

Georgia Southern University Digital Commons@Georgia Southern

Mathematical Sciences Faculty Publications

Mathematical Sciences, Department of

12-2013

Multiple Solutions of a $p(x)$ -Laplacian Equation Involving Critical Nonlinearities

Yuan Liang
Zhejiang Wanli University

Xianbin Wu
Zhejiang Wanli University

Qihu Zhang
Zhengzhou University of Light Industry

Chunshan Zhao
Georgia Southern University, czhao@georgiasouthern.edu

Follow this and additional works at: <https://digitalcommons.georgiasouthern.edu/math-sci-facpubs>

 Part of the [Mathematics Commons](#)

Recommended Citation

Liang, Yuan, Xianbin Wu, Qihu Zhang, Chunshan Zhao. 2013. "Multiple Solutions of a $p(x)$ -Laplacian Equation Involving Critical Nonlinearities." *Taiwanese Journal of Mathematics*, 17 (6): 2083-2100. doi: 10.11650/tjm.17.2013.3074 source:
<http://journal.taiwanmathsoc.org.tw/index.php/TJM/article/view/3074>
<https://digitalcommons.georgiasouthern.edu/math-sci-facpubs/253>

This article is brought to you for free and open access by the Mathematical Sciences, Department of at Digital Commons@Georgia Southern. It has been accepted for inclusion in Mathematical Sciences Faculty Publications by an authorized administrator of Digital Commons@Georgia Southern. For more information, please contact digitalcommons@georgiasouthern.edu.

MULTIPLE SOLUTIONS OF A $p(x)$ -LAPLACIAN EQUATION INVOLVING CRITICAL NONLINEARITIES

Yuan Liang, Xianbin Wu*, Qihu Zhang* and Chunshan Zhao

Abstract. In this paper, we consider the existence of multiple solutions for the following $p(x)$ -Laplacian equations with critical Sobolev growth conditions

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) + |u|^{p(x)-2} u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We show the existence of infinitely many pairs of solutions by applying the Fountain Theorem and the Dual Fountain Theorem respectively. We also present a variant of the concentration-compactness principle, which is of independent interest.

1. INTRODUCTION

In recent years, there are a lot of interest in the study of various mathematical problems with variable exponent (see [2-6, 13, 14, 16-25, 30-34, 36-39, 42, 44-51, 54-58] and references therein). We refer readers to [17, 51] for an overview of this research area and [2, 13, 49, 58] for the background of these problems. Recently, people are also interested in the applications of variable exponent analysis to image restoration [29, 30, 34, 38]. The most typical differential equation with variable exponent is the $p(x)$ -Laplacian equation, which is a generalization of the usual p -Laplacian equation with the constant exponent p being replaced by a variable exponent $p(x)$. For Sobolev spaces with variable exponent which have been used to study the $p(x)$ -Laplacian equations, we refer readers to [16, 19, 37]. On the existence of solutions of elliptic equations with variable exponent and subcritical growth conditions, we refer readers to [5, 6, 21,

Received March 18, 2013, accepted May 23, 2013.

Communicated by Yosikazu Giga.

2010 *Mathematics Subject Classification*: 35J20, 35J25, 35J60.

Key words and phrases: $p(x)$ -Laplacian, Integral functional, Variable exponent Sobolev space, Critical exponent.

Supported by the National Science Foundation of China (10701066 & 10971087).

*Corresponding author.

23, 31, 54]. However, to the best of our knowledge, results on elliptic equations with variable exponent and critical growth condition are rare (see [24, 25]).

In this paper, we consider the existence of multiple solutions of the following equations with critical Sobolev growth conditions

$$(P) \quad \begin{cases} -div(|\nabla u|^{p(x)-2} \nabla u) + |u|^{p(x)-2} u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2} \nabla u)$ is called the $p(x)$ -Laplacian; $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $p(x) \in C(\overline{\Omega})$ is Lipschitz continuous and $1 < \inf_{x \in \Omega} p(x) \leq \sup_{x \in \Omega} p(x) < N$; f satisfies the following condition.

$$|f(x, t)| \leq C(1 + |t|^{p^*(x)-1}), \forall (x, t) \in \overline{\Omega} \times \mathbb{R},$$

where $p^*(x) = \begin{cases} Np(x)/(N - p(x)), & p(x) < N, \\ \infty, & p(x) \geq N. \end{cases}$

Because of its non-homogeneity, the $p(x)$ -Laplacian possesses more complicated nonlinearity than the p -Laplacian. Many results for p -Laplacian problems do not hold for $p(x)$ -Laplacian problems anymore. For examples,

(1⁰) If $\Omega \subset \mathbb{R}^N$ is an open bounded domain, the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx}{\int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx}$$

is zero in general. Only under some special conditions, we have $\lambda_{p(x)} > 0$. For example, $\lambda_{p(x)} > 0$ if and only if $p(x)$ is monotone in one dimensional case (i.e. $N = 1$) (see [22]). It is well known that the fact that $\lambda_p > 0$ is very important in the study of p -Laplacian problems.

