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A NEW CLASS OF GENERALIZED POWER LINDLEY DISTRIBUTION WITH APPLICATION TO LIFETIME DATA

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Abstract. In this paper, a new class of generalized distribution called the Kumaraswamy Power Lindley (KPL) distribution is proposed and studied. This class of distributions contains the Kumaraswamy Lindley (KL), exponentiated power Lindley (EPL), power Lindley (PL), generalized or exponentiated Lindley (GL), and Lindley (L) distributions as special cases. Series expansion of the density is obtained. Statistical properties of this class of distributions, including hazard function, reverse hazard function, monotonicity property, shapes, moments, reliability, quantile function, mean deviations, Bonferroni and Lorenz curves, entropy and Fisher information are derived. Method of maximum likelihood is used to estimate the parameters of this new class of distributions. Finally, a real data example is discussed to illustrate the applicability of this class of distribution.

1. Introduction

Lindley [11] used a mixture of exponential and length-biased exponential distributions to illustrate the difference between fiducial and posterior distributions. This mixture is called the Lindley (L) distribution. Ghitany et al. [7] studied the statistical properties of the Lindley distribution. Sankaran [14] obtained and studied the Poisson-Lindley distribution. Jones [9] explored the background and genesis of the Kumaraswamy (Kum) distribution (Kumaraswamy [10]) and, more importantly, made clear some similarities and differences between the beta and Kum distributions. Among the advantages are: simple normalizing constant; the distribution and quantile functions have simple explicit formula which do not involve special functions; explicit formula for moments of order statistics and L-moments. However, compared to Kum distribution, the beta distribution has the following advantages: simpler formula for moments and moment generating function (mgf); a one-parameter subfamily of symmetric distributions; simpler moment estimation and more ways of generating the distribution via physical processes. Gupta and Kundu [8] provided a review and recent developments on the exponentiated exponential distribution. Cordeiro et al. [4] studied the Kumaraswamy Weibull (KW) distribution and applied it to failure time data.

Motivated by the advantages of the generalized or exponentiated Lindley distribution with respect to having a hazard function that exhibits increasing, decreasing and bathtub shapes,

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as well as the versatility and flexibility of the Kum distribution in modeling lifetime data, we propose and study a new class of distributions that inherit these very important and desirable properties, and also contains several sub-models with quite a number of shapes.

In this article, we propose a new distribution, called Kumaraswamy Power Lindley (KPL) distribution. We discuss some structural properties of this distribution, derive the Fisher information matrix and estimate the parameters via the method of maximum likelihood. In section 2, we present some generalized Lindley distributions including the power Lindley distribution, Kum distribution, Kum-G distribution, and the corresponding probability density functions (pdf). Section 3 contains results on the generalized and KPL distributions, including the hazard and reverse hazard functions, monotonicity property, and the sub-models. In section 4, we present the moment of the KPL distribution. Reliability and quantile function are given in sections 5 and 6, respectively. Mean deviations are presented in section 7. Section 8 contains results on Bonferroni and Lorenz curves. Measures of uncertainty, Fisher information and distribution of order statistics are presented in section 9. Maximum likelihood estimates of the model parameters and asymptotic confidence intervals are given in section 10. Section 11 contains an application of the proposed model to real data, followed by concluding remarks in section 12.

2. Some Generalized Lindley Distributions

In this section, some recent generalizations of the Lindley distribution are given. The Kumaraswamy-G distribution is also presented. A useful series representation is given below. For $|\omega| < 1$ and $b > 0$ a real non-integer, we have the series representation

$$(1-\omega)^{b-1} = \sum_{j=0}^{\infty}(-1)^j \binom{b-1}{j}\omega^j.$$

The one parameter cdf of the Lindley distribution [11] is given by

$$G_L(x;\lambda) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}, \quad \text{for } x > 0, \text{ and } \lambda > 0. \tag{1}$$

The corresponding Lindley pdf is given by

$$g_L(x;\lambda) = \frac{\lambda^2(1+x)}{1+\lambda}e^{-\lambda x}, \quad \text{for } x > 0, \text{ and } \lambda > 0. \tag{2}$$

Lindley distribution is a mixture of exponential and gamma distributions, that is $f(x;\lambda) = (1-p)f_G(x;\lambda) + pf_E(x;\lambda)$ with $p = \frac{\lambda}{1+\lambda}$, where $f_G(x;\lambda) \equiv GAM(2,\lambda)$, and $f_E(x;\lambda) \equiv EXP(\lambda)$. Now, let Y_1 and Y_2 be two independently gamma distributed random variables with parameters (α,λ) and $(\alpha+1,\lambda)$, respectively. For $\gamma > 0$, let $X = Y_1$ with probability $\frac{\lambda}{\lambda+\gamma}$ and $X = Y_2$ with probability $\frac{\gamma}{\lambda+\gamma}$, then the pdf of X (see Zakerzadeh and Dolati [16]) is given by

$$f_{GL}(x;\alpha,\lambda,\gamma) = \frac{\lambda^2(\lambda x)^{\alpha-1}(\alpha+\gamma x)e^{-\lambda x}}{(\lambda+\gamma)\Gamma(\alpha+1)},$$

for $x > 0$, $\lambda > 0$, $\alpha > 0$, $\gamma > 0$. Note that when $\alpha = \gamma = 1$, we obtain the Lindley pdf given in equation (2). When $\gamma = 0$ we have the gamma pdf with parameters α and λ. If $\alpha = 1$ and $\gamma = 0$ the resulting pdf is the exponential pdf with parameter λ.

2.1. **Generalized Lindley Distributions.** Nadarajah et al. [12] studied the mathematical and statistical properties of the generalized Lindley (GL) distribution. The cumulative distribution function (cdf) of the GL distribution is given by

$$G_{GL}(x;\alpha,\lambda) = \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\alpha}, \tag{3}$$

and the corresponding GL probability density function (pdf) is given by

$$g_{GL}(x;\alpha,\lambda) = \frac{\alpha\lambda^2}{1+\lambda}(1+x)\left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\alpha-1}\exp(-\lambda x), \tag{4}$$

for $x > 0$, $\lambda > 0$, $\alpha > 0$. This distribution is essentially the exponentiated Lindley distribution. Zakerzadeh and Dolati [16] presented and studied another generalization of the Lindley distribution. These generalizations of the Lindley distribution are considered to be useful life distributions and are suitable for modeling data with different types of hazard rate functions: increasing, decreasing, bathtub and unimodal. These models constitute flexible family of distributions in terms of the varieties of shapes and hazard functions.

2.2. **Power Lindley Distribution.** Ghitany et al. [6] presented results on a two-parameter Lindley distribution and referred to model as the power Lindley distribution. Considering the power transformation $X = T^{1/\alpha}$, the cdf and pdf of the power Lindley (PL) distribution are given by

$$G_{PL}(x;\alpha,\lambda) = 1 - \frac{1+\lambda+\lambda x^{\alpha}}{1+\lambda}e^{-\lambda x^{\alpha}}, \tag{5}$$

and

$$g_{PL}(x;\alpha,\lambda) = \frac{\alpha\lambda^2}{1+\lambda}(1+x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}, \tag{6}$$

for $x > 0$, $\alpha > 0$, $\lambda > 0$, respectively.

2.3. **Kumaraswamy-Generalized Distribution.** Kumaraswamy [10] in his paper proposed a two-parameter distribution (Kum distribution) defined in (0, 1). Its cdf and pdf are given by:

$$F(x;a,b) = 1 - (1-x^a)^b, \quad \text{and} \quad f(x;a,b) = abx^{a-1}(1-x^a)^{b-1},$$

respectively, for $x \in (0,1)$, $a > 0$, $b > 0$. The parameters a and b are the shape parameters. Let $G(x)$, be an arbitrary baseline cdf in the interval $(0,1)$. The Kum-G cdf $F(x;a,b)$ and pdf $f(x;a,b)$ are defined by

$$F(x;a,b) = 1 - (1-[G(x)]^a)^b, \tag{7}$$

and

$$f(x;a,b) = abg(x)[G(x)]^{a-1}(1-[G(x)]^a)^{b-1} \quad \text{for } a>0, b>0, \tag{8}$$

respectively, where $g(x) = \frac{dG(x)}{dx}$ is the pdf corresponding to the baseline cdf $G(x)$.

3. Kumaraswamy Power Lindley Distribution

Now, with the choice of $G(x)$ in the Kumaraswamy generalized distribution as the power Lindley distribution, we obtain the Kumaraswamy Power Lindley (KPL) distribution. The four-parameter KPL cdf and pdf are given by

$$F_{KPL}(x;\alpha,\lambda,a,b) = 1 - \left\{1 - \left[1 - \frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^a\right\}^b, \tag{9}$$

and

$$\begin{aligned} f_{KPL}(x;\alpha,\lambda,a,b) &= ab[G_{PL}(x)]^{a-1}[1-[G_{PL}(x)]^a]^{b-1}g_{PL}(x) \qquad (10)\\ &= \frac{ab\alpha\lambda^2}{1+\lambda}(1+x^\alpha)x^{\alpha-1}\exp(-\lambda x^\alpha)\\ &\times \left[1 - \frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^{a-1}\\ &\times \left\{1 - \left[1 - \frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^a\right\}^{b-1}, \qquad (11) \end{aligned}$$

for $x>0$, $\alpha>0$, $\lambda>0$, $a>0$, $b>0$, respectively. Figure 1 illustrates some possible shapes of the pdf of the KPL distribution.

3.1. **Expansion of Density.** In this section, the series expansion of the KPL pdf is presented. When $b>0$ is real non-integer, we use the series representation

$$[1-[G_{PL}(x)]^a]^{b-1} = \sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}[G_{PL}(x)]^{ai},$$

where $G_{PL}(x;\lambda) = 1 - \frac{1+\lambda+\lambda x^\alpha}{1+\lambda}e^{-\lambda x^\alpha}$. If a is an integer, from the above expansion and equation (10), we can write the KPL density as

$$f_{KPL}(x;\alpha,\lambda,a,b) = abg_{PL}(x)\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}[G_{PL}(x)]^{a(1+i)-1} \tag{12}$$

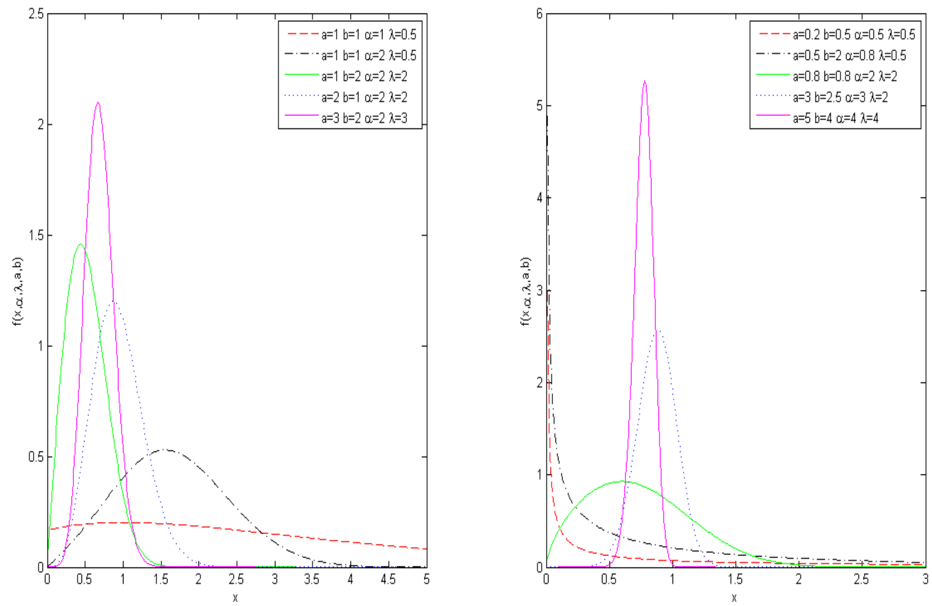

FIGURE 1. Plots of the pdf of KPL distribution for selected values of the parameters

$$
\begin{aligned}
&= ab\frac{\alpha\lambda^2}{1+\lambda}(1+x^\alpha)x^{\alpha-1}\exp(-\lambda x^\alpha) \\
&\times \sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\left[1-\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^{a(1+i)-1} \\
&= \frac{\alpha\lambda^2}{1+\lambda}(1+x^\alpha)x^{\alpha-1}\exp(-\lambda x^\alpha) \\
&\times \sum_{i=0}^{\infty}d_i\left[1-\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^{a(1+i)-1}, \qquad (13)
\end{aligned}
$$

where the coefficients d_i are

$$d_i = d_i(a,b) = (-1)^i ab\binom{b-1}{i},$$

and $\sum_{i=0}^{\infty} d_i = 1$, for $x>0$, $\alpha>0$, $\lambda>0$, $a>0$, $b>0$.

