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# Long Path Lemma Concerning Connectivity and Independence Number

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# Long path lemma concerning connectivity and independence number

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**Abstract**

We show that, in a $k$-connected graph $G$ of order $n$ with $\alpha(G) = \alpha$, between any pair of vertices, there exists a path $P$ joining them with $|P| \geq \min\left\{n, \frac{(k-1)(n-k)}{\alpha} + k\right\}$. This implies that, for any edge $e \in E(G)$, there is a cycle containing $e$ of length at least $\min\left\{n, \frac{(k-1)(n-k)}{\alpha} + k\right\}$. Moreover, we generalize our result as follows: for any choice $S$ of $s \leq k$ vertices in $G$, there exists a tree $T$ whose set of leaves is $S$ with $|T| \geq \min\left\{n, \frac{(k-s+1)(n-k)}{\alpha} + k\right\}$.

## 1 Introduction

In this work, we present a tool which we believe will be useful in many applications. Much work has been devoted to finding long paths and cycles in graphs. In particular, in [4], O, West and Wu recently proved a conjecture by Fouquet and Jolivet [3] stated as follows.

**Theorem 1 ([4])** *Let $k \geq 2$ and let $G$ be a $k$-connected graph of order $n$ with $\alpha(G) = \alpha$. Then there is a cycle in $G$ of length at least* $\min\{n, \frac{k(n+\alpha-k)}{\alpha}\}$.

In various situations including this work, it often becomes necessary to find a long path between a chosen pair of vertices. For this reason, O, West and Wu proved the following theorem which they used in their proof of the conjecture.

**Theorem 2 ([4])** *Let $G$ be a $k$-connected graph for $k \geq 1$. If $H \subseteq G$ and $u$ and $v$ are distinct vertices in $G$, then $G$ contains a $u, v$-path $P$ such that $V(H) \subseteq V(P)$ or $\alpha(H - P) \leq \alpha(H) - (k-1)$.*

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We also use this theorem and, following the proofs presented in [4], we prove the following lemma which is our main result.

**Lemma 1** *Let $k \geq 1$ be an integer and let $G$ be a graph of order $n$ with $\kappa(G) = k$ and $\alpha(G) = \alpha$. Then for any pair of vertices $u$, $v$ in $G$, there exists a $u, v$-path of order at least* $\min\{n, \frac{(k-1)(n-k)}{\alpha} + k\}$.

Our hope is that this lemma may be applied to produce other results like Theorem 3, which follows immediately from Lemma 1 by choosing $u$ and $v$ to be the ends of $e$.

**Theorem 3** *Let $k \geq 2$ be an integer and let $G$ be a $k$-connected graph of order $n$ with $\alpha(G) = \alpha$. Then for any edge $e \in E(G)$, there exists a cycle of length at least* $\min\left\{n, \frac{(k-1)(n-k)}{\alpha} + k\right\}$ *in $G$ containing the edge $e$.*

Lemma 1 can be generalized to the following result concerning large trees with specified sets of leaves. Let $\ell(T)$ denote the set of leaves in a tree $T$.

**Theorem 4** *Let $k$ and $s$ be integers with $2 \leq s \leq k$ and let $G$ be a $k$-connected graph of order $n$ with $\alpha(G) = \alpha$. Then for any set of $s$ vertices $V_s = \{v_1, \ldots, v_s\} \subseteq G$, there exists a tree $T \subseteq G$ with $V_s = \ell(T)$ and $|T| \geq \min\left\{n, \frac{(k-s+1)(n-k)}{\alpha} + k\right\}$.*

The proofs of Lemma 1 and Theorem 4 are presented in Section 3. As we will observe in Section 4, our results are all best possible.

# 2 Preliminaries

In our proof, we use the following corollary to break the problem into cases. We also state and prove a path version of Theorem 6. Both of these results come from [4].

**Corollary 5 ([4])** *If a graph $G$ admits no vertex partition $(V_1, V_2)$ such that $\alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2])$, then $G$ is 2-connected or $G \in \{K_1, K_2\}$. Also, for distinct vertices $u, v \in G$, there is a $u, v$-path $P$ such that $\alpha(G - P) < \alpha(G)$.*

**Theorem 6 ([4])** *Let $k$ be an integer greater than 1. If $C$ is a cycle with size at least $k$ in a $k$-connected graph $G$, then for any non-empty subgraph $H \subseteq G - C$, there exists a cycle $C'$ in $G$ such that $|C - C'| \leq \frac{|C|}{k} - 1$ and $\alpha(H - C') \leq \alpha(H) - 1$.*

We will also make use of the following classical result of Chvátal and Erdős [2]. A graph is said to be *hamiltonian connected* if, between any pair of vertices, there exists a path covering the entire graph.