(2⁰) The norm in $L^{p(\cdot)}(\Omega)$ is of Luxemburg type (we will explain later in the second section). It is easy to see that $\int_{\Omega} |u|^{p(x)} dx = |u|_{p(\cdot)}^{p(\xi)}$ for some $\xi \in \overline{\Omega}$. Hence the integral and the norm can not keep the constant exponent relationship. It implies that we will have more difficulties in the study of $p(x)$ -Laplacian problems. For example, it is very difficult to get the best Sobolev imbedding constant when we deal with the critical Sobolev exponent problems. Even if the best Sobolev imbedding constant could be obtained, it is also hard to be applied to study the critical exponent problems.

In [8], Brézis and Nirenberg initially studied the equations involving critical Sobolev exponents. In [40, 41], Lions discovered the concentration-compactness principles which have been proved to be very effective in variational problems involving critical

Sobolev growth conditions. These principles are currently named as the first and second concentration-compactness principles (CCP1, CCP2). The proof of these concentration-compactness principles can also be found in [52, 53]. In [11], Chabrowski formulated a variant of these two principles, namely, the concentration-compactness principle at infinity (CCP $_{\infty}$) for both critical and subcritical cases. As to p -Laplacian problems with critical growth conditions, there are many results (see [1, 7-10, 15-18, 26-28, 35, 43-47] and the references therein). But results on the $p(x)$ -Laplacian problems with critical growth conditions are rare.

In [24], Fu gave the concentration-compactness principle in $L^{p(\cdot)}(\Omega)$ space, and discussed the existence of at least one nontrivial solution. Our aim here is to deal with the existence of multiple solutions for $p(x)$ -Laplacian problems involving critical growth conditions. We obtain the existence of infinitely many pairs of solutions by the Fountain Theorem and the Dual Fountain Theorem. Especially, we give a variant of concentration-compactness principle. These results are extension of results of p -Laplacian problems.

This paper is organized as follows. In Section 2, we introduce some basic properties of the variable exponent Sobolev spaces, and also present a variant of concentration-compactness principle. In Section 3, several important properties of $p(x)$ -Laplacian are presented. Finally, we give the main results and the proofs in Section 4.

2. WEIGHTED VARIABLE EXPONENT LEBESGUE AND SOBOLEV SPACES

In order to discuss the problem (P), we need the functional space $W^{1,p(\cdot)}(\Omega)$ which is called variable exponent Sobolev space. To deal with critical nonlinearities, we also need a variant of the concentration-compactness principle. Let $S(\Omega)$ be the set of all measurable real valued functions defined on Ω . Denote

$$\begin{aligned} h^+ &= \operatorname{ess\,sup}_{x \in \overline{\Omega}} h(x), \quad h^- = \operatorname{ess\,inf}_{x \in \overline{\Omega}} h(x), \quad \text{for any } h \in S(\overline{\Omega}), \\ C_+(\overline{\Omega}) &= \{h \mid h \in C(\overline{\Omega}), h^- \geq 1 \text{ for } x \in \overline{\Omega}\}, \\ L^{p(\cdot)}(\Omega) &= \left\{u \mid u \in S(\Omega), \int_{\Omega} |u(x)|^{p(x)} dx < \infty\right\}. \end{aligned}$$

The Luxemburg norm on $L^{p(\cdot)}(\Omega)$ is defined by

$$|u|_{p(\cdot)} = \inf \left\{ \lambda > 0 \mid \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Under the norm as above ($L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)}$) becomes a Banach space, which is called variable exponent Lebesgue space.

The space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) \mid |\nabla u| \in L^{p(\cdot)}(\Omega) \right\},$$

in which the norm is defined by

$$\|u\|_{p(\cdot)} = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}, \forall u \in W^{1,p(\cdot)}(\Omega).$$

Let $r^0(\cdot)$ be the conjugate function of $r(\cdot)$, namely $r^0(x) = \begin{cases} \frac{r(x)}{r(x)-1}, r \in C(\overline{\Omega}) \\ 1, r = \infty \end{cases}$.

Proposition 2.1 . (see [19]).