If a is real non-integer, we can expand $[G_{PL}(x)]^{a(1+i)-1}$ as follows

$$
\begin{aligned}
[G_{PL}(x)]^{a(1+i)-1} &= \{1-[1-G_{PL}(x)]\}^{a(1+i)-1} \\
&= \sum_{j=0}^{\infty}(-1)^j\binom{a(1+i)-1}{j}[1-G_{PL}(x)]^j,
\end{aligned}
$$

with

$$[1-G_{PL}(x)]^j = \sum_{r=0}^{j}(-1)^r\binom{j}{r}[G_{PL}(x)]^r,$$

so that

$$[G_{PL}(x)]^{a(1+i)-1} = \sum_{j=0}^{\infty}\sum_{r=0}^{j}(-1)^{j+r}\binom{a(1+i)-1}{j}\binom{j}{r}[G_{PL}(x)]^{r}. \tag{14}$$

From equations (12) and (14), the KPL density can be rearranged in the form

$$f_{KPL}(x;\alpha,\lambda,a,b) = g_{PL}(x)\sum_{i,j=0}^{\infty}\sum_{r=0}^{j} d_{i,j,r}[G_{PL}(x)]^{r} \tag{15}$$

$$\begin{aligned} &= \frac{\alpha\lambda^2}{1+\lambda}(1+x^{\alpha})x^{\alpha-1}\exp(-\lambda x^{\alpha})\sum_{i,j=0}^{\infty}\sum_{r=0}^{j} d_{i,j,r} \\ &\times \left[1-\frac{1+\lambda+\lambda x^{\alpha}}{1+\lambda}\exp(-\lambda x^{\alpha})\right]^{r}, \end{aligned} \tag{16}$$

where the coefficient $d_{i,j,r}$ is

$$d_{i,j,r} = d_{i,j,r}(a,b) = (-1)^{i+j+r}ab\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r},$$

and $\sum_{i,j=0}^{\infty}\sum_{r=0}^{j} d_{i,j,r} = 1$, for $x>0$, $\alpha>0$, $\lambda>0$, $a>0$, $b>0$. Hence, for any real non-integer a, the KPL density is given by three (two infinite and one finite) weighted power series sums of the baseline cdf $G_{PL}(x)$. By changing $\sum_{j=0}^{\infty}\sum_{r=0}^{j}$ to $\sum_{r=0}^{\infty}\sum_{j=r}^{\infty}$ in equation (16), we obtain

$$\begin{aligned} f_{KPL}(x;\alpha,\lambda,a,b) &= g_{PL}(x)\sum_{i,r=0}^{\infty} c_i[G_{PL}(x)]^{r} \\ &= \frac{\alpha\lambda^2}{1+\lambda}(1+x^{\alpha})x^{\alpha-1}\exp(-\lambda x^{\alpha}) \\ &\times \sum_{i,r=0}^{\infty} c_i\left[1-\frac{1+\lambda+\lambda x^{\alpha}}{1+\lambda}\exp(-\lambda x^{\alpha})\right]^{r}, \end{aligned} \tag{17}$$

where the coefficient c_i is

$$c_i = c_i(a,b) = (-1)^{i}ab\binom{b-1}{i}e_r(a(1+i)-1),$$

with

$$e_r = e_r(a(1+i)-1) = \sum_{j=r}^{\infty}(-1)^{j+r}\binom{a(1+i)-1}{j}\binom{j}{r},$$

for $x>0$, $\alpha>0$, $\lambda>0$, $a>0$, $b>0$, respectively. Note that the KPL density is given by three infinite weighted power series sums of the baseline distribution function $G_{PL}(x)$. When $b>0$ is an integer, the index i in the previous series representation stops at $b-1$.

3.2. **Some Sub-models of KPL Distribution.** In this section, we present the sub-models of KPL distribution for selected values of the parameters α, a, and b.

(1) $\boldsymbol{b=1}$

If $b=1$, this is the exponentiated power Lindley (EPL) distribution with cdf and pdf given by

$$F_{EPL}(x;\alpha,\lambda,a)=\left[1-\frac{1+\lambda+\lambda x^{\alpha}}{1+\lambda}\exp(-\lambda x^{\alpha})\right]^{a},$$

and

$$\begin{aligned} f_{EPL}(x;\alpha,\lambda,a) &= \frac{a\alpha\lambda^2}{1+\lambda}(1+x^{\alpha})x^{\alpha-1}\exp(-\lambda x^{\alpha}) \\ &\times \left[1-\frac{1+\lambda+\lambda x^{\alpha}}{1+\lambda}\exp(-\lambda x^{\alpha})\right]^{a-1}, \end{aligned}$$

for $x>0$, $\alpha>0$, $\lambda>0$, $a>0$, respectively.

(2) $\boldsymbol{a=1}$

If $a=1$, the KPL cdf and pdf reduce to:

$$F_{KPL}(x;\alpha,\lambda,b)=1-\left[\frac{1+\lambda+\lambda x^{\alpha}}{1+\lambda}\exp(-\lambda x^{\alpha})\right]^{b},$$

and

$$\begin{aligned} f_{KPL}(x;\alpha,\lambda,b) &= \frac{b\alpha\lambda^2}{1+\lambda}(1+x^{\alpha})x^{\alpha-1}\exp(-\lambda x^{\alpha}) \\ &\times \left[\frac{1+\lambda+\lambda x^{\alpha}}{1+\lambda}\exp(-\lambda x^{\alpha})\right]^{b-1}, \end{aligned}$$

for $x>0$, $\alpha>0$, $\lambda>0$, $b>0$, respectively.

(3) $\boldsymbol{a=b=1}$

If $a=b=1$, this is the power Lindley (PL) distribution given by equation (7).

(4) $\boldsymbol{\alpha=1}$

If $\alpha=1$, this is the Kum Lindley (KL) distribution with cdf and pdf given by

$$F_{KL}(x;\lambda,a,b)=1-\left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{a}\right\}^{b},$$

and

$$\begin{aligned} f_{KL}(x;\lambda,a,b) &= \frac{ab\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x) \\ &\times \left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{a-1} \\ &\times \left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{a}\right\}^{b-1}, \end{aligned}$$

for $x>0$, $\lambda>0$, $a>0$, $b>0$, respectively.

(5) $\boldsymbol{\alpha = a = 1}$

If $\alpha = a = 1$, then the KL cdf and pdf are given by

$$F_{KL}(x;\lambda,b) = 1 - \left[\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^b,$$

and

$$f_{KL}(x;\lambda,b) \quad = \quad \frac{b\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x)\left[\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{b-1},$$

for $x > 0$, $\lambda > 0$, $b > 0$, respectively.

(6) $\boldsymbol{\alpha = b = 1}$

If $\alpha = b = 1$, this is the generalized Lindley (GL) or exponentiated Lindley (EL) distribution (Nadarajah et al. [12]) given by equation (1).

(7) $\boldsymbol{\alpha = a = b = 1}$

If $\alpha = a = b = 1$, this is the Lindley (L) distribution (Lindley [11]). The Lindley cdf and pdf are given by

$$F_L(x;\lambda) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x),$$

and

$$f_L(x;\lambda) \quad = \quad \frac{\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x),$$

for $x > 0$, $\lambda > 0$, respectively.

(8) $\boldsymbol{a = b = 1, \alpha = 2}$

If $a = b = 1, \alpha = 2$, then the KPL cdf and pdf reduce to:

$$F_{PL}(x;\lambda) = 1 - \frac{1+\lambda+\lambda x^2}{1+\lambda}e^{-\lambda x^2},$$

and

$$f_{PL}(x;\lambda) = \frac{2\lambda^2}{1+\lambda}(1+x^2)xe^{-\lambda x^2},$$

for $x > 0$, $\lambda > 0$, respectively.

(9) $\boldsymbol{a = 1, \alpha = 2}$

If $a = 1, \alpha = 2$, then the KPL cdf and pdf reduces to:

$$F_{KPL}(x;\lambda,b) = 1 - \left[\frac{1+\lambda+\lambda x^2}{1+\lambda}\exp(-\lambda x^2)\right]^b,$$

and

$$\begin{aligned} f_{KPL}(x;\lambda,b) \quad &= \quad \frac{2b\lambda^2}{1+\lambda}(1+x^2)x\exp(-\lambda x^2) \\ &\times \quad \left[\frac{1+\lambda+\lambda x^2}{1+\lambda}\exp(-\lambda x^2)\right]^{b-1}, \end{aligned}$$

for $x > 0$, $\lambda > 0$, $b > 0$, respectively.

(10) $\boldsymbol{b = 1, \alpha = 2}$
If $b = 1, \alpha = 2$, then the EPL cdf and pdf are given by

$$F_{EPL}(x; \lambda, a) = \left[1 - \frac{1 + \lambda + \lambda x^2}{1 + \lambda} \exp(-\lambda x^2)\right]^a,$$

and

$$\begin{aligned} f_{EPL}(x; \lambda, a) &= \frac{2a\lambda^2}{1+\lambda}(1 + x^2)x \exp(-\lambda x^2) \\ &\times \left[1 - \frac{1 + \lambda + \lambda x^2}{1 + \lambda} \exp(-\lambda x^2)\right]^{a-1}, \end{aligned}$$

for $x > 0$, $\lambda > 0$, $a > 0$, respectively.

3.3. **Hazard and Reverse Hazard Functions.** In this section, the hazard and reverse hazard functions of the KPL distribution are presented. Graphs of these functions for selected values of the parameters α, λ, a, and b are also presented. The hazard and reverse hazard functions of the KPL distribution are given by

$$\begin{aligned} h_{KPL}(x; \alpha, \lambda, a, b) &= \frac{f_{KPL}(x; \alpha, \lambda, a, b)}{\bar{F}_{KPL}(x; \alpha, \lambda, a, b)} \\ &= \frac{ab\alpha\lambda^2}{1+\lambda}(1 + x^\alpha)x^{\alpha-1} \exp(-\lambda x^\alpha) \\ &\times \left[1 - \frac{1 + \lambda + \lambda x^\alpha}{1 + \lambda} \exp(-\lambda x^\alpha)\right]^{a-1} \\ &\times \left\{1 - \left[1 - \frac{1 + \lambda + \lambda x^\alpha}{1 + \lambda} \exp(-\lambda x^\alpha)\right]^a\right\}^{-1}, \end{aligned}$$

and

$$\begin{aligned} \tau_{KPL}(x; \alpha, \lambda, a, b) &= \frac{f_{KPL}(x; \alpha, \lambda, a, b)}{F_{KPL}(x; \alpha, \lambda, a, b)} \\ &= \frac{ab\alpha\lambda^2}{1+\lambda}(1 + x^\alpha)x^{\alpha-1} \exp(-\lambda x^\alpha) \\ &\times \left[1 - \frac{1 + \lambda + \lambda x^\alpha}{1 + \lambda} \exp(-\lambda x^\alpha)\right]^{a-1} \\ &\times \left\{1 - \left[1 - \frac{1 + \lambda + \lambda x^\alpha}{1 + \lambda} \exp(-\lambda x^\alpha)\right]^a\right\}^{b-1} \\ &\times \left\{1 - \left\{1 - \left[1 - \frac{1 + \lambda + \lambda x^\alpha}{1 + \lambda} \exp(-\lambda x^\alpha)\right]^a\right\}^b\right\}^{-1}, \end{aligned}$$

for $x > 0$, $\alpha > 0$, $\lambda > 0$, $a > 0$, $b > 0$, respectively. The graphs of hazard function of KPL distribution are shown in Figure 2. These graphs show the variety of shapes for the KPL hazard function including bathtub, upside down bathtub, deceasing, and increasing hazard rate functions. This attractive flexibility makes the KPL hazard rate function useful and desirable for non-monotonic empirical hazard behaviors that are more likely to be encountered in real life or practice.

