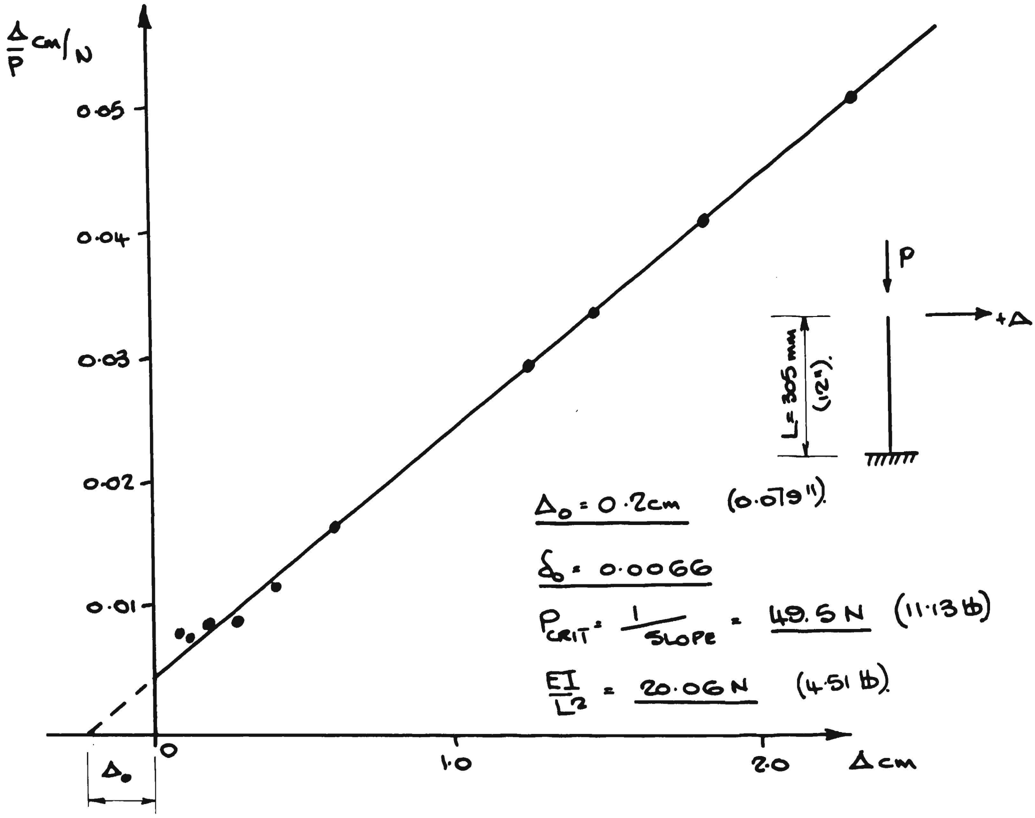


Fig.(12) SOUTHWELL PLOT

On the basis of the assumption that Δ_o so measured could be effectively taken to be equal to Δ_1 - the critical value of

$$H = H_{CRIT} = \frac{\pi^4}{32} \times \frac{EI}{L^2} \times \frac{\Delta_o}{L} = \frac{\pi^4}{32} \times 4.51 \times \frac{0.079}{12} = 0.09\ lb_f$$
$$(0.40N)$$

and as $p_{CRIT\ (1)} = 11.13\ lb_f$ (49.5N)

the maximum allowable coefficient of friction in a test system of the type shown in fig.(1a) would be

$$\lambda_{CRIT} = \frac{0.09}{11.1} = 0.008 \doteq 8\%$$

The last of the tests conducted gave the highest collapse load of

$p_o = 56.2\ lb_f$ (250N)

while $H_o = 0.97\ lb_f$ (4.31N)

so that for an observed coefficient of friction $\lambda = \frac{0.97}{56.2} = 1.7\%$

the collapse load was increased by a factor of five above the theoretical load, whilst λ was only about twice its limiting value.