(i) If $q \in L^\infty(\Omega)$, $1 < q^- \leq q^+ < \infty$, then the space $(L^{q(\cdot)}(\Omega), |\cdot|_{q(\cdot)})$ is a separable, uniformly convex Banach space, and it's conjugate space is $L^{q^0(\cdot)}(\Omega)$ where $\frac{1}{q(x)} + \frac{1}{q^0(x)} \equiv 1$. For any $u \in L^{q(\cdot)}(\Omega)$ and $v \in L^{q^0(\cdot)}(\Omega)$, we have

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{q^-} + \frac{1}{(q^0)^-} \right) |u|_{q(\cdot)} |v|_{q^0(\cdot)}.$$

(ii) If $\Omega \subset \mathbb{R}^N$ is open bounded, $1 \leq p_1, p_2 \in C(\overline{\Omega})$, $p_1(x) \leq p_2(x)$ for any $x \in \Omega$, then $L^{p_2(\cdot)}(\Omega) \subset L^{p_1(\cdot)}(\Omega)$, and the imbedding is continuous.

Proposition 2.2. (see [19]). If we denote

$$\rho(u) = \int_{\Omega} |u|^{p(x)} dx, \forall u \in L^{p(\cdot)}(\Omega),$$

then

- (i) $|u|_{p(\cdot)} < 1 (= 1; > 1) \iff \rho(u) < 1 (= 1; > 1)$;
- (ii) $|u|_{p(\cdot)} > 1 \implies |u|_{p(\cdot)}^{p^-} \leq \rho(u) \leq |u|_{p(\cdot)}^{p^+}$; $|u|_{p(\cdot)} < 1 \implies |u|_{p(\cdot)}^{p^-} \geq \rho(u) \geq |u|_{p(\cdot)}^{p^+}$;
- (iii) $|u|_{p(\cdot)} \rightarrow 0 \iff \rho(u) \rightarrow 0$; $|u|_{p(\cdot)} \rightarrow \infty \iff \rho(u) \rightarrow \infty$.

Proposition 2.3. (see [19]). If $u, u_n \in L^{p(\cdot)}(\Omega)$, $n = 1, 2, \dots$, then the following statements are equivalent to each other:

- (1) $\lim_{k \rightarrow \infty} |u_k - u|_{p(\cdot)} = 0$;
- (2) $\lim_{k \rightarrow \infty} \rho(u_k - u) = 0$;
- (3) $u_k \rightarrow u$ in measure in Ω and $\lim_{k \rightarrow \infty} \rho(u_k) = \rho(u)$.

Proposition 2.4. (see [19]).

- (i) $W^{1,p(\cdot)}(\Omega)$ is a separable reflexive Banach space;
- (ii) If $1 \leq q \in C(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding from $W^{1,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact;
- (iii) If p is Lipschitz continuous, q is measurable and satisfies $1 \leq q(x) \leq p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding from $W^{1,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is continuous.

Proposition 2.5. (see [19]). *If Ω is an open subset of \mathbb{R}^N , $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function and satisfies*

$$|f(x, s)| \leq a(x) + b |s|^{p_1(x)/p_2(x)} \text{ for any } x \in \Omega, s \in \mathbb{R},$$

where $1 \leq p_1, p_2 \in C(\Omega)$, $a(\cdot) \in L^{p_2(\cdot)}(\Omega)$, $a(x) \geq 0$, $b \geq 0$ is a constant, then the Nemytsky operator from $L^{p_1(\cdot)}(\Omega)$ to $L^{p_2(\cdot)}(\Omega)$ defined by $(N_f u)(x) = f(x, u(x))$ is a continuous and bounded operator.

Denote $\|u\|_{p(\cdot)}' = \inf\{\lambda > 0 \mid \int_{\Omega} |\frac{\nabla u}{\lambda}|^{p(x)} dx + \int_{\Omega} |\frac{u}{\lambda}|^{p(x)} dx \leq 1\}$, then it is easy to see that $\|\cdot\|_{p(\cdot)}'$ is an equivalence norm of $\|\cdot\|_{p(\cdot)}$ on $W^{1,p(\cdot)}(\Omega)$. In the following, we will use $\|\cdot\|_{p(\cdot)}'$ instead of $\|\cdot\|_{p(\cdot)}$ on $W^{1,p(\cdot)}(\Omega)$.

Let $\mathbb{M}(\overline{\Omega})$ denote the class of nonnegative Borel measures of finite total mass, and $\mu_\varepsilon \xrightarrow{*} \mu$ in $\mathbb{M}(\overline{\Omega})$ is defined by $\int_{\overline{\Omega}} \eta d\mu_\varepsilon \rightarrow \int_{\overline{\Omega}} \eta d\mu$ for every test function $\eta \in C(\overline{\Omega})$.