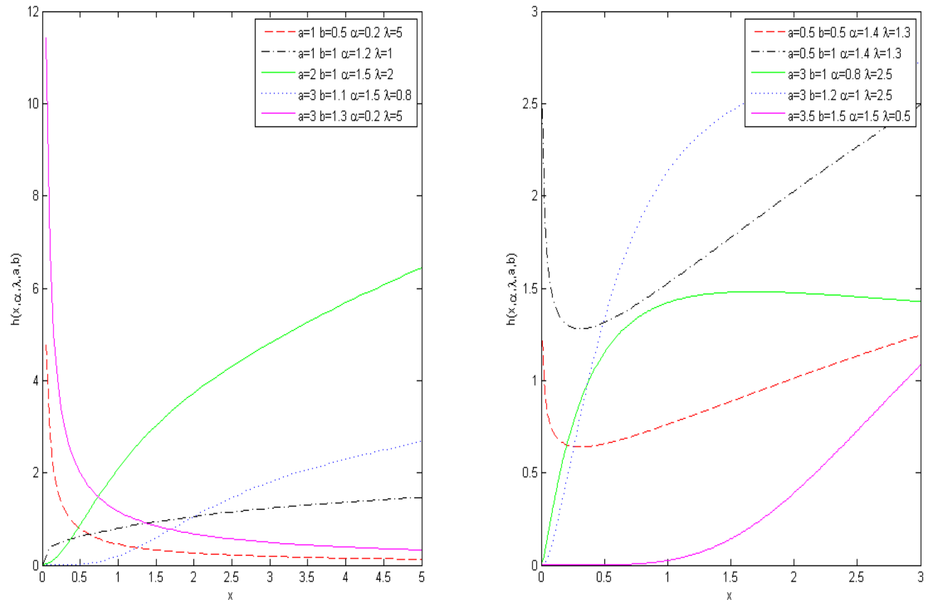

FIGURE 2. Plots of the hazard function of KPL distribution for selected values of the parameters

3.4. **Monotonicity Property.** In this section, we discuss the monotonicity properties of the KPL distribution. Let

$$W(x) = G_{PL}(x;\alpha,\lambda) = 1 - \frac{1+\lambda+\lambda x^{\alpha}}{1+\lambda}\exp(-\lambda x^{\alpha}), \tag{18}$$

then from equation (11), we can rewrite KPL pdf as

$$\begin{aligned} f_{KPL}(x;\alpha,\lambda,a,b) &= \frac{ab\alpha\lambda^2}{1+\lambda}(1+x^{\alpha})x^{\alpha-1}\exp(-\lambda x^{\alpha}) \\ &\times [W(x)]^{a-1}\left\{1-[W(x)]^{a}\right\}^{b-1}, \end{aligned}$$

for $x > 0$, $\alpha > 0$, $\lambda > 0$, $a > 0$, $b > 0$. It follows that

$$\begin{aligned} \log\, f_{KPL}(x) &= \log\left(\frac{ab\alpha\lambda^2}{1+\lambda}\right) + \log(1+x^{\alpha}) + (\alpha-1)\log\, x - \lambda x^{\alpha} \\ &+ (a-1)\log W(x) + (b-1)\log\left[1-[W(x)]^{a}\right], \end{aligned} \tag{19}$$

and

$$\begin{aligned}
\frac{\mathrm{d}\,\log\, f_{KPL}(x)}{\mathrm{d}x} &= \frac{\alpha x^{\alpha-1}}{1+x^{\alpha}} + \frac{\alpha-1}{x} - \lambda\alpha x^{\alpha-1} \\
&+ (a-1)\frac{W'(x)}{W(x)} + a(1-b)\frac{[W(x)]^{a-1}W'(x)}{1-[W(x)]^{a}} \\
&= \frac{(1-\lambda)\alpha x^{\alpha-1} - \lambda\alpha x^{2\alpha-1}}{1+x^{\alpha}} + \frac{\alpha-1}{x} \\
&+ (a-1)\frac{W'(x)}{W(x)} + a(1-b)\frac{[W(x)]^{a-1}W'(x)}{1-[W(x)]^{a}}, \qquad (20)
\end{aligned}$$

where $W'(x) = \frac{\mathrm{d}\,W(x)}{\mathrm{d}x} = \frac{\lambda^2\alpha}{1+\lambda}(1+x^{\alpha})x^{\alpha-1}\exp(-\lambda x^{\alpha})$.

Analysis: We know that $x > 0$, $\alpha > 0$, $\lambda > 0$, $a > 0$, and $b > 0$, so that

$$W'(x) = \frac{\mathrm{d}\,W(x)}{\mathrm{d}x} = \frac{\lambda^2\alpha}{1+\lambda}(1+x^{\alpha})x^{\alpha-1}\exp(-\lambda x^{\alpha}) > 0, \forall\, x > 0.$$

If $x \to 0$,

$$W(x) = 1 - \frac{1+\lambda+\lambda x^{\alpha}}{1+\lambda}\exp(-\lambda x^{\alpha}) \to 0.$$

If $x \to \infty$,

$$W(x) = 1 - \frac{1+\lambda+\lambda x^{\alpha}}{1+\lambda}\exp(-\lambda x^{\alpha}) \to 1,$$

since

$$\lim_{x\to\infty}\frac{1+\lambda+\lambda x^{\alpha}}{(1+\lambda)\exp(\lambda x^{\alpha})} = \lim_{x\to\infty}\frac{\lambda\alpha x^{\alpha-1}}{(1+\lambda)\exp(\lambda x^{\alpha})(\lambda\alpha x^{\alpha-1})} = 0.$$

Thus, $W(x)$ is monotonically increasing from 0 to 1. Now, since $0 < W(x) < 1$, $0 < [W(x)]^{a} < 1, \forall\, a > 0$, $[W(x)]^{a-1} > 0, \forall\, a > 0$, $0 < 1-[W(x)]^{a} < 1, \forall\, a > 0$, and $W'(x) > 0$. Then we have $\frac{W'(x)}{W(x)} > 0$, and $\frac{[W(x)]^{a-1}W'(x)}{1-[W(x)]^{a}} > 0$. Also, from $x > 0$, we have $x^{\alpha-1} > 0$ and $x^{2\alpha-1} > 0$.

If $\lambda > 1$, $\alpha < 1$, $a < 1$, and $b > 1$, we get $\frac{\mathrm{d}\,\log\, f_{KPL}(x)}{\mathrm{d}x} < 0$, since $(1-\lambda)\alpha x^{\alpha-1} - \lambda\alpha x^{2\alpha-1} < 0$. In this case, $f_{KPL}(x;\alpha,\lambda,a,b)$ is monotonically decreasing for all x.

If $\lambda \leq 1$, $f_{KPL}(x;\alpha,\lambda,a,b)$ could attain a maximum, a minimum or a point of inflection according to whether

$$\frac{\mathrm{d}^2\,\log\, f_{KPL}(x)}{\mathrm{d}x^2} < 0, \frac{\mathrm{d}^2\,\log\, f_{KPL}(x)}{\mathrm{d}x^2} > 0, \text{or } \frac{\mathrm{d}^2\,\log\, f_{KPL}(x)}{\mathrm{d}x^2} = 0,$$

respectively.

3.5. **Shape of Hazard Function.** Note that if $x \to \infty$, then $\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha) \to 0$. Also,

$$\begin{aligned}\left[1-\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^{a-1} &= \sum_{i=0}^{\infty}\binom{a-1}{i} \\ &\times \left[-\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^{i} \\ &\approx 1-(a-1)\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha).\end{aligned}$$

Consequently,

$$f_{KPL}(x;\alpha,\lambda,a,b) \sim \frac{a^b b\alpha\lambda^{b+1}}{(1+\lambda)^b}x^{(b+1)\alpha-1}\exp(-\lambda b x^\alpha). \tag{21}$$

If $x \to 0$, then

$$f_{KPL}(x;\alpha,\lambda,a,b) \sim \frac{ab\alpha\lambda^{2a}}{(1+\lambda)^a}x^{(a+1)\alpha-1}. \tag{22}$$

The cdf of KPL distribution is

$$F_{KPL}(x;\alpha,\lambda,a,b) = 1-\left\{1-\left[1-\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^a\right\}^b,$$

for $x > 0$, $\alpha > 0$, $\lambda > 0$, $a > 0$, $b > 0$.

If $x \to \infty$, then $\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha) \to 0$. Also,

$$\begin{aligned}\left[1-\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^{a} &= \sum_{i=0}^{\infty}\binom{a}{i}\left[-\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^{i} \\ &\approx 1-a\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha),\end{aligned}$$

so that

$$F_{KPL}(x;\alpha,\lambda,a,b) \approx 1-\left[a\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^b.$$

Also,

$$\begin{aligned}1-F_{KPL}(x;\alpha,\lambda,a,b) &\approx \left[a\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^b \\ &= \frac{a^b}{(1+\lambda)^b}(1+\lambda+\lambda x^\alpha)^b\exp(-\lambda b x^\alpha) \\ &\sim \frac{a^b\lambda^b}{(1+\lambda)^b}x^{b\alpha}\exp(-\lambda b x^\alpha). \qquad (23)\end{aligned}$$

If $x \to 0$, then $\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha) \to 1$, and $1-\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha) \to 0$, so that

$$\left[1-\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^a \to 0.$$

Thus,

$$
\begin{aligned}
F_{KPL}(x;\alpha,\lambda,a,b) &\approx 1-\left\{1-b\left[1-\frac{1+\lambda+\lambda x^{\alpha}}{1+\lambda}\exp(-\lambda x^{\alpha})\right]^{a}\right\} \\
&= b\left[1-\frac{1+\lambda+\lambda x^{\alpha}}{1+\lambda}\exp(-\lambda x^{\alpha})\right]^{a} \\
&\approx b\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}(1-\lambda x^{\alpha})\right]^{a} \\
&= b\left(\frac{\lambda^2 x^{\alpha}+\lambda^2 x^{2\alpha}}{1+\lambda}\right)^{a} \\
&\sim \frac{b\lambda^{2a}}{(1+\lambda)^a}x^{a\alpha}. \qquad (24)
\end{aligned}
$$

The hazard functions of KPL is given by

$$
h_{KPL}(x;\alpha,\lambda,a,b) = \frac{f_{KPL}(x;\alpha,\lambda,a,b)}{\bar{F}_{KPL}(x;\alpha,\lambda,a,b)} = \frac{f_{KPL}(x;\alpha,\lambda,a,b)}{1-F_{KPL}(x;\alpha,\lambda,a,b)}, \qquad (25)
$$

for $x>0$, $\alpha>0$, $\lambda>0$, $a>0$, $b>0$. If $x\to\infty$, with equations (21) and (23) in equation (25), we get

$$
\begin{aligned}
h_{KPL}(x;\alpha,\lambda,a,b) &\sim \frac{a^b b\alpha\lambda^{b+1}x^{(b+1)\alpha-1}\exp(-\lambda b x^{\alpha})/(1+\lambda)^b}{a^b\lambda^b x^{b\alpha}\exp(-\lambda b x^{\alpha})/(1+\lambda)^b} \\
&= b\alpha\lambda x^{\alpha-1}.
\end{aligned}
$$

If $x\to 0$, with equations (22) and (24) in equation (25), we get

$$
\begin{aligned}
h_{KPL}(x;\alpha,\lambda,a,b) &\sim \frac{ab\alpha\lambda^{2a}x^{(a+1)\alpha-1}/(1+\lambda)^a}{1-b\lambda^{2a}x^{a\alpha}/(1+\lambda)^a} \\
&\sim \frac{ab\alpha\lambda^{2a}}{(1+\lambda)^a}x^{(a+1)\alpha-1},
\end{aligned}
$$

since $\frac{b\lambda^{2a}}{(1+\lambda)^a}x^{a\alpha}\to 0$, as $x\to 0$.

4. Moments of KPL Distribution

In this section, moments of the KPL distribution are presented. The following lemma is proved by using the result given by Nadarajah et al. [12].