**Theorem 7 ([2])** *For any graph $G$, if $\kappa(G) > \alpha(G)$, then $G$ is hamiltonian connected.*

Following the notation of [4], let $P$ be a path and $u$ and $v$ be vertices in $P$. Define $P(u, v)$ to be the subpath of $P$ strictly between (not including) $u$ and $v$. Also, for a vertex $v$ and a set of vertices or subgraph $A$, define a $(v, A)$ *$k$-fan* to be a set of $k$ paths from $v$ to $A$ which are all pairwise vertex disjoint except at $v$. All other standard notation comes from [1].

## 3 Proofs of our Main Results

We begin by proving a key lemma used to obtain our main result. The main idea of the proof is based on that of Theorem 6.

**Lemma 2** *Let $k \geq 2$ be an integer, and suppose $G$ is a $k$-connected graph containing vertices $u, v$. If $P$ is a $u, v$-path of order at least $k$ in $G$, then for any non-empty subgraph $H \subseteq G \setminus P$, there is a $u, v$-path $P'$ in $G$ such that $|P \setminus P'| \leq \frac{|P|-k}{k-1}$ and $\alpha(H \setminus P') \leq \alpha(H) - 1$.*

**Proof:** Suppose there exists a subgraph $H$ for which there is no desired path $P'$ and choose $H$ to be the smallest such subgraph. By Corollary 5, either

(1) $H$ can be bipartitioned into non-empty subgraphs $H_1$ and $H_2$ so that $\alpha(H) = \alpha(H_1) + \alpha(H_2)$, or

(2) $H$ is 2-connected or $H \in \{K_1, K_2\}$. Also, for any distinct vertices $x, y \in H$, there exists an $x, y$-path $P_{xy}$ in $H$ such that $\alpha(H \setminus P_{xy}) < \alpha(H)$.

If (1) holds, we simply apply Lemma 2 on $H_1$ (since $H$ was the smallest counterexample) and obtain a path $P'$ satisfying the desired conditions. Hence we may assume (2) holds.

Let $B$ be the block of $G \setminus P$ containing $H$. First we assume $|B| \geq k$. By Menger's Theorem, there exist $k$ vertex-disjoint paths from $P$ to $B$. Choose the shortest such set of paths, meaning that each path contains exactly one vertex of $B$ and one vertex of $P$. This means that there must exist a pair of these paths, say $P_1 = p_1 \ldots b_1$ and $P_2 = p_2 \ldots b_2$ for $p_i \in V(P)$ and $b_i \in V(B)$ such that there are at most $\frac{|P|-k}{k-1}$ vertices between $p_1$ and $p_2$ on $P$. Since $B$ is 2-connected, there exist vertex-disjoint paths $P_{b_i}$ in $B$ from $b_i$ to $h_i \in V(H)$ for $i = 1, 2$. Note that $h_1 = h_2$ is only possible if $|H| = 1$. (Suppose $P_{b_i} \cap H = h_i$.) By (2), there is a path $P_H$ in $H$ from $h_1$ to $h_2$ for which $\alpha(H \setminus P_H) < \alpha(H)$. Then $P' = (P \setminus P(p_1, p_2)) \cup (P_1 \cup P_{b_1} \cup P_H \cup P_{b_2} \cup P_2)$ is the desired path. Hence, we may assume $|B| < k$.

Let $V(B) = \{b_1, \ldots, b_\ell\}$, where we have assumed $\ell < k$. Note that we may possibly have $\ell = 1$. Let $C$ be the component of $G \setminus P$ containing $B$. Let $S = \{p_1, \ldots, p_m\}$ be the set of vertices of $P$ (in order along $P$) with at least one neighbor in $C$. Note that, by Menger's Theorem, $m \geq k$.

For each edge $e$ from $p_i$ to $C$, there exists a unique vertex $b_j \in B$ such that there is a unique path $Q_{i,j}$ from $b_j$ to $p_i$ containing $e$ with all interior vertices in $C \setminus B$. Let $X_j$ be the set of vertices $p_i$ for which such a path $Q_{i,j}$ exists. Note that the sets $\{X_j\}$ are not necessarily disjoint. Also note that, since $B$ is a block, $Q_{i,j}$ and $Q_{i',j'}$ are internally disjoint when $j \neq j'$. Call a segment $P(p_i, p_j)$ for $i < j$ *large* if $p_i \in X_{i'}$ and $p_j \in X_{j'}$ for some $i' \neq j'$. Otherwise, as long as the segment $P(p_i, p_j)$ is not contained in a large segment, it will be called *small*.

Using the same argument as above, the following fact is immediate.