Proposition 2.6. (see [24]). *Assume Ω is an open bounded domain in \mathbb{R}^N , p is Lipschitz continuous on $\overline{\Omega}$ and satisfy $1 < p(x) < N$. Let $\{\omega_\varepsilon\}$ be a sequence in $W_0^{1,p(\cdot)}(\Omega)$ of norm $\|\nabla \omega_\varepsilon\|_{p(\cdot)} \leq 1$ such that*

$$\omega_\varepsilon \rightharpoonup \omega \text{ in } W_0^{1,p(\cdot)}(\Omega), |\nabla \omega_\varepsilon|^{p(x)} \xrightarrow{*} \mu \text{ in } \mathbb{M}(\overline{\Omega}), |\omega_\varepsilon|^{p^*(x)} \xrightarrow{*} \nu \text{ in } \mathbb{M}(\overline{\Omega}).$$

Denote

$$C^* = \sup\left\{ \int_{\Omega} |\omega_\varepsilon|^{p^*(x)} dx \mid \omega_\varepsilon \in W_0^{1,p(\cdot)}(\Omega), \|\nabla \omega_\varepsilon\|_{p(\cdot)} \leq 1 \right\}$$

and then $0 < C^* < +\infty$. The limit measures are of the form

$$\mu = |\nabla \omega|^{p(x)} + \sum_{j \in J} \mu_j \delta_{x_j} + \tilde{\mu}, \mu(\overline{\Omega}) \leq 1, \nu = |\omega|^{p^*(x)} + \sum_{j \in J} \nu_j \delta_{x_j}, \nu(\overline{\Omega}) \leq C^*,$$

where $x_j \in \overline{\Omega}$, J is a countable set, $\tilde{\mu} \in \mathbb{M}(\overline{\Omega})$ is a nonatomic positive measure. The atoms and the regular part satisfy the generalized Sobolev inequality

$$\nu(\overline{\Omega}) \leq C^* \max\left\{ \mu(\overline{\Omega})^{\frac{p^*+}{p^-}}, \mu(\overline{\Omega})^{\frac{p^*-}{p^+}} \right\}, \nu_j \leq C^* \max\left\{ \mu_j^{\frac{p^*+}{p^-}}, \mu_j^{\frac{p^*-}{p^+}} \right\}.$$

We have the following version of the concentration-compactness principle:

Theorem 2.7. *Assume Ω is an open bounded domain in \mathbb{R}^N , p is Lipschitz continuous on $\overline{\Omega}$ satisfying $1 < p(x) < N$, $1 < q(x) \leq p^*(x)$, $\varepsilon_n \rightarrow 0^+$ as $n \rightarrow +\infty$. Let $\{\omega_n\}$ be a sequence in $W_0^{1,p(\cdot)}(\Omega)$ of norm $\|\nabla \omega_n\|_{p(\cdot)} \leq 1$ such that*

$$\omega_n \rightharpoonup \omega \text{ in } W_0^{1,p(\cdot)}(\Omega), |\nabla \omega_n|^{p(x)} \xrightarrow{*} \mu \text{ in } \mathbb{M}(\overline{\Omega}), |\omega_n|^{q(x)-\varepsilon_n} \xrightarrow{*} \nu^\# \text{ in } \mathbb{M}(\overline{\Omega}).$$

Denote

$$C_q^* = \sup\{|\omega|_{q(\cdot)}^{q^+} + 1 \mid \omega \in W_0^{1,p(\cdot)}(\Omega), |\nabla\omega_\varepsilon|_{p(\cdot)} \leq 1\},$$

and then $0 < C_q^* < +\infty$. The limit measures are of the form

$$\begin{aligned} \mu &= |\nabla\omega|^{p(x)} + \sum_{j \in J} \mu_j \delta_{x_j} + \tilde{\mu}, \mu(\overline{\Omega}) \leq 1, \\ v^\# &= |\omega|^{q(x)} + \sum_{j \in J} v_j^\# \delta_{x_j}, v^\#(\overline{\Omega}) \leq C_q^* + 1, \\ v^\#(\{x\}) &\leq v(\{x\}), \forall x \in \Omega, \end{aligned}$$

where $x_j \in \overline{\Omega}$, J is a countable set, and $\tilde{\mu} \in \mathbb{M}(\overline{\Omega})$ is a non-atomic positive measure. The atoms and the regular part satisfy the generalized Sobolev inequality

$$v^\#(\overline{\Omega}) \leq C_q^* \max\{\mu(\overline{\Omega})^{\frac{q^+}{p^-}}, \mu(\overline{\Omega})^{\frac{q^-}{p^+}}\}, v_j^\# \leq C_q^* \max\{\mu_j^{\frac{q^+}{p^-}}, \mu_j^{\frac{q^-}{p^+}}\}.$$

In order to prove the Theorem 2.7, we need the following Lemma.