Lemma 1

Let

$$
K(m,n,p,q) = \int_0^{\infty} x^p(1+x)\left[1-\frac{1+n+nx}{1+n}\exp(-nx)\right]^{m-1}\exp(-qx)\,\mathrm{d}x.
$$

1. If m is non-integer, we have

$$
K(m,n,p,q) = \sum_{l=0}^{\infty}\sum_{k=0}^{l}\sum_{w=0}^{k+1}\binom{m-1}{l}\binom{l}{k}\binom{k+1}{w}\frac{(-1)^l n^k\Gamma(p+w+1)}{(1+n)^l(nl+q)^{p+w+1}}.
$$

2. If m is an integer, we have

$$K(m,n,p,q) = \sum_{l=0}^{m-1}\sum_{k=0}^{l}\sum_{w=0}^{k+1}\binom{m-1}{l}\binom{l}{k}\binom{k+1}{w}\frac{(-1)^l n^k\Gamma(p+w+1)}{(1+n)^l(nl+q)^{p+w+1}}.$$

Proof. (1) If m is non-integer, then

$$\left[1-\frac{1+n+nx}{1+n}\exp(-nx)\right]^{m-1} = \sum_{l=0}^{\infty}\binom{m-1}{l}(-1)^l \times \left(\frac{1+n+nx}{1+n}\exp(-nx)\right)^l,$$

and

$$K(m,n,p,q) = \sum_{l=0}^{\infty}\binom{m-1}{l}\frac{(-1)^l}{(1+n)^l} \times \int_0^\infty x^p(1+x)(1+n+nx)^l\exp[-(nl+q)x]\,\mathrm{d}x.$$

Furthermore, l is an integer, so that

$$(1+n+nx)^l = \sum_{k=0}^{l}\binom{l}{k}(n+nx)^k = \sum_{k=0}^{l}\binom{l}{k}n^k(1+x)^k,$$

and

$$K(m,n,p,q) = \sum_{l=0}^{\infty}\binom{m-1}{l}\frac{(-1)^l}{(1+n)^l}\sum_{k=0}^{l}\binom{l}{k}n^k \times \int_0^\infty x^p(1+x)^{k+1}\exp[-(nl+q)x]\,\mathrm{d}x.$$

Now, k is an integer, so that

$$(1+x)^{k+1} = \sum_{w=0}^{k+1}\binom{k+1}{w}x^w,$$

and

$$\begin{aligned} K(m,n,p,q) &= \sum_{l=0}^{\infty}\binom{m-1}{l}\frac{(-1)^l}{(1+n)^l}\sum_{k=0}^{l}\binom{l}{k}n^k\sum_{w=0}^{k+1}\binom{k+1}{w} \\ &\times \int_0^\infty x^{p+w}\exp[-(nl+q)x]\,\mathrm{d}x \\ &= \sum_{l=0}^{\infty}\sum_{k=0}^{l}\sum_{w=0}^{k+1}\binom{m-1}{l}\binom{l}{k}\binom{k+1}{w}\frac{(-1)^l n^k\Gamma(p+w+1)}{(1+n)^l(nl+q)^{p+w+1}}. \end{aligned}$$

(26)

(2) If m is an integer, the index l in equation (26) stops at $m-1$, so that

$$K(m,n,p,q) = \sum_{l=0}^{m-1}\sum_{k=0}^{l}\sum_{w=0}^{k+1}\binom{m-1}{l}\binom{l}{k}\binom{k+1}{w}\frac{(-1)^l n^k\Gamma(p+w+1)}{(1+n)^l(nl+q)^{p+w+1}}.$$

□

The s^{th} moment of the KPL distribution, say μ'_s, is given by

$$\mu'_s = \int_0^\infty x^s f_{KPL}(x;\alpha,\lambda,a,b)\,\mathrm{d}x.$$

Let $b>0$ and $a>0$ be real non-integer, then from equation (16), we obtain

$$\begin{aligned}\mu'_s &= \frac{ab\alpha\lambda^2}{1+\lambda}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}(-1)^{i+j+r}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r}\\ &\times \int_0^\infty(1+x^\alpha)x^{s+\alpha-1}\exp(-\lambda x^\alpha)\left[1-\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^r\mathrm{d}x\\ &= \frac{ab\lambda^2}{1+\lambda}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}(-1)^{i+j+r}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r}\\ &\times \int_0^\infty(1+x^\alpha)x^{s}\exp(-\lambda x^\alpha)\left[1-\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^r\mathrm{d}x^\alpha.\end{aligned}\tag{27}$$

Let $y=x^\alpha$, then equation (27) changes to

$$\begin{aligned}\mu'_s &= \frac{ab\lambda^2}{1+\lambda}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}(-1)^{i+j+r}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r}\\ &\times \int_0^\infty(1+y)y^{\frac{s}{\alpha}}\exp(-\lambda y)\left[1-\frac{1+\lambda+\lambda y}{1+\lambda}\exp(-\lambda y)\right]^r\mathrm{d}y.\end{aligned}$$

Applying Lemma 1, with $m=r+1$, $n=\lambda$, $p=\frac{s}{\alpha}$, $q=\lambda$, we have

$$\begin{aligned}\mu'_s &= \frac{ab\lambda^2}{1+\lambda}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}(-1)^{i+j+r}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r}\\ &\times K\left(r+1,\lambda,\frac{s}{\alpha},\lambda\right).\end{aligned}\tag{28}$$

As r is integer, then $r+1$ is integer, then the s^{th} moment of the KPL is given by

$$\begin{aligned}\mu'_s &= ab\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}\sum_{l=0}^{r}\sum_{k=0}^{l}\sum_{w=0}^{k+1}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r}\binom{r}{l}\binom{l}{k}\binom{k+1}{w}\\ &\times \frac{(-1)^{i+j+r+l}\Gamma(\frac{s}{\alpha}+w+1)}{(1+\lambda)^{l+1}\lambda^{\frac{s}{\alpha}+w-k-1}(1+l)^{\frac{s}{\alpha}+w+1}}.\end{aligned}\tag{29}$$

Note here that we have considered the case when $b>0$ and $a>0$ are non-integer, however the other cases can be similarly derived.

5. Reliability

In reliability and related areas, the stress-strength model describes the life of a component with random strength X, that is subjected to a random stress Y. The component will fail at the instant that the applied stress exceeds the strength, and the component will function satisfactorily whenever $X > Y$. We derive $R = P(X > Y)$, a measure of component reliability, when X and Y have independent $KPL(\alpha_1, \lambda_1, a_1, b_1)$ and $KPL(\alpha_2, \lambda_2, a_2, b_2)$ distributions, respectively. Note from equations (9) and (11) that

$$\begin{aligned} R &= P(X > Y) \\ &= \int_0^\infty f_X(x; \alpha_1, \lambda_1, a_1, b_1) F_Y(x; \alpha_2, \lambda_2, a_2, b_2)\,\mathrm{d}x \\ &= \int_0^\infty \frac{a_1 b_1 \alpha_1 \lambda_1^2}{1+\lambda_1}(1+x^{\alpha_1})x^{\alpha_1-1}\exp(-\lambda_1 x^{\alpha_1}) \\ &\times \left[1 - \frac{1+\lambda_1+\lambda_1 x^{\alpha_1}}{1+\lambda_1}\exp(-\lambda_1 x^{\alpha_1})\right]^{a_1-1} \\ &\times \left\{1 - \left[1 - \frac{1+\lambda_1+\lambda_1 x^{\alpha_1}}{1+\lambda_1}\exp(-\lambda_1 x^{\alpha_1})\right]^{a_1}\right\}^{b_1-1} \mathrm{d}x \\ &- \int_0^\infty \frac{a_1 b_1 \alpha_1 \lambda_1^2}{1+\lambda_1}(1+x^{\alpha_1})x^{\alpha_1-1}\exp(-\lambda_1 x^{\alpha_1}) \\ &\times \left[1 - \frac{1+\lambda_1+\lambda_1 x^{\alpha_1}}{1+\lambda_1}\exp(-\lambda_1 x^{\alpha_1})\right]^{a_1-1} \\ &\times \left\{1 - \left[1 - \frac{1+\lambda_1+\lambda_1 x^{\alpha_1}}{1+\lambda_1}\exp(-\lambda_1 x^{\alpha_1})\right]^{a_1}\right\}^{b_1-1} \\ &\times \left\{1 - \left[1 - \frac{1+\lambda_2+\lambda_2 x^{\alpha_2}}{1+\lambda_2}\exp(-\lambda_2 x^{\alpha_2})\right]^{a_2}\right\}^{b_2} \mathrm{d}x. \end{aligned} \tag{30}$$

Applying the series expansions

$$\begin{aligned} \left[1 - \frac{1+\lambda_1+\lambda_1 x^{\alpha_1}}{1+\lambda_1}\exp(-\lambda_1 x^{\alpha_1})\right]^{a_1-1} &= \sum_{k=0}^\infty \sum_{m=0}^k \binom{a_1-1}{k}\binom{k}{m} \\ &\times \frac{(-1)^k \lambda_1^m x^{m\alpha_1}\exp(-\lambda_1 k x^{\alpha_1})}{(1+\lambda_1)^m}, \end{aligned} \tag{31}$$

$$\begin{aligned} \left\{1 - \left[1 - \frac{1+\lambda_1+\lambda_1 x^{\alpha_1}}{1+\lambda_1}\exp(-\lambda_1 x^{\alpha_1})\right]^{a_1}\right\}^{b_1-1} &= \sum_{l,p=0}^\infty \sum_{n=0}^p \binom{b_1-1}{l}\binom{a_1 l}{p}\binom{p}{n} \\ &\times \frac{(-1)^{l+p}\lambda_1^n x^{n\alpha_1}\exp(-\lambda_1 p x^{\alpha_1})}{(1+\lambda_1)^n}, \end{aligned} \tag{32}$$

$$\left\{1-\left[1-\frac{1+\lambda_2+\lambda_2 x^{\alpha_2}}{1+\lambda_2}\exp(-\lambda_2 x^{\alpha_2})\right]^{a_2}\right\}^{b_2}$$

$$= \sum_{q,t=0}^{\infty}\sum_{h=0}^{t}\binom{b_2}{q}\binom{a_2 q}{t}\binom{t}{h}\frac{(-1)^{q+t}\lambda_2^h x^{h\alpha_2}\exp(-\lambda_2 t x^{\alpha_2})}{(1+\lambda_2)^h}, \tag{33}$$

and substituting equations (31), (32), and (33) into equation (30), we get

$$\begin{aligned}
R &= a_1 b_1 \alpha_1 \sum_{k,l,p=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{p}\binom{a_1-1}{k}\binom{k}{m}\binom{b_1-1}{l}\binom{a_1 l}{p}\binom{p}{n} \\
&\times \frac{(-1)^{k+l+p}\lambda_1^{m+n+2}}{(1+\lambda_1)^{m+n+1}}\int_0^{\infty}(1+x^{\alpha_1})x^{(m+n+1)\alpha_1-1}\exp(-\lambda_1(k+p+1)x^{\alpha_1})\,\mathrm{d}x \\
&- a_1 b_1 \alpha_1 \sum_{k,l,p,q,t=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{p}\sum_{h=0}^{t}\binom{a_1-1}{k}\binom{k}{m}\binom{b_1-1}{l}\binom{a_1 l}{p}\binom{p}{n}\binom{b_2}{q} \\
&\times \binom{a_2 q}{t}\binom{t}{h}\frac{(-1)^{k+l+p+q+t}\lambda_1^{m+n+2}\lambda_2^h}{(1+\lambda_1)^{m+n+1}(1+\lambda_2)^h} \\
&\times \int_0^{\infty}(1+x^{\alpha_1})x^{(m+n+1)\alpha_1+h\alpha_2-1}\exp(-\lambda_1(k+p+1)x^{\alpha_1}-\lambda_2 t x^{\alpha_2})\,\mathrm{d}x.
\end{aligned} \tag{34}$$

Note that

$$\begin{aligned}
&\int_0^{\infty}(1+x^{\alpha_1})x^{(m+n+1)\alpha_1-1}\exp(-\lambda_1(k+p+1)x^{\alpha_1})\,\mathrm{d}x \\
&= \frac{1}{\alpha_1}\int_0^{\infty}x^{(m+n)\alpha_1}\exp(-\lambda_1(k+p+1)x^{\alpha_1})\,\mathrm{d}x^{\alpha_1} \\
&+ \frac{1}{\alpha_1}\int_0^{\infty}x^{(m+n+1)\alpha_1}\exp(-\lambda_1(k+p+1)x^{\alpha_1})\,\mathrm{d}x^{\alpha_1} \\
&= \frac{1}{\alpha_1}\int_0^{\infty}y^{m+n}\exp(-\lambda_1(k+p+1)y)\,\mathrm{d}y \\
&+ \frac{1}{\alpha_1}\int_0^{\infty}y^{m+n+1}\exp(-\lambda_1(k+p+1)y)\,\mathrm{d}y \\
&= \frac{\Gamma(m+n+1)}{\alpha_1[\lambda_1(1+k+p)]^{m+n+1}}+\frac{\Gamma(m+n+2)}{\alpha_1[\lambda_1(1+k+p)]^{m+n+2}} \\
&= \frac{(m+n)!}{\alpha_1[\lambda_1(1+k+p)]^{m+n+1}}\left[1+\frac{m+n+1}{\lambda_1(1+k+p)}\right],
\end{aligned} \tag{35}$$

where we let $y = x^{\alpha_1}$. Also, by using $\exp(-\lambda_2 t x^{\alpha_2}) = \sum_{i=0}^{\infty} \frac{(-1)^i \lambda_2^i t^i x^{i\alpha_2}}{i!}$, we get