**Fact 1** *For any large segment $P(p_i, p_j)$, we have*

$$|P(p_i, p_j)| > \frac{|P| - k}{k - 1}.$$

Let $t$ be the number of segments $P(p_i, p_{i+1})$ for $1 \leq i \leq m$ which are large. Since large segments contain at least $\frac{|P|-k+1}{k-1}$ vertices, we see that

$$|P| \geq t\left(\frac{|P| - k + 1}{k - 1}\right) + k,$$

which implies that $t < k - 1$. For each $b_i \in B$, there exists a $(b_i - P)$ $k$-fan. Choose such a fan so that each path intersects $P$ in exactly one vertex. Let $v_1, \ldots, v_k$ (in this order on $P$) be the vertices of $P$ at the ends of this fan. For each pair $v_j, v_{j+1}$, we already know that $v_j, v_{j+1} \in X_i$, but if one of these is also in $X_{i'}$ for some $i' \neq i$, then $P(v_j, v_{j+1})$ must be a large segment of $P$. This means that, for each vertex in $B$, there are at least $k - 1 - t$ corresponding small segments of $P$. Since the ends of these small segments corresponding to $b_i$ are all in $X_i$, these segments must then be disjoint from all small segments corresponding to $b_j$ for $j \neq i$ since the ends of those segments would be in $X_j$. Therefore there are $(k - 1 - t)\ell$ small segments *all* pairwise disjoint. This implies that the average order of small segments is at most

$$\frac{|P| - t\left(\frac{|P|-k+1}{k-1}\right) - k}{(k - 1 - t)\ell}.$$

By the pigeonhole principle, if we choose the shortest small segment corresponding to each vertex $b_i \in B$, then the sum of the orders of these shortest segments is at most

$$\frac{|P| - t\left(\frac{|P|-k+1}{k-1}\right) - k}{(k - 1 - t)} \leq \frac{|P| - k}{k - 1}.$$

We now replace each of these small segments with the corresponding $b_i$ using the paths $Q_{i,j}$ and $Q_{i,j+1}$ for the appropriate choice of $j$. This creates a new $u, v$-path $P'$ such that $H \subseteq B \subseteq P'$ and $|P \setminus P'| \leq \frac{|P|-k}{k-1}$. □

Before our next lemma, we observe an easy fact without proof.

**Fact 2** *Let $G$ be a $k$-connected graph for $k \geq 2$ and let $u$ and $v$ be two distinct vertices in $G$. Then for any $u, v$-path $P$ with $|P| < k$, there is another $u, v$-path $P'$ with $|P'| \geq k$ such that $P \subseteq P'$.*

**Lemma 3** *Let $G$ be a graph with $\kappa(G) = k$ and $\alpha(G) = \alpha$. If $u$, $v$ are two vertices in $G$, $\ell$ is an integer satisfying $0 \leq \ell \leq \alpha - k + 1$, then there exists a set of $u, v$-paths $P_0, \ldots, P_\ell$ satisfying:*

1. $\alpha\left(G \setminus \bigcup_{i=0}^{\ell} P_i\right) \leq \alpha - k + 1 - \ell$

2. $\left| P_i \setminus \bigcup_{j=0}^{j-1} P_j \right| \le \frac{|P_0|-k}{k-1}$ *for* $1 \le i \le \ell$

**Proof:** Induct on $\ell$. If $\ell = 0$, Theorem 2 gives a $u, v$-path $P_0$ with $\alpha(G \setminus P_0) \le \alpha - k + 1$. Now suppose we have $u, v$-paths $P_0, \ldots, P_{\ell-1}$ satisfying Properties 1 and 2 for $\ell - 1$.

Let $H = G \setminus \cup_{i=0}^{\ell-1} P_i$ be so that $\alpha(H) \le \alpha - k + 1 - (\ell - 1)$. Assume $\alpha(H) \ge 1$ since otherwise we could simply set $P_\ell = P_0$. By Lemma 2 with $P_0 = P$ (note that Fact 2 implies we may assume $|P_0| \ge k$), there is a $u, v$-path $P'$ such that $|P_0 \setminus P'| \le \frac{|P_0|-k}{k-1}$ and $\alpha(H \setminus P') \le \alpha(H) - 1 \le \alpha - k + 1 - \ell$.

**Case 1** $|P'| \le |P_0|$

Then $\left| P' \setminus \cup_{i=0}^{\ell-1} P_i \right| \le |P' \setminus P_0| \le |P_0 \setminus P'| \le \frac{|P_0|-k}{k-1}$, so we can set $P' = P_\ell$ to satisfy the desired properties.