Lemma 2.8. *Assume Ω is an open bounded domain in \mathbb{R}^N , $\{f_n\}$ is bounded in $L^{p(\cdot)}(\Omega)$ and $f_n \rightarrow f \in L^{p(\cdot)}(\Omega)$ a.e. on Ω . If $1 < p(x) < N$, and $\varepsilon_n \rightarrow 0^+$ as $n \rightarrow +\infty$, then*

$$\lim_{n \rightarrow \infty} \left\{ \int_\Omega |f_n|^{p(x)-\varepsilon_n} dx - \int_\Omega |f_n - f|^{p(x)-\varepsilon_n} dx \right\} = \int_\Omega |f|^{p(x)} dx.$$

Proof. Without loss of generality, we may assume that $\varepsilon_n \leq \frac{1}{2}$, $n = 1, 2, \dots$. It is easy to see that

$$\lim_{n \rightarrow \infty} \int_\Omega |f|^{p(x)-\varepsilon_n} dx = \int_\Omega |f|^{p(x)} dx.$$

Now it suffices to show that

$$(1) \quad \lim_{n \rightarrow \infty} \left\{ \int_\Omega |f_n|^{p(x)-\varepsilon_n} dx - \int_\Omega |f_n - f|^{p(x)-\varepsilon_n} dx - \int_\Omega |f|^{p(x)-\varepsilon_n} dx \right\} = 0.$$

Denote

$$W_{\varepsilon,n}(x) = \left[\left| |f_n|^{p(x)-\varepsilon_n} - |f_n - f|^{p(x)-\varepsilon_n} - |f|^{p(x)-\varepsilon_n} \right| - \varepsilon |f_n - f|^{p(x)-\varepsilon_n} \right]_+,$$

where $[a]_+ = \max\{a, 0\}$. Obviously, $W_{\varepsilon,n}(x) \rightarrow 0$ a.e. on Ω , as $n \rightarrow \infty$.

Similarly to the proof of Lemma 2.1 of [24], we have

$$\left| |f_n|^{p(x)-\varepsilon_n} - |f_n - f|^{p(x)-\varepsilon_n} - |f|^{p(x)-\varepsilon_n} \right| \leq \varepsilon |f_n - f|^{p(x)-\varepsilon_n} + C(\varepsilon) |f|^{p(x)-\varepsilon_n},$$

where ε and $C(\varepsilon)$ are independent of n .

Therefore, $W_{\varepsilon,n}(x) \leq C(\varepsilon) |f|^{p(x)-\varepsilon_n} \leq C(\varepsilon)(|f|^{p(x)}+1) \in L^1(\Omega)$. By Lebesgue's Dominated Convergence Theorem, we have $\int_{\Omega} W_{\varepsilon,n}(x)dx \rightarrow 0$, as $n \rightarrow \infty$. Similarly to the proof of Lemma 2.1 of [24], we can see that (1) holds. ■

Proof of Theorem 2.7. There exists a subsequence of $\{u_n\}$ (for simplicity we still denote it as $\{u_n\}$) such that $u_n(x) \rightarrow u(x)$ a.e. on Ω . By Lemma 2.8, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\Omega'} |u_n|^{q(x)-\varepsilon_n} dx - \int_{\Omega'} |u|^{q(x)} dx &= \liminf_{n \rightarrow \infty} \int_{\Omega'} |u_n - u|^{q(x)-\varepsilon_n} dx, \\ \liminf_{n \rightarrow \infty} \int_{\Omega} |u_n|^{q(x)-\varepsilon_n} \eta dx - \int_{\Omega} |u|^{q(x)} \eta dx &= \liminf_{n \rightarrow \infty} \int_{\Omega} |u_n - u|^{q(x)-\varepsilon_n} \eta dx, \end{aligned}$$

for every sub-domain $\Omega' \subset \Omega$ and $\eta \in C(\overline{\Omega})$. Thus

$$\bar{v}^\# = v^\# - |u|^{q(x)} = \sum_{j \in J} v_j^\# \delta_{x_j} + \tilde{v}^\#,$$

with non-atomics $\tilde{v}^\# \in \mathbb{M}(\overline{\Omega})$. Similarly to the proof of Theorem 3.1 of [24], we can see that $\tilde{v}^\# = 0$. Thus $v^\# = |\omega|^{q(x)} + \sum_{j \in J} v_j^\# \delta_{x_j}$.

For any $x_0 \in \overline{\Omega}$ and $\forall \varepsilon > 0$, let $\phi_\varepsilon \in C(\mathbb{R}^N)$ with $\phi_\varepsilon(x) \geq 0, \phi_\varepsilon(x_0) = 1, \phi_\varepsilon(x) = 0$ when $|x - x_0| \geq \varepsilon$. We have

$$\begin{aligned} v^\#(\{x_0\}) &\leq \int_{\Omega} \phi_\varepsilon(x) v^\# dx = \lim_{n \rightarrow \infty} \int_{\Omega} \phi_\varepsilon(x) |u_n|^{q(x)-\varepsilon_n} dx \\ &\leq \lim_{n \rightarrow \infty} \int_{\Omega} \phi_\varepsilon(x) \{|u_n|^{q(x)} + 1\} dx \leq \int_{\Omega} \phi_\varepsilon(x) v dx + C\varepsilon^N. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we have $v^\#(\{x_0\}) \leq v(\{x_0\})$. ■

3. PROPERTIES OF OPERATORS AND VARIATIONAL PRINCIPLE

In the following, we will discuss the properties of the $p(x)$ -Laplacian operator and Nemytsky operator. Also, we will present several variational principles.