$$
\begin{aligned}
& \int_0^\infty (1 + x^{\alpha_1}) x^{(m+n+1)\alpha_1 + h\alpha_2 - 1} \exp(-\lambda_1(k+p+1)x^{\alpha_1} - \lambda_2 t x^{\alpha_2})\,\mathrm{d}x \\
= & \int_0^\infty x^{(m+n+1)\alpha_1 + h\alpha_2 - 1} \exp(-\lambda_1(k+p+1)x^{\alpha_1} - \lambda_2 t x^{\alpha_2})\,\mathrm{d}x \\
+ & \int_0^\infty x^{(m+n+2)\alpha_1 + h\alpha_2 - 1} \exp(-\lambda_1(k+p+1)x^{\alpha_1} - \lambda_2 t x^{\alpha_2})\,\mathrm{d}x \\
= & \sum_{i=0}^{\infty} \frac{(-1)^i \lambda_2^i t^i}{i!} \int_0^\infty x^{(m+n+1)\alpha_1 + (h+i)\alpha_2 - 1} \exp(-\lambda_1(k+p+1)x^{\alpha_1})\,\mathrm{d}x \\
+ & \sum_{i=0}^{\infty} \frac{(-1)^i \lambda_2^i t^i}{i!} \int_0^\infty x^{(m+n+2)\alpha_1 + (h+i)\alpha_2 - 1} \exp(-\lambda_1(k+p+1)x^{\alpha_1})\,\mathrm{d}x \\
= & \sum_{i=0}^{\infty} \frac{(-1)^i \lambda_2^i t^i}{i!\alpha_1} \int_0^\infty x^{(m+n)\alpha_1 + (h+i)\alpha_2} \exp(-\lambda_1(k+p+1)x^{\alpha_1})\,\mathrm{d}x^{\alpha_1} \\
+ & \sum_{i=0}^{\infty} \frac{(-1)^i \lambda_2^i t^i}{i!\alpha_1} \int_0^\infty x^{(m+n+1)\alpha_1 + (h+i)\alpha_2} \exp(-\lambda_1(k+p+1)x^{\alpha_1})\,\mathrm{d}x^{\alpha_1} \\
= & \sum_{i=0}^{\infty} \frac{(-1)^i \lambda_2^i t^i}{i!\alpha_1} \int_0^\infty y^{m+n+(h+i)\frac{\alpha_2}{\alpha_1}} \exp(-\lambda_1(k+p+1)y)\,\mathrm{d}y \\
+ & \sum_{i=0}^{\infty} \frac{(-1)^i \lambda_2^i t^i}{i!\alpha_1} \int_0^\infty y^{m+n+1+(h+i)\frac{\alpha_2}{\alpha_1}} \exp(-\lambda_1(k+p+1)y)\,\mathrm{d}y \\
= & \sum_{i=0}^{\infty} \frac{(-1)^i \lambda_2^i t^i}{i!\alpha_1} \\
\times & \left\{ \frac{\Gamma(m+n+1+(h+i)\frac{\alpha_2}{\alpha_1})}{[\lambda_1(1+k+p)]^{m+n+1+(h+i)\frac{\alpha_2}{\alpha_1}}} + \frac{\Gamma(m+n+2+(h+i)\frac{\alpha_2}{\alpha_1})}{[\lambda_1(1+k+p)]^{m+n+2+(h+i)\frac{\alpha_2}{\alpha_1}}} \right\}, \qquad (36)
\end{aligned}
$$

where we let $y = x^{\alpha_1}$. Substituting equations (35) and (36) into equation (34) to get

$$
\begin{aligned}
R \;=\; & a_1 b_1 \sum_{k,l,p=0}^{\infty} \sum_{m=0}^{k} \sum_{n=0}^{p} \binom{a_1 - 1}{k} \binom{k}{m} \binom{b_1 - 1}{l} \binom{a_1 l}{p} \binom{p}{n} \\
\times\; & \frac{(-1)^{k+l+p} \lambda_1 (m+n)!}{(1+\lambda_1)^{m+n+1} (1+k+p)^{m+n+1}} \left[1 + \frac{m+n+1}{\lambda_1(1+k+p)} \right]
\end{aligned}
$$

$$
\begin{aligned}
- \quad & a_1 b_1 \sum_{k,l,p,q,t,i=0}^{\infty} \sum_{m=0}^{k} \sum_{n=0}^{p} \sum_{h=0}^{t} \binom{a_1-1}{k} \binom{k}{m} \binom{b_1-1}{l} \binom{a_1 l}{p} \binom{p}{n} \binom{b_2}{q} \\
\times \quad & \binom{a_2 q}{t} \binom{t}{h} \frac{(-1)^{k+l+p+q+t+i} \lambda_1^{1-(h+i)\frac{\alpha_2}{\alpha_1}} \lambda_2^{h+i} t^i}{(1+\lambda_1)^{m+n+1} (1+\lambda_2)^h (1+k+p)^{m+n+1+(h+i)\frac{\alpha_2}{\alpha_1}} i!} \\
\times \quad & \left\{ \Gamma\left(m+n+1+(h+i)\frac{\alpha_2}{\alpha_1}\right) + \frac{\Gamma(m+n+2+(h+i)\frac{\alpha_2}{\alpha_1})}{1+k+p} \right\}.
\end{aligned}
$$

6. Quantile Function

The quantile function, say $Q(p)$, is defined by $F(Q(p)) = p$. Now, from the cdf of the KPL distribution, we have

$$
F_{KPL}(Q(p)) = 1 - \left\{ 1 - \left[1 - \frac{1+\lambda+\lambda[Q(p)]^\alpha}{1+\lambda} \exp(-\lambda[Q(p)]^\alpha) \right]^a \right\}^b = p,
$$

and we can obtain $Q(p)$ as the root of the following equation

$$
-\frac{1+\lambda+\lambda[Q(p)]^\alpha}{1+\lambda} \exp(-\lambda[Q(p)]^\alpha) = \left[1-(1-p)^{\frac{1}{b}}\right]^{\frac{1}{a}} - 1, \tag{37}
$$

for $0 < p < 1$. Substituting $Z(p) = -(1+\lambda+\lambda[Q(p)]^\alpha)$, we can rewrite equation (37) as

$$
\frac{Z(p)}{1+\lambda} \exp(1+\lambda+Z(p)) = \left[1-(1-p)^{\frac{1}{b}}\right]^{\frac{1}{a}} - 1,
$$

so that

$$
Z(p)\exp(Z(p)) = (1+\lambda)\exp(-1-\lambda) \left\{ \left[1-(1-p)^{\frac{1}{b}}\right]^{\frac{1}{a}} - 1 \right\},
$$

for $0 < p < 1$. As the defining equation for Lambert W function $W(x)$ is $x = W(x)\exp(W(x))$, we get

$$
Z(p) = W\left((1+\lambda)\exp(-1-\lambda) \left\{ \left[1-(1-p)^{\frac{1}{b}}\right]^{\frac{1}{a}} - 1 \right\} \right),
$$

for $0 < p < 1$. Then, we obtain

$$
Q(p) = \left(\frac{-1-\lambda-W\left((1+\lambda)\exp(-1-\lambda) \left\{ \left[1-(1-p)^{\frac{1}{b}}\right]^{\frac{1}{a}} - 1 \right\} \right)}{\lambda} \right)^{\frac{1}{\alpha}},
$$

for $0 < p < 1$.

7. Mean Deviations

The mean deviation about the mean and the mean deviation about the median are defined by

$$\delta_1(X) = \int_0^\infty |x-\mu| f(x)\,\mathrm{d}x, \quad \text{and} \quad \delta_2(X) = \int_0^\infty |x-M| f(x)\,\mathrm{d}x,$$

respectively, where $\mu = E(X)$, and $M = Median(X)$ denotes the median. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated as follows:

$$\delta_1(X) = 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty x f(x)\,\mathrm{d}x, \tag{38}$$

and

$$\delta_2(X) = -\mu + 2\int_M^\infty x f(x)\,\mathrm{d}x, \tag{39}$$

respectively. By using the moments for KPL distribution and the results in Lemma 2 (Nadarajah et al. [12]), we can calculate equations (38) and (39). Note that

$$K(m,n,p,q) = \int_0^\infty x^p(1+x)\left[1 - \frac{1+n+nx}{1+n}\exp(-nx)\right]^{m-1} \exp(-qx)\,\mathrm{d}x,$$

and

$$L(m,n,p,q,t) = \int_t^\infty x^p(1+x)\left[1 - \frac{1+n+nx}{1+n}\exp(-nx)\right]^{m-1} \exp(-qx)\,\mathrm{d}x.$$

We consider the case when b is real non-integer and a is non-integer. From equation (28) and Lemma 2 (Nadarajah et al. [12]), we know that

$$\begin{aligned}\mu &= \frac{ab\lambda^2}{1+\lambda}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}(-1)^{i+j+r}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r} \\ &\times K\left(r+1,\lambda,\frac{1}{\alpha},\lambda\right),\end{aligned} \tag{40}$$

$$\begin{aligned}\int_\mu^\infty x f(x)\,\mathrm{d}x &= \frac{ab\lambda^2}{1+\lambda}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}(-1)^{i+j+r}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r} \\ &\times L\left(r+1,\lambda,\frac{1}{\alpha},\lambda,\mu\right),\end{aligned} \tag{41}$$

and

$$\begin{aligned}\int_M^\infty x f(x)\,\mathrm{d}x &= \frac{ab\lambda^2}{1+\lambda}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}(-1)^{i+j+r}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r} \\ &\times L\left(r+1,\lambda,\frac{1}{\alpha},\lambda,M\right),\end{aligned}$$

so that

$$\begin{aligned}\delta_1(X) &= 2\mu F(\mu) - 2\mu + \frac{2ab\lambda^2}{1+\lambda}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}(-1)^{i+j+r}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r} \\ &\times\ L\left(r+1,\lambda,\frac{1}{\alpha},\lambda,\mu\right),\end{aligned}$$

and

$$\begin{aligned}\delta_2(X) &= -\mu + \frac{2ab\lambda^2}{1+\lambda}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}(-1)^{i+j+r}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r} \\ &\times\ L\left(r+1,\lambda,\frac{1}{\alpha},\lambda,M\right).\end{aligned}$$

Note here that we have considered the case when a and b are non-integer, however the other cases can be similarly derived.

8. Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves are defined by

$$B(p) = \frac{1}{p\mu}\int_0^q x f(x)\,\mathrm{d}x, \quad \text{and} \quad L(p) = \frac{1}{\mu}\int_0^q x f(x)\,\mathrm{d}x,$$

respectively, where $\mu = E(X)$, and $q = F^{-1}(p)$. Now, we obtain Bonferroni and Lorenz curves for KPL distribution as follows: If b is real non-integer and a is non-integer, then from equations (40) and (41), we have

$$\begin{aligned}\int_q^{\infty} x f(x)\,\mathrm{d}x &= \frac{ab\lambda^2}{1+\lambda}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}(-1)^{i+j+r}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r} \\ &\times\ L\left(r+1,\lambda,\frac{1}{\alpha},\lambda,q\right),\end{aligned}$$

so that Bonferroni and Lorenz curves are

$$\begin{aligned}B(p) &= \frac{1}{p\mu}\left[\int_0^{\infty} x f(x)\,\mathrm{d}x - \int_q^{\infty} x f(x)\,\mathrm{d}x\right] \\ &= \frac{1}{p} - \frac{ab\lambda^2}{p\mu(1+\lambda)}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}(-1)^{i+j+r}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r} \\ &\times\ L\left(r+1,\lambda,\frac{1}{\alpha},\lambda,q\right),\end{aligned}$$

and

$$
\begin{aligned}
L(p) &= 1-\frac{ab\lambda^2}{\mu(1+\lambda)}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}(-1)^{i+j+r}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r} \\
&\times\ L\left(r+1,\lambda,\frac{1}{\alpha},\lambda,q\right),
\end{aligned}
$$

respectively.

Note here that we have considered the case when a and b are non-integer, however the other cases can be similarly derived.

9. Order Statistics, Measures of Uncertainty, and Information

In this section, the distribution of the k^{th} order statistic, measures of uncertainty, and information for the KPL distribution are presented. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty.