**Case 2** $|P'| > |P_0|$

Relabel the paths as follows: $P'_0 = P'$ and $P'_i = P_{i-1}$ for $1 \le i \le \ell$. This new labelling gives $\alpha\left(G \setminus \cup_{i=0}^{\ell} P'_i\right) \le \alpha - k + 1 - \ell$ so Property 1 is satisfied. For Property 2, first consider the case $i = 1$. $|P'_i \setminus P'_0| = |P_0 \setminus P'| \le \frac{|P'|-k}{k-1}$ as desired. For $2 \le i \le \ell$, we have

$$\left| P'_i \setminus \bigcup_{j=0}^{i-1} P'_j \right| \le \left| P_{i-1} \setminus \bigcup_{j=0}^{i-2} P_j \right| \le \frac{|P_0| - k}{k-1} \le \frac{|P'_0| - k}{k-1}$$

so this labelling satisfies Properties 1 and 2, and we have our desired result. □

Using these lemmas, the proof of our main result is easy.

**Proof of Lemma 1:** For $k = 1$, the result is trivial so we will assume $k \ge 2$. When $k > \alpha$, the assertion holds by Theorem 7. Thus, we may also assume $\alpha \ge k$.

Set $\ell = \alpha - k + 1$ and apply Lemma 3. By Property 1, the set of paths $P_0, \ldots, P_\ell$ must cover all of $V(G)$. Using Property 2, this implies

$$n = |P_0| + \sum_{i=1}^{\ell} \left| P_i \setminus \bigcup_{j=0}^{i-1} P_j \right| \le |P_0| + (\alpha - k + 1) \left( \frac{|P_0| - k}{k-1} \right).$$

Solving for $|P_0|$, we get get the desired result $|P_0| \ge \frac{(k-1)(n-k)}{\alpha} + k$. □

**Proof of Theorem 4:** This proof is by induction on $s$. If $s = 2$, the result follows immediately from Lemma 1. Now suppose $s > 3$ and consider $G \setminus v_s$. This graph has $\kappa(G \setminus v_s) \ge k - 1$ and we will assume $\alpha(G \setminus v_s) = \alpha(G)$ (otherwise a stronger result is possible). By induction on $s$, there exists a tree $T_{s-1} \subseteq G$ with $\ell(T_{s-1}) = \{v_1, \ldots, v_{s-1}\}$ and

$$
\begin{aligned}
|T_{s-1}| &\geq \min\left\{n-1, \frac{(k-s+1)(n-k)}{\alpha}+k-1, \frac{(k-s+2)(n-k-1)}{\alpha}+k\right\} \\
&\geq \min\left\{n-1, \frac{(k-s+1)(n-k)}{\alpha}+k-1\right\}
\end{aligned}
$$

as long as $n \geq 2k+2-s-\alpha$. Otherwise, if we assume $n < 2k+2-s-\alpha$, then since $n \geq k+1$, if we let $H = G \setminus \{v_3, v_4, \ldots, v_s\}$, we have $\kappa(H) \geq \alpha + 1$. By Theorem 7, this means that $H$ is hamiltonian connected so we can find a path $P$ from $v_1$ to $v_2$ using all of $H$. Since $G$ is $k$-connected, each vertex $v_i$ for $3 \leq i \leq s$ has at least $k$ paths to $P$. Since $k \geq s$, there is an edge from each $v_i$ to $P \setminus \{v_1, v_2\}$, forming the desired tree of order $n$. Hence, we may suppose the above inequality holds.

In $G$, there are $k$ disjoint (except at $v_s$) paths from $v_s$ to $T_{s-1}$ so there is at least one such path $Q$ which avoids the set $\{v_1, \ldots, v_{s-1}\}$. Hence, the tree $T = T_{s-1} \cup Q$ is the desired tree with $|T| \geq |T_{s-1}| + 1$. □

# 4 Conclusion

The results contained in this work are all sharp by the following example. Let $C = K_k$ and let $H_i = K_{\frac{n-k}{\alpha}}$ for $1 \leq i \leq \alpha$ where we assume $\alpha$ divides $n-k$. Let $G = C + (\cup H_i)$ where $+$ is the standard join operation such that $V(A+B) = V(A) \cup V(B)$ and $E(A+B) = E(A) \cup E(B) \cup \{u, v : u \in A, v \in B\}$. Choose $u, v \in C$ and let $P$ be a $u,v$-path that uses all vertices of $C$ and all of $H_1, \ldots, H_{k-1}$. This is the longest $u,v$-path in $G$, which shows that Lemma 1 is sharp. The same example, with the inclusion of the edge $uv$ to complete a cycle, shows that Theorem 3 is sharp.

For Theorem 4, choose $v_1, \ldots, v_s$ from $C$ to obtain the desired bound. In this situation, because these vertices must be leaves of the constructed tree, we may use the vertices of at most $k-s+1$ components $H_i$ in building $T$. Note also that if $s > k$, a similar result cannot hold because, if we choose all of $C$ and at least one vertex of $G \setminus C$, at least one vertex of $C$ must not be a leaf of a tree including these vertices.

The authors hope that the results contained in this work may be applied in other works. Like Theorems 3 and 4 we believe that many results will follow from this work and perhaps other proofs may be simplified through use of Lemma 1.

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