From now on, the letters $c, c_i, C, C_i, i = 1, 2, \dots$, denote positive constants which may vary from line to line but are independent of the terms which take part in any limit process. Denote $X := W_0^{1,p(\cdot)}(\Omega)$. We Consider the following functional

$$J(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx, u \in X.$$

Obviously (see [12]), $J \in C^1(X, \mathbb{R})$ and it is weak lower semi-continuous. Denote $L = J' : X \rightarrow X^*$, then we have

$$(L(u), v) = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\Omega} |u|^{p(x)-2} uv dx, \forall u, v \in X.$$

Proposition 3.1. (see [21]).

- (i) $L : X \rightarrow X^*$ is continuous, bounded and strictly monotone;
- (ii) L is a mapping of type (S_+) , i.e. if $u_n \rightharpoonup u$ in X and $\overline{\lim}_{n \rightarrow \infty} (L(u_n) - L(u), u_n - u) \leq 0$, then $u_n \rightarrow u$ in X ;
- (iii) $L : X \rightarrow X^*$ is a homeomorphism.

Denote $F(x, u) = \int_0^u f(x, t) dt$, $\Psi(u) = \int_{\Omega} F(x, u) dx$, $(\Psi'(u), v) = \int_{\Omega} f(x, u) v dx$. The corresponding functional of (P) is

$$\varphi(u) = J(u) - \Psi(u), \forall u \in X.$$

Then we have the following theorem.

Theorem 3.2.

- (i) $\Psi \in C^1(X, \mathbb{R})$;
- (ii) If $f(x, u)/|u|^{p^*(x)-1} \rightarrow 0$ as $|u| \rightarrow \infty$, then Ψ and Ψ' are weak-strong continuous, i.e., $u_n \rightharpoonup u$ implies $\Psi(u_n) \rightarrow \Psi(u)$ and $\Psi'(u_n) \rightarrow \Psi'(u)$.

Proof.

- (i) From the continuity of the Nemytsky operator, we can see that both Ψ and Ψ' are continuous.
- (ii) Since $u_n \rightharpoonup u$, we have $|u_n - u|_{p(\cdot)} \rightarrow 0$ and $u_n \rightarrow u$ a.e. on $\overline{\Omega}$. Thus, $F(x, u_n(x)) \rightarrow F(x, u(x))$ a.e. on $\overline{\Omega}$. Clearly,

$$\int_U |F(x, u_n)| dx \leq \int_U [\varepsilon |u_n|^{p^*(x)} + C(\varepsilon)] dx, \forall U \subset \Omega,$$

then $\{|F(x, u_n)|\}$ is uniformly integrable, and then $\{|F(x, u_n) - F(x, u)|\}$ is uniformly integrable. Noticing the boundedness of the domain Ω , we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} |F(x, u_n) - F(x, u)| dx = \int_{\Omega} \lim_{n \rightarrow \infty} |F(x, u_n) - F(x, u)| dx = 0.$$

Similarly, we can get the weak-strong continuity of Ψ' . ■

Since X is a reflexive and separable Banach space, there are sequences $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\text{span}}\{e_j, j = 1, 2, \dots\}, \quad X^* = \overline{\text{span}}^{w^*}\{e_j^*, j = 1, 2, \dots\},$$

and $\langle e_j^*, e_i \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$

For convenience, we write

(2) $X_j = \text{span}\{e_j\}, Y_k = \bigoplus_{j=1}^k X_j, Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$

Definition 3.3. (i) We say φ satisfies (PS) condition in X , if any sequence $\{u_n\} \subset X$ such that $\{\varphi(u_n)\}$ is bounded and $\|\varphi'(u_n)\|_{X^*} \rightarrow 0$ as $n \rightarrow \infty$, has a convergent subsequence; (ii) We say φ satisfies $(PS)_c^*$ condition in X , if any sequence $\{u_{n_j}\} \subset X$ such that $n_j \rightarrow \infty, u_{n_j} \in Y_{n_j}, \varphi(u_{n_j}) \rightarrow c$ and $(\varphi|_{Y_{n_j}})'(u_{n_j}) \rightarrow 0$, contains a subsequence converging to a critical point of φ .