9.1. **Distribution of Order Statistics.** Suppose that $X_1,\cdots,X_n$ is a random sample of size n from a continuous pdf, $f(x)$. Let $X_{1:n}<X_{2:n}<\cdots<X_{n:n}$ denote the corresponding order statistics. If $X_1,\cdots,X_n$ is a random sample from KPL distribution, it follows from the equations (9) and (11) that the pdf of the k^{th} order statistic, say $Y_k=X_{k:n}$, is given by

$$
\begin{aligned}
f_k(y_k) &= \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l[F_{KPL}(y_k)]^{k-1+l}f_{KPL}(y_k) \\
&= \frac{ab\alpha\lambda^2 n!(1+y_k^{\alpha})y_k^{\alpha-1}\exp(-\lambda y_k^{\alpha})}{(1+\lambda)(k-1)!(n-k)!}\sum_{l=0}^{n-k}\sum_{p=0}^{k-1+l}\sum_{q,i,j=0}^{\infty}\sum_{r=0}^{j} \\
&\times\ \binom{n-k}{l}\binom{k-1+l}{p}\binom{bp}{q}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r} \\
&\times\ (-1)^{l+p+q+i+j+r}[W(y_k)]^{aq+r},
\end{aligned}
$$

where $W(y_k)=G_{PL}(y_k;\alpha,\lambda)=1-\frac{1+\lambda+\lambda y_k^{\alpha}}{1+\lambda}\exp(-\lambda y_k^{\alpha})$. The corresponding cdf of Y_k is

$$
F_k(y_k) = \sum_{j=k}^{n}\sum_{l=0}^{n-j}\sum_{p=0}^{j+l}\sum_{q=0}^{\infty}\binom{n}{j}\binom{n-j}{l}\binom{j+l}{p}\binom{bp}{q}(-1)^{l+p+q}[W(y_k)]^{aq}.
$$

The s^{th} moment of the k^{th} order statistic Y_k from KPL distribution is obtained as follows: If b is real non-integer and a is non-integer, then

$$\begin{aligned}
E(Y_k^s) &= \int_0^\infty y_k^s f_k(y_k;\alpha,\lambda,a,b)\,\mathrm{d}y_k \\
&= \frac{ab\alpha\lambda^2 n!(-1)^{l+p+q+i+j+r}}{(1+\lambda)(k-1)!(n-k)!}\sum_{l=0}^{n-k}\sum_{p=0}^{k-1+l}\sum_{q,i,j=0}^{\infty}\sum_{r=0}^{j}\binom{n-k}{l}\binom{k-1+l}{p} \\
&\times \binom{bp}{q}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r} \\
&\times \int_0^\infty (1+y_k^\alpha)y_k^{s+\alpha-1}\exp(-\lambda y_k^\alpha)[W(y_k)]^{aq+r}\,\mathrm{d}y_k \\
&= \frac{ab\alpha\lambda^2 n!(-1)^{l+p+q+i+j+r}}{(1+\lambda)(k-1)!(n-k)!}\sum_{l=0}^{n-k}\sum_{p=0}^{k-1+l}\sum_{q,i,j=0}^{\infty}\sum_{r=0}^{j}\binom{n-k}{l}\binom{k-1+l}{p} \\
&\times \binom{bp}{q}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r} \\
&\times \frac{1}{\alpha}\int_0^\infty (1+y_k^\alpha)y_k^{s}\exp(-\lambda y_k^\alpha)\left[1-\frac{1+\lambda+\lambda y_k^\alpha}{1+\lambda}\exp(-\lambda y_k^\alpha)\right]^{aq+r}\,\mathrm{d}y_k^\alpha
\end{aligned}$$

$$\begin{aligned}
&= \frac{ab\lambda^2 n!(-1)^{l+p+q+i+j+r}}{(1+\lambda)(k-1)!(n-k)!}\sum_{l=0}^{n-k}\sum_{p=0}^{k-1+l}\sum_{q,i,j=0}^{\infty}\sum_{r=0}^{j}\binom{n-k}{l}\binom{k-1+l}{p} \\
&\times \binom{bp}{q}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r} \\
&\times \int_0^\infty (1+t)t^{\frac{s}{\alpha}}\exp(-\lambda t)\left[1-\frac{1+\lambda+\lambda t}{1+\lambda}\exp(-\lambda t)\right]^{aq+r}\,\mathrm{d}t \\
&= \frac{ab\lambda^2 n!(-1)^{l+p+q+i+j+r}}{(1+\lambda)(k-1)!(n-k)!}\sum_{l=0}^{n-k}\sum_{p=0}^{k-1+l}\sum_{q,i,j=0}^{\infty}\sum_{r=0}^{j}\binom{n-k}{l}\binom{k-1+l}{p} \\
&\times \binom{bp}{q}\binom{a(1+i)-1}{j}\binom{b-1}{i}\binom{j}{r} \\
&\times K\left(aq+r+1,\lambda,\frac{s}{\alpha},\lambda\right),
\end{aligned}$$

where we let $t = y_k^\alpha$.

Note here that we have considered the case when a and b are non-integer, however the other cases can be similarly derived.

9.2. **Renyi Entropy.** Renyi entropy [13] is an extension of Shannon entropy. Renyi entropy is defined to be $H_\gamma(f_{KPL}(x)) = H_\gamma(f_{KPL}(x;\alpha,\lambda,a,b)) = \frac{\log\left(\int_0^\infty f_{KPL}^\gamma(x;\alpha,\lambda,a,b)\,dx\right)}{1-\gamma}$, where $\gamma > 0$, and $\gamma \neq 1$. Renyi entropy tends to Shannon entropy as $\gamma \to 1$. Now,

$$\begin{aligned}\int_0^\infty f_{KPL}^{\gamma}(x)\,\mathrm{d}x &= \left(\frac{ab\alpha\lambda^2}{1+\lambda}\right)^{\gamma}\\ &\times \int_0^\infty (1+x^\alpha)^{\gamma} x^{\gamma(\alpha-1)}\exp(-\lambda\gamma x^\alpha)\\ &\times [W(x)]^{a\gamma-\gamma}[1-[W(x)]^a]^{b\gamma-\gamma}\,\mathrm{d}x.\end{aligned} \tag{42}$$

Note that

$$\begin{aligned}[W(x)]^{a\gamma-\gamma} &= \left[1-\frac{1+\lambda+\lambda x^\alpha}{1+\lambda}\exp(-\lambda x^\alpha)\right]^{a\gamma-\gamma}\\ &= \sum_{k=0}^{\infty}(-1)^k\binom{a\gamma-\gamma}{k}\frac{\sum_{j=0}^{k}\binom{k}{j}\lambda^j(1+x^\alpha)^j}{(1+\lambda)^k}\exp(-\lambda k x^\alpha),\end{aligned} \tag{43}$$

and

$$\begin{aligned}[1-[W(x)]^a]^{b\gamma-\gamma} &= \sum_{m=0}^{\infty}(-1)^m\binom{b\gamma-\gamma}{m}\sum_{n=0}^{\infty}(-1)^n\binom{am}{n}\\ &\times \frac{\sum_{t=0}^{n}\binom{n}{t}\lambda^t(1+x^\alpha)^t}{(1+\lambda)^n}\exp(-\lambda n x^\alpha).\end{aligned} \tag{44}$$

Substituting equations (43) and (44) into equation (42), we get

$$\begin{aligned}\int_0^\infty f_{KPL}^{\gamma}(x)\,\mathrm{d}x &= \left(\frac{ab\alpha\lambda^2}{1+\lambda}\right)^{\gamma}\sum_{k,m,n=0}^{\infty}\sum_{j=0}^{k}\sum_{t=0}^{n}\\ &\times \binom{a\gamma-\gamma}{k}\binom{k}{j}\binom{b\gamma-\gamma}{m}\binom{am}{n}\binom{n}{t}\frac{(-1)^{k+m+n}\lambda^{j+t}}{(1+\lambda)^{k+n}}\\ &\times \int_0^\infty (1+x^\alpha)^{\gamma+j+t}x^{\gamma(\alpha-1)}\exp(-\lambda(\gamma+k+n)x^\alpha)\,\mathrm{d}x.\end{aligned} \tag{45}$$

By using $(1+x^\alpha)^{\gamma+j+t} = \sum_{i=0}^{\infty}\binom{\gamma+j+t}{i}x^{i\alpha}$ in equation (45), we get

$$\begin{aligned}&\int_0^\infty (1+x^\alpha)^{\gamma+j+t}x^{\gamma(\alpha-1)}\exp(-\lambda(\gamma+k+n)x^\alpha)\,\mathrm{d}x\\ &= \frac{1}{\alpha}\sum_{i=0}^{\infty}\binom{\gamma+j+t}{i}\int_0^\infty x^{(i+\gamma-1)\alpha-\gamma+1}\exp(-\lambda(\gamma+k+n)x^\alpha)\,\mathrm{d}x^\alpha\\ &= \frac{1}{\alpha}\sum_{i=0}^{\infty}\binom{\gamma+j+t}{i}\int_0^\infty y^{i+\gamma-1+\frac{1-\gamma}{\alpha}}\exp(-\lambda(\gamma+k+n)y)\,\mathrm{d}y\\ &= \frac{1}{\alpha}\sum_{i=0}^{\infty}\binom{\gamma+j+t}{i}\frac{\Gamma(i+\gamma+\frac{1-\gamma}{\alpha})}{[\lambda(\gamma+k+n)]^{i+\gamma+\frac{1-\gamma}{\alpha}}},\end{aligned}$$

where we let $y = x^\alpha$. Now, equation (45) simplifies to

$$\begin{aligned} \int_0^\infty f_{KPL}^\gamma(x)\,\mathrm{d}x &= (ab)^\gamma \alpha^{\gamma-1} \sum_{k,m,n,i=0}^{\infty} \sum_{j=0}^{k} \sum_{t=0}^{n} \binom{a\gamma-\gamma}{k}\binom{k}{j} \\ &\times \binom{b\gamma-\gamma}{m}\binom{am}{n}\binom{n}{t}\binom{\gamma+j+t}{i} \\ &\times \frac{(-1)^{k+m+n}\Gamma(i+\gamma+\frac{1-\gamma}{\alpha})}{(1+\lambda)^{\gamma+k+n}\lambda^{i-\gamma-j-t+\frac{1-\gamma}{\alpha}}(\gamma+k+n)^{i+\gamma+\frac{1-\gamma}{\alpha}}}. \end{aligned} \tag{46}$$

Consequently, Renyi entropy for KPL distribution reduces to :

$$\begin{aligned} H_\gamma(f_{KPL}(x)) &= \frac{1}{1-\gamma}\log((ab)^\gamma\alpha^{\gamma-1}) \\ &+ \frac{1}{1-\gamma}\log\Bigg\{ \sum_{k,m,n,i=0}^{\infty} \sum_{j=0}^{k} \sum_{t=0}^{n} \binom{a\gamma-\gamma}{k}\binom{k}{j} \\ &\times \binom{b\gamma-\gamma}{m}\binom{am}{n}\binom{n}{t}\binom{\gamma+j+t}{i} \\ &\times \frac{(-1)^{k+m+n}\Gamma(i+\gamma+\frac{1-\gamma}{\alpha})}{(1+\lambda)^{\gamma+k+n}\lambda^{i-\gamma-j-t+\frac{1-\gamma}{\alpha}}(\gamma+k+n)^{i+\gamma+\frac{1-\gamma}{\alpha}}}\Bigg\}, \end{aligned}$$

for $\gamma > 0$, and $\gamma \neq 1$.

Note here that we have considered the case when a and b are non-integer, however the other cases can be similarly derived.

9.3. ***s*-Entropy.** The s-entropy for KPL distribution is defined by

$$H_s(f_{KPL}(x;\alpha,\lambda,a,b)) = \begin{cases} \frac{1}{s-1}[1-\int_0^\infty f_{KPL}^s(x;\alpha,\lambda,a,b)\,\mathrm{d}x] & \text{if } s\neq 1,\ s>0, \\ E[-\log f(X)] & \text{if } s=1. \end{cases}$$

Consequently, if $s \neq 1, s > 0$, then from equation (46), we have

$$\begin{aligned} H_s(f_{KPL}(x;\alpha,\lambda,a,b)) &= \frac{1}{s-1} - \frac{(ab)^s\alpha^{s-1}}{s-1} \sum_{k,m,n,i=0}^{\infty} \sum_{j=0}^{k} \sum_{t=0}^{n} \binom{as-s}{k}\binom{k}{j} \\ &\times \binom{bs-s}{m}\binom{am}{n}\binom{n}{t}\binom{s+j+t}{i} \\ &\times \frac{(-1)^{k+m+n}\Gamma(i+s+\frac{1-s}{\alpha})}{(1+\lambda)^{s+k+n}\lambda^{i-s-j-t+\frac{1-s}{\alpha}}(s+k+n)^{i+s+\frac{1-s}{\alpha}}}. \end{aligned}$$

If $s = 1$, then s-entropy is Shannon entropy.

9.4. **Fisher Information Matrix.** This section presents a measure for the amount of information. This information measure can be used to obtain bounds on the variance of estimators, and as well as approximate the sampling distribution of an estimator obtained from a large sample. Furthermore, it can used to obtain an approximate confidence interval in case of large sample.