Taking $\Delta_o' = 1.5\Delta_1,\ \Delta_2 = 0$

and
$$\frac{h_o^2}{\delta_o} = \frac{H_o L^2}{EI} \times \frac{L}{\Delta_o} = \frac{0.97}{4.51} \times \frac{12}{0.079} \doteq 33$$

from the experimental observations, and using this data in equations (9) and (10), the predicted collapse load is

$$p_o^2 = \frac{P_o L^2}{EI} \doteq 15$$

i.e. $P_o = 67.4\ lb_f$ (300N) that is the collapse load is approximately 6 times the theoretical collapse load. The difference between this projection and the observed load is due to erroneous assumptions madeabout the nature of the imperfections in the system and the

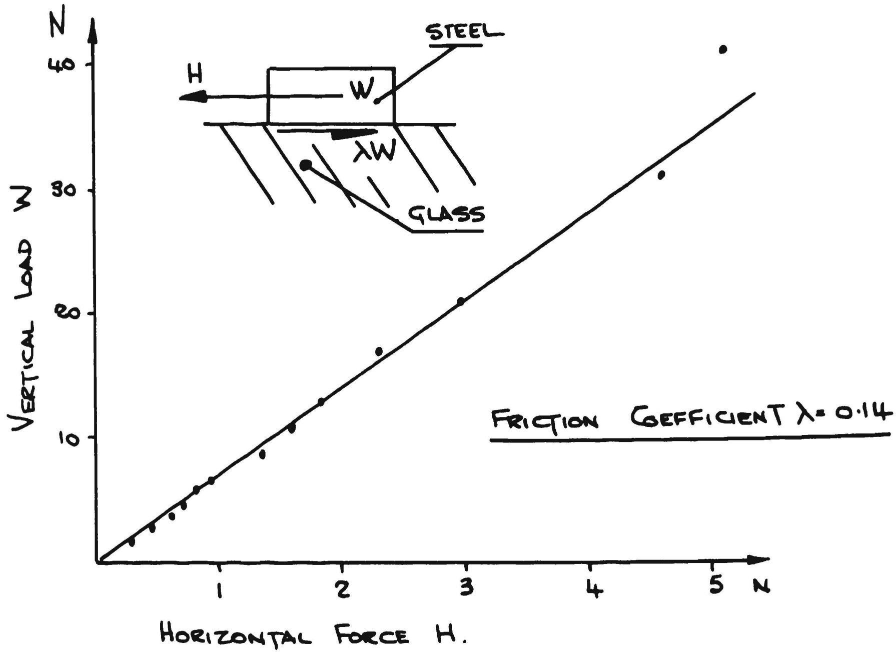


Fig.(13) DETERMINATION OF GLASS-STEEL FRICTION COEFFICIENT behaviour of the friction device. Fig.(13) shows the results for the experimental determination for the coefficient of friction between the steel plate and glass surface used in the experiment.

CONCLUSIONS

The paper demonstrated by experiment as well as theory that the presence of small frictional restraints to a simple structure whose lowest mode is a sway mode, can significantly alter the collapse behaviour of the structure. The work is concerned with elastic systems only, and the relatively simple approach could be applied to an investigation of the behaviour of similar structures. Its relevance to the racking structures is indirect, but important to an analysis of systems such as Drive-In which may benefit in their behaviour from the presence of incipient frictional restraints developing between pallet and support rail.

APPENDIX A: NOTATION

E:	Elastic Modulus for steel. lb_f/ins^2 or N/mm^2
H:	Horizontal reaction or force. lb_f or N
H_o:	Limiting horizontal reaction. lb_f or N
I:	Second moment of area about axis of bending ins^4 or mm^4
L:	Length of strut. ins or mm
P:	Vertical load on strut lb_f or N

P_o, P_{o1}, P_{o2}:	Breakaway loads lb_f or N
$P_{CRIT_{(1)}}$ $P_{CRIT_{(2)}}$ $P_{CRIT_{(3)}}$:	Critical loads for the strut
W:	Load on steel disc. lb_f of N
X:) Y:)	Cartesian co-ordinates ins or mm
Δ'_o, Δ'_{o1}, Δ'_{o2}	Initial misalignments, ins or mm
λ	Friction coefficient, non dimensional

Δ_o, Δ_1, Δ_2.

Also, the following non-dimensional veriables are used:

$$h = \sqrt{\frac{HL^2}{EI}} \quad , \quad p = \sqrt{\frac{PL^2}{EI}} \quad , \quad x = \frac{X}{L} \quad , \quad y = \frac{Y}{L} \quad , \quad \delta = \frac{\Delta}{L} \quad .$$

APPENDIX B: REFERENCES

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