Let $f(x, t) = g(x, t) + h(x, t)$. Denote

$$G(x, t) = \int_0^t g(x, s)ds, H(x, t) = \int_0^t h(x, s)ds.$$

We assume

(B₁). There exist a positive constant M and a function $\theta(\cdot) \in C^1(\overline{\Omega})$ satisfying

$$p(x) < \theta(x) \leq p^*(x), \forall x \in \overline{\Omega},$$

such that h satisfies

$$0 < H(x, s) \leq \frac{s}{\theta(x)}h(x, s), \forall x \in \overline{\Omega}, |s| \geq M.$$

(B₂). For the function $\theta(\cdot)$ in **(B₁)**, there exists a small positive constant δ such that g satisfies

$$|g(x, s)| \leq |s|^{\frac{\theta(x)}{1+\delta}-1}, \forall x \in \overline{\Omega}, |s| \geq M.$$

Lemma 3.4. *If **(B₁)** and **(B₂)** are satisfied, then every (PS) sequence of φ in X is bounded.*

Proof. The conditions **(B₁)** and **(B₂)** together imply that

$$H(x, s) \geq |s|^{\theta(x)}, \forall x \in \overline{\Omega}, \text{ when } |s| \text{ is large enough,}$$

$$|G(x, s)| + |sg(x, s)| \leq (1 + \theta(x)) |s|^{\frac{\theta(x)}{1+\delta}} + C, \forall (x, s) \in \overline{\Omega} \times \mathbb{R}.$$

Denote

$$l_1 = \min_{x \in \overline{\Omega}} \left(\frac{1}{p(x)} - \frac{1 + \delta}{\theta(x)} \right),$$

where the positive constant δ is small enough such that $l_1 > 0$.

Let $\{u_n\}$ be a (PS) sequence. Since $\theta \in C^1(\overline{\Omega})$, we have

$$\begin{aligned}
& c + \|u_n\|_{p(\cdot)} \\
& \geq \varphi(u_n) - (\varphi'(u_n), \frac{1+\delta}{\theta(x)}u_n) \\
& = \int_{\Omega} \frac{1}{p(x)} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) dx - \int_{\Omega} F(x, u_n) dx \\
& \quad - \int_{\Omega} \frac{1+\delta}{\theta(x)} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) dx \\
& \quad + \int_{\Omega} \frac{1+\delta}{\theta(x)} u_n f(x, u_n) dx + \int_{\Omega} \frac{1+\delta}{\theta^2(x)} u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \theta(x) dx \\
& \geq \int_{\Omega} (\frac{1}{p(x)} - \frac{1+\delta}{\theta(x)}) (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) dx + \int_{\Omega} \frac{\delta}{\theta(x)} H(x, u_n) dx \\
& \quad - \int_{\Omega} \frac{(1+\delta)|\nabla \theta(x)|}{\theta^2(x)} |u_n| |\nabla u_n|^{p(x)-1} dx - \int_{\Omega} C_1 |u_n|^{\frac{\theta(x)}{1+\delta}} dx - C_2 \\
& \geq l_1 \int_{\Omega} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) dx + \int_{\Omega} \frac{\delta}{\theta(x)} H(x, u_n) dx \\
& \quad - \int_{\Omega} \frac{(1+\delta)|\nabla \theta(x)|}{\theta^2(x)} |u_n| |\nabla u_n|^{p(x)-1} dx - \int_{\Omega} C_1 |u_n|^{\frac{\theta(x)}{1+\delta}} dx - C_2.
\end{aligned}$$

Since $\theta \in C(\bar{\Omega})$ it follows that

$$\begin{aligned}
& \frac{(1+\delta)|\nabla \theta(x)|}{\theta^2(x)} |u_n| |\nabla u_n|^{p(x)-1} \\
& \leq C_4 \frac{1}{p(x)} (\frac{1}{\varepsilon_1} |u_n|)^{p(x)} + C_4 \frac{p(x)-1}{p(x)} (\varepsilon_1 |\nabla u_n|^{p(x)-1})^{\frac{p(x)}{p(x)-1}} \\
& = C_4 \frac{1}{p(x)} \frac{1}{\varepsilon_1^{p(x)}} |u_n|^{p(x)} + C_4 \frac{p(x)-1}{p(x)} \varepsilon_1^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)}.
\end{aligned}$$

It is not hard to see that

$$\begin{aligned}
\int_{\Omega} C_1 |u_n|^{\frac{\theta(x)}{1+\delta}} dx & \leq \int_{\Omega} \left\{ \frac{\delta}{1+\delta} (\frac{C_1}{\varepsilon_1})^{\frac{1+\delta}{\delta}} + \frac{1}{1+\delta} (\varepsilon_1 |u_n|^{\frac{\theta(x)}{1+\delta}})^{1+\delta} \right\} dx \\
& = C_5 + \int_{\Omega} \frac{\varepsilon_1}{1+\delta} |u_n|^{\theta(x)} dx.
\end{aligned}$$