Let X be a random variable (rv) with the KPL pdf $f_{KPL}(.;\mathbf{\Theta})$, where $\mathbf{\Theta}=(\theta_1,\theta_2,\theta_3,\theta_4)^T=(\alpha,\lambda,a,b)^T$. Then Fisher information matrix (FIM) is the 4×4 symmetric matrix with elements:

$$\mathbf{I}_{ij}(\mathbf{\Theta})=E_{\mathbf{\Theta}}\left[\frac{\partial\log(f_{KPL}(X;\mathbf{\Theta}))}{\partial\theta_i}\frac{\partial\log(f_{KPL}(X;\mathbf{\Theta}))}{\partial\theta_j}\right].$$

If the density $f_{KPL}(.;\mathbf{\Theta})$ has second derivative for all i and j, then an alternative expression for $\mathbf{I}_{ij}(\mathbf{\Theta})$ is

$$\mathbf{I}_{ij}(\mathbf{\Theta})=-E_{\mathbf{\Theta}}\left[\frac{\partial^2\log(f_{KPL}(X;\mathbf{\Theta}))}{\partial\theta_i\partial\theta_j}\right]. \tag{47}$$

For the KPL distribution, all second derivatives exist, therefore the formula above is appropriate and, most importantly significantly simplifies the computations. The elements of the observed information matrix of the KPL distribution are given in Appendix A.

10. Maximum Likelihood Estimators

In this section, the maximum likelihood estimates (MLEs) of the parameters α, λ, a, and b of the KPL distribution are presented. If $x_1,\cdots,x_n$ is a random sample from KPL distribution, then the log-likelihood function is given by

$$\begin{aligned}\log(L(\alpha,\lambda,a,b)) &= n\log\left(\frac{ab\alpha\lambda^2}{1+\lambda}\right)+\sum_{i=1}^{n}\log(1+x_i^{\alpha})+(\alpha-1)\sum_{i=1}^{n}\log x_i\\ &- \lambda\sum_{i=1}^{n}x_i^{\alpha}+(a-1)\sum_{i=1}^{n}\log(W(x_i))\\ &+ (b-1)\sum_{i=1}^{n}\log[1-[W(x_i)]^a],\end{aligned} \tag{48}$$

where $W(x_i)=1-\frac{1+\lambda+\lambda x_i^{\alpha}}{1+\lambda}\exp(-\lambda x_i^{\alpha})$.

The partial derivatives of $\log L(\alpha,\lambda,a,b)$ with respect to the parameters α, λ, a, and b are:

$$\frac{\partial\log L(\alpha,\lambda,a,b)}{\partial a} = \frac{n}{a}+\sum_{i=1}^{n}\log(W(x_i))+(1-b)\sum_{i=1}^{n}\frac{[W(x_i)]^a\log(W(x_i))}{1-[W(x_i)]^a},$$

$$\frac{\partial\log L(\alpha,\lambda,a,b)}{\partial b} = \frac{n}{b}+\sum_{i=1}^{n}\log[1-[W(x_i)]^a],$$

$$\begin{aligned}\frac{\partial \log L(\alpha,\lambda,a,b)}{\partial\alpha} &= \frac{n}{\alpha}+\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{1+x_i^{\alpha}}+\sum_{i=1}^{n}\log x_i-\lambda\sum_{i=1}^{n}x_i^{\alpha}\log x_i\\ &+ (a-1)\sum_{i=1}^{n}\frac{\partial W(x_i)/\partial\alpha}{W(x_i)}\\ &+ a(1-b)\sum_{i=1}^{n}\frac{[W(x_i)]^{a-1}(\partial W(x_i)/\partial\alpha)}{1-[W(x_i)]^{a}},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L(\alpha,\lambda,a,b)}{\partial\lambda} &= \frac{n(2+\lambda)}{\lambda(1+\lambda)}-\sum_{i=1}^{n}x_i^{\alpha}+(a-1)\sum_{i=1}^{n}\frac{\partial W(x_i)/\partial\lambda}{W(x_i)}\\ &+ a(1-b)\sum_{i=1}^{n}\frac{[W(x_i)]^{a-1}(\partial W(x_i)/\partial\lambda)}{1-[W(x_i)]^{a}},\end{aligned}$$

where

$$\frac{\partial W(x_i)}{\partial\alpha}=\frac{\lambda^2(1+x_i^{\alpha})x_i^{\alpha}\log x_i\exp(-\lambda x_i^{\alpha})}{1+\lambda},$$

and

$$\frac{\partial W(x_i)}{\partial\lambda} = \left[\frac{1+\lambda+\lambda x_i^{\alpha}}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]x_i^{\alpha}\exp(-\lambda x_i^{\alpha}).$$

When all the parameters are unknown, numerical methods must be used to obtain estimates of the model parameters since the system does not admit any explicit solution, therefore the MLE $(\hat{\alpha},\hat{\lambda},\hat{a},\hat{b})$ of (α,λ,a,b) can be obtained only by means of numerical procedures. The MLEs of the parameters, denoted by $\hat{\mathbf{\Theta}}$ is obtained by solving the nonlinear equation $(\frac{\partial\log L}{\partial\alpha},\frac{\partial\log L}{\partial\lambda},\frac{\partial\log L}{\partial a},\frac{\partial\log L}{\partial b})^T=\mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix given by $\mathbf{I}(\mathbf{\Theta})=[\mathbf{I}_{\theta_i,\theta_j}]_{4X4}=E(-\frac{\partial^2\log L}{\partial\theta_i\partial\theta_j})$, $i,j=1,2,3,4$, can be numerically obtained by MATLAB or MAPLE software. The total Fisher information matrix $\mathbf{I}_n(\mathbf{\Theta})=n\mathbf{I}(\mathbf{\Theta})$ can be approximated by

$$\mathbf{J}_n(\hat{\mathbf{\Theta}})\approx\left[-\left.\frac{\partial^2\log L}{\partial\theta_i\partial\theta_j}\right|_{\mathbf{\Theta}=\hat{\mathbf{\Theta}}}\right]_{4X4},\quad i,j=1,2,3,4. \tag{49}$$

For real data, the matrix given in equation (49) is obtained after the convergence of the Newton-Raphson procedure in MATLAB or R software.

10.1. **Asymptotic Confidence Intervals.** In this section, we present the asymptotic confidence intervals for the parameters of the KPL distribution. The expectations in the FIM can be obtained numerically. Let $\hat{\mathbf{\Theta}}=(\hat{\alpha},\hat{\lambda},\hat{a},\hat{b})^T$ be the MLE of $\mathbf{\Theta}=(\alpha,\lambda,a,b)^T$. Under the conditions that the parameters are in the interior of the parameter space, but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\mathbf{\Theta}}-\mathbf{\Theta})$ is $N_4(\mathbf{0},\mathbf{I}^{-1}(\mathbf{\Theta}))$.

The multivariate normal distribution with mean vector $(0,0,0,0)^T$ and covariance matrix $\mathbf{I}^{-1}(\mathbf{\Theta})$ can be used to construct confidence intervals for the model parameters. That is, the approximate $100(1-\eta)\%$ two-sided confidence intervals for α, λ, a and b are given by:

$$\widehat{\alpha} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}^{-1}_{\alpha\alpha}(\widehat{\mathbf{\Theta}})}, \quad \widehat{\lambda} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}^{-1}_{\lambda\lambda}(\widehat{\mathbf{\Theta}})}, \quad \widehat{a} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}^{-1}_{aa}(\widehat{\mathbf{\Theta}})}, \quad \text{and} \quad \widehat{b} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}^{-1}_{bb}(\widehat{\mathbf{\Theta}})},$$

respectively, where $\mathbf{I}^{-1}_{\alpha\alpha}(\widehat{\mathbf{\Theta}})$, $\mathbf{I}^{-1}_{\lambda\lambda}(\widehat{\mathbf{\Theta}})$, $\mathbf{I}^{-1}_{aa}(\widehat{\mathbf{\Theta}})$, and $\mathbf{I}^{-1}_{bb}(\widehat{\mathbf{\Theta}})$ are the diagonal elements of $\mathbf{I}^{-1}_{n}(\hat{\mathbf{\Theta}}) = (n\mathbf{I}(\hat{\mathbf{\Theta}}))^{-1}$, and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

11. Application

In this section, application of the KPL distribution including the estimation of the parameters via the method of maximum likelihood and likelihood ratio (LR) test for comparison of the KPL distribution with its sub-models for given sets of data are presented. The examples illustrate the flexibility of the KPL distribution in contrast to other models including the Kumaraswamy Lindley (KL), power Lindley (PL), GL, L, Kumaraswamy Weibull (KW), Weibull (W), and gamma (GAM) distributions for data modeling.

The MLEs of the KPL parameters α, λ, a, and b are computed by maximizing the objective function via the subroutine NLMIXED in SAS. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\log(L)$, Bayesian Information Criterion, $BIC = p\log(n) - 2\log(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\Theta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented in Table 3. The KPL distribution is fitted to the data sets and these fits are compared to the fits using the KL, PL, GL, L, W, and GAM distributions.

We can use the LR test to compare the fit of the KPL distribution with its sub-models for a given data set. For example, to test $a = b = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{\alpha}, \hat{\lambda}, \hat{a}, \hat{b})) - \ln(L(\tilde{\alpha}, \tilde{\lambda}, 1, 1))]$, where $\hat{\alpha}$, $\hat{\lambda}$, $\hat{a}$, and $\hat{b}$, are the unrestricted estimates, and $\tilde{\alpha}$, and $\tilde{\lambda}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_d$, where χ^2_d denote the upper $100d\%$ point of the χ^2 distribution with 2 degrees of freedom.

Specifically, we consider a maintainance data set. The set of data is the maintenance data with 46 observations reported on active repair times (hours) for an airborne communication transceiver discussed by Alven [1], Chhikara and Folks [3], and Dimitrakopoulou et al. [5]. The data set is given in Table 1. The MLEs of the parameters with standard errors in parenthesis and the values of the statistics (-2ln(L), AIC, AICC and BIC) are given in Table 3. The starting points of the iterative processes for the data sets for the $KPL(\alpha, \lambda, a, b)$ distribution are (1, 0.115, 0.026, 0.1).

Probability plots (Chambers et al [2]) consists of plots of the observed probabilities against the probabilities predicted by the fitted model are also presented in Figures 3 and 4. For

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7	0.7
0.8	0.8	1.0	1.0	1.0	1.0	1.1	1.3	1.5	1.5	1.5
1.5	2.0	2.0	2.2	2.5	2.7	3.0	3.0	3.3	3.3	4.0
4.0	4.5	4.7	5.0	5.4	5.4	7.0	7.5	8.8	9.0	10.3
22.0	24.5	-	-	-	-	-	-	-	-	-

Table 1. Maintenance Data [1], [3], [5]

Data set	Model	α	λ	a	b	$-2\ln(L)$	AIC	$AICC$	BIC	SS
I (n=46)	$KPL(\alpha,\lambda,a,b)$	0.6679	4.4222	5.6016	0.1488	200.6	208.6	209.6	215.9	0.06484707
		(0.05909)	(0.8266)	(2.7768)	(0.02612)					
	$KL(1,\lambda,a,b)$	1	1.6091	1.0576	0.2029	212.7	218.7	219.2	224.2	0.4270972
			(0.03375)	(0.3385)	(0.03180)					
	$PL(\alpha,\lambda,1,1)$	0.7581	0.6757	1	1	210.0	214.0	214.3	217.7	0.1185674
		(0.07424)	(0.1016)							
	$GL(1,\lambda,a,1)$	1	0.3677	0.6643	1	215.7	219.7	220.0	223.4	0.2635925
			(0.06442)	(0.1352)						
	$L(1,\lambda,1,1)$	1	0.4664	1	1	220.0	222.0	222.1	223.8	0.5676042
			(0.04990)							
	$W(\alpha,\lambda,1,1)$	0.8986	0.2949	1	1	208.9	212.9	213.2	216.6	0.1156807
		(0.09576)	(0.05138)							
	$GAM(\alpha,\lambda)$	0.9323	0.2585	-	-	209.9	213.9	214.1	217.5	0.1716121
		(0.1701)	(0.06150)							

Table 2. Parameters Estimates, Log-likelihood, AIC, AICC, BIC, and SS

the KPL distribution, we plotted for example,

$$F_{KPL}(y_k;\hat{\alpha},\hat{\lambda},\hat{a},\hat{b}) = 1-\left\{1-\left[1-\frac{1+\hat{\lambda}+\hat{\lambda}y_k^{\hat{\alpha}}}{1+\hat{\lambda}}\exp(-\hat{\lambda}y_k^{\hat{\alpha}})\right]^{\hat{a}}\right\}^{\hat{b}}, \tag{50}$$

against $\frac{k-0.375}{n+0.25}$, $k=1,2,\cdots,n$, where y_k are the ordered values of the observed data. A measure of closeness of the plot to the diagonal line given by the sum of squares

$$SS=\sum_{k=1}^{n}\left[F_{KPL}(y_k;\hat{\alpha},\hat{\lambda},\hat{a},\hat{b})-\left(\frac{k-0.375}{n+0.25}\right)\right]^2,$$

was calculated for each plot. The plot with the smallest SS corresponds to the model with points that are closer to the diagonal line. The KPL model performs very well in this regard.

For the maintenance data, the LR statistics for the test of the hypotheses $H_0: KL(1,\lambda,a,b)$ against $H_a: KPL(\alpha,\lambda,a,b)$, $H_0: PL(\alpha,\lambda,1,1)$ against $H_a: KPL(\alpha,\lambda,a,b)$, $H_0: GL(1,\lambda,a,1)$ against $H_a: KPL(\alpha,\lambda,a,b)$, and $H_0: L(1,\lambda,1,1)$ against $H_a: KPL(\alpha,\lambda,a,b)$ are 12.1 ($p-value = 5.04\times10^{-4} < 0.001$), 9.4 ($p-value = 9.095\times10^{-3} < 0.01$), 15.1 ($p-value = 5.2611\times10^{-4} < 0.001$), and 19.4 ($p-value = 2.2597\times10^{-4} < 0.001$), respectively. Consequently, we reject the null hypothesis in favor of the KPL distribution and conclude that the KPL distribution is significantly better than the KL, PL, GL, and L distributions based on the LR statistic. The KPL distribution is also better than the Weibull and Gamma distributions based on the values of the statistics AIC, AICC and BIC. The plots of the fitted KPL distribution and sub-models are shown in Figure 3.

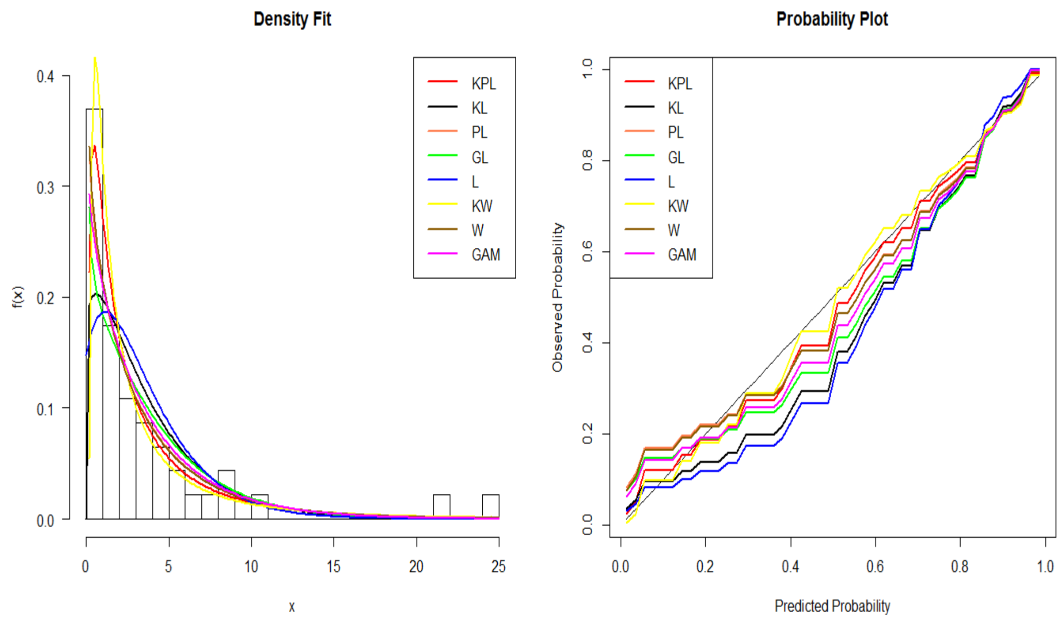

FIGURE 3. Fitted densities and probability plots of KPL distribution and sub-models for maintenance data

Based on the values of these statistics, we conclude that the KPL distribution provides a better fit than the KL, PL, GL, L, and GAM distributions. For the maintenance data, KPL distribution is far better than its sub-models, and a pretty good competitor to the KW distribution. The KPL model can provide better fits than other common multi parameter lifetime models.

12. Concluding Remarks

A new class of generalized Lindley distribution referred to as Kumaraswamy power Lindley (KPL) distribution with flexible and desirable properties is proposed. Properties of the KPL distribution and sub-distributions were presented. The pdf, cdf, moments, hazard function, reverse hazard function, reliability, quantile function, mean deviations, Bonferroni and Lorenz curves were presented. Entropy measures including Renyi entropy, s- entropy as well as Fisher information matrix for KPL distribution were also derived. Estimate of the model parameters via the method of maximum likelihood obtained and application to illustrate the usefulness of the model to real data given.

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Appendix Appendix A FIM for KPL distribution

Let $\ell = L(\alpha,\lambda,a,b)$, and $W(x) = G_{PL}(x;\alpha,\lambda) = 1 - \frac{1+\lambda+\lambda x^{\alpha}}{1+\lambda}\exp(-\lambda x^{\alpha})$. Elements of the observed information matrix of the KPL distribution are given by

$$\frac{\partial^2 \ell}{\partial a^2} = -\frac{n}{a^2} + (1-b)\sum_{i=1}^{n} \frac{[W(x_i)]^a[\log(W(x_i))]^2}{[1-[W(x_i)]^a]^2},$$

$$\frac{\partial^2 \ell}{\partial a \partial b} = -\sum_{i=1}^{n} \frac{[W(x_i)]^a \log(W(x_i))}{1-[W(x_i)]^a},$$

$$\begin{aligned}\frac{\partial^2 \ell}{\partial a \partial \alpha} &= \sum_{i=1}^{n} \frac{\lambda^2(1+x_i^{\alpha})x_i^{\alpha}\log x_i \exp(-\lambda x_i^{\alpha})}{(1+\lambda)W(x_i)} \\ &+ (1-b)\sum_{i=1}^{n} \frac{\lambda^2(1+x_i^{\alpha})x_i^{\alpha}\log x_i \exp(-\lambda x_i^{\alpha})}{(1+\lambda)[1-[W(x_i)]^a]^2} \\ &\times [W(x_i)]^{a-1}[a\log W(x_i) + 1 - [W(x_i)]^a],\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ell}{\partial a \partial \lambda} &= \sum_{i=1}^{n} \frac{x_i^{\alpha}\exp(-\lambda x_i^{\alpha})}{W(x_i)}\left[\frac{1+\lambda+\lambda x_i^{\alpha}}{1+\lambda} - \frac{1}{(1+\lambda)^2}\right] \\ &+ (1-b)\sum_{i=1}^{n} \frac{x_i^{\alpha}\exp(-\lambda x_i^{\alpha})[W(x_i)]^{a-1}[a\log W(x_i) + 1 - [W(x_i)]^a]}{[1-[W(x_i)]^a]^2} \\ &\times \left[\frac{1+\lambda+\lambda x_i^{\alpha}}{1+\lambda} - \frac{1}{(1+\lambda)^2}\right],\end{aligned}$$

$$\frac{\partial^2 \ell}{\partial b^2} = -\frac{n}{b^2},$$

$$\frac{\partial^2 \ell}{\partial b \partial \alpha} = -a\lambda^2 \sum_{i=1}^{n} \frac{(1+x_i^{\alpha})x_i^{\alpha}\log x_i \exp(-\lambda x_i^{\alpha})[W(x_i)]^{a-1}}{(1+\lambda)[1-[W(x_i)]^a]},$$

$$\frac{\partial^2 \ell}{\partial b \partial \lambda} = -a\sum_{i=1}^{n} \frac{x_i^{\alpha}\exp(-\lambda x_i^{\alpha})[W(x_i)]^{a-1}}{1-[W(x_i)]^a}\left[\frac{1+\lambda+\lambda x_i^{\alpha}}{1+\lambda} - \frac{1}{(1+\lambda)^2}\right],$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial \alpha^2} &= -\frac{n}{\alpha^2} + \sum_{i=1}^{n} \frac{x_i^\alpha (\log x_i)^2}{(1+x_i^\alpha)^2} - \lambda \sum_{i=1}^{n} x_i^\alpha (\log x_i)^2 \\
&+ (a-1) \sum_{i=1}^{n} \frac{\lambda^2 x_i^\alpha (\log x_i)^2 \exp(-\lambda x_i^\alpha)}{(1+\lambda)[W(x_i)]^2} \\
&\times \left\{ [1 + 2x_i^\alpha - \lambda(1+x_i^\alpha)x_i^\alpha] W(x_i) - \frac{\lambda^2 (1+x_i^\alpha)^2 x_i^\alpha \exp(-\lambda x_i^\alpha)}{1+\lambda} \right\} \\
&+ a(1-b) \sum_{i=1}^{n} \frac{\lambda^2 x_i^\alpha (\log x_i)^2 \exp(-\lambda x_i^\alpha) [W(x_i)]^{a-2}}{(1+\lambda)[1-[W(x_i)]^a]^2} \\
&\times \left\{ [1 + 2x_i^\alpha - \lambda(1+x_i^\alpha)x_i^\alpha] W(x_i) [1-[W(x_i)]^a] \right. \\
&+ \left. \frac{\lambda^2 (1+x_i^\alpha)^2 x_i^\alpha \exp(-\lambda x_i^\alpha)[a-1+[W(x_i)]^a]}{1+\lambda} \right\},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} &= -\sum_{i=1}^{n} x_i^\alpha \log x_i \\
&+ (a-1) \sum_{i=1}^{n} \frac{\lambda(1+x_i^\alpha) x_i^\alpha \log x_i \exp(-\lambda x_i^\alpha)}{(1+\lambda) W(x_i)} \left[\frac{2+\lambda}{1+\lambda} - \lambda x_i^\alpha \right] \\
&- (a-1) \sum_{i=1}^{n} \frac{\lambda^2 (1+x_i^\alpha) x_i^{2\alpha} \log x_i \exp(-2\lambda x_i^\alpha)}{(1+\lambda)[W(x_i)]^2} \left[\frac{1+\lambda+\lambda x_i^\alpha}{1+\lambda} - \frac{1}{(1+\lambda)^2} \right] \\
&+ a(1-b) \sum_{i=1}^{n} \frac{\lambda^2 (1+x_i^\alpha) x_i^{2\alpha} \log x_i \exp(-2\lambda x_i^\alpha) [W(x_i)]^{a-2}}{(1+\lambda)[1-[W(x_i)]^a]^2} \\
&\times \left[\frac{1+\lambda+\lambda x_i^\alpha}{1+\lambda} - \frac{1}{(1+\lambda)^2} \right] [a-1+[W(x_i)]^a] \\
&+ a(1-b) \sum_{i=1}^{n} \frac{\lambda(1+x_i^\alpha) x_i^\alpha \log x_i \exp(-\lambda x_i^\alpha) [W(x_i)]^{a-1} [1-[W(x_i)]^a]}{(1+\lambda)[1-[W(x_i)]^a]^2} \\
&\times \left[\frac{2+\lambda}{1+\lambda} - \lambda x_i^\alpha \right],
\end{aligned}
$$

and

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \lambda^2} &= -n\frac{\lambda^2+4\lambda+2}{\lambda^2(1+\lambda)^2} \\
&+ (a-1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\exp(-\lambda x_i^{\alpha})}{W(x_i)}\left[\frac{2x_i^{\alpha}}{(1+\lambda)^2}+\frac{2}{(1+\lambda)^3}-\frac{(1+\lambda+\lambda x_i^{\alpha})x_i^{\alpha}}{1+\lambda}\right] \\
&- (a-1)\sum_{i=1}^{n}\frac{x_i^{2\alpha}\exp(-2\lambda x_i^{\alpha})}{[W(x_i)]^2}\left[\frac{1+\lambda+\lambda x_i^{\alpha}}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]^2 \\
&+ a(1-b)\sum_{i=1}^{n}\frac{x_i^{2\alpha}\exp(-2\lambda x_i^{\alpha})[W(x_i)]^{a-2}}{[1-[W(x_i)]^a]^2} \\
&\times \left[\frac{1+\lambda+\lambda x_i^{\alpha}}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]^2[a-1+[W(x_i)]^a] \\
&+ a(1-b)\sum_{i=1}^{n}\frac{x_i^{\alpha}\exp(-\lambda x_i^{\alpha})[W(x_i)]^{a-1}}{1-[W(x_i)]^a} \\
&\times \left[\frac{2x_i^{\alpha}}{(1+\lambda)^2}+\frac{2}{(1+\lambda)^3}-\frac{(1+\lambda+\lambda x_i^{\alpha})x_i^{\alpha}}{1+\lambda}\right].
\end{aligned}$$

Note here that we have considered the case when $b > 0$ and $a > 0$ are non-integer, however the other cases can be similarly derived.

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