When the positive constants ε_1 is small enough, we have

$$\begin{aligned}
c + \|u_n\|_{p(\cdot)} & \geq \varphi(u_n) - (\varphi'(u_n), \frac{1+\delta}{\theta(x)}u_n) \\
& \geq \frac{2l_1}{3} \int_{\Omega} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) dx - C_6.
\end{aligned}$$

Thus $\{\|u_n\|_{p(\cdot)}\}$ is bounded. ■

Lemma 3.5. *If (B_1) and (B_2) are satisfied, $\{u_n\}$ is a bounded (PS) sequence of φ , then there exists a small enough positive constant C_0 such that, if f satisfies*

$$|f(x, s)| \leq C + C_0 |s|^{p^*(x)-1}, \quad \forall x \in \overline{\Omega},$$

then $\{u_n\}$ has a convergent subsequence in X .

Proof. Let $\{u_n\}$ be a (PS) sequence of φ , i.e.

$$\varphi(u_n) \rightarrow c, \varphi'(u_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since $\{u_n\}$ is bounded, there exists a $u \in X$, such that $u_n \rightharpoonup u$ in X . By Proposition 2.6, we may assume that there exist $\mu, \nu \in \mathbb{M}(\overline{\Omega})$ and sequence $\{x_j\}_{j \in J}$ in $\overline{\Omega}$ such that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } W_0^{1,p(\cdot)}(\Omega), \\ |\nabla u_n|^{p(x)} &\overset{*}{\rightharpoonup} \mu = |\nabla u|^{p(x)} + \sum_{j \in J} \mu_j \delta_{x_j} + \tilde{\mu}, \text{ in } \mathbb{M}(\overline{\Omega}), \\ |u_n|^{p^*(x)} &\overset{*}{\rightharpoonup} \nu = |u|^{p^*(x)} + \sum_{j \in J} \nu_j \delta_{x_j}, \text{ in } \mathbb{M}(\overline{\Omega}), \\ \nu_j &\leq C_{p^*}^* \max\left\{\mu_j^{\frac{p^+}{p^-}}, \mu_j^{\frac{p^-}{p^+}}\right\}, \end{aligned}$$

where

$$C_{p^*}^* = \sup\{|\omega|_{p^*(\cdot)}^{p^{*+}} + 1 \mid \omega \in W_0^{1,p(x)}(\Omega), |\nabla \omega|_{p(\cdot)} \leq 1\}$$

and $0 < C_{p^*}^* < +\infty$.

Next we will complete the proof of this Theorem in three steps.

Step 1. We claim that $\mu(\{x_j\}) = \nu(\{x_j\}) = 0$ for all $j = 1, 2, \dots$.

Obviously, there exists $r_n > 0$ such that

$$\begin{aligned} p^-(x_n) &:= \inf_{y \in B_r(x_n) \cap \overline{\Omega}} p(y) \leq p^+(x_n) := \sup_{y \in B_r(x_n) \cap \overline{\Omega}} p(y) \\ &< p^{*-}(x_n) := \inf_{y \in B_r(x_n) \cap \overline{\Omega}} p^*(y) \leq p^{*+}(x_n) := \sup_{y \in B_r(x_n) \cap \overline{\Omega}} p^*(y), \quad \forall r \in (0, r_n]. \end{aligned}$$

For every $\varepsilon > 0$, we set $\phi_\varepsilon(x) = \phi(\frac{x-x_1}{\varepsilon})$, $x \in \Omega$, where $\phi \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $B_1\{0\}$ and $\phi \equiv 0$ in $\mathbb{R}^N \setminus B_2\{0\}$ and $|\nabla \phi| \leq 2$. Noting that $\varphi'(u_n) \rightarrow 0$ in X^* as $n \rightarrow \infty$ and $\{u_n\}$ is bounded, we have

$$\begin{aligned}
& \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla(\phi_{\varepsilon} u_n) dx + \int_{\Omega} |u_n|^{p(x)-2} u_n \phi_{\varepsilon} u_n dx \\
&= \int_{\Omega} f(x, u_n) \phi_{\varepsilon} u_n dx + o(1) \\
&\leq \int_{\Omega} (C |u_n| + C_0 |u_n|^{p^*(x)}) \phi_{\varepsilon} dx + o(1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(3) \quad & \int_{\Omega} \phi_{\varepsilon} |\nabla u_n|^{p(x)} dx + \int_{\Omega} |u_n|^{p(x)} \phi_{\varepsilon} dx + \int_{\Omega} u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_{\varepsilon} dx \\
&\leq \int_{\Omega} (C |u_n| + C_0 |u_n|^{p^*(x)}) \phi_{\varepsilon} dx + o(1).
\end{aligned}$$

Due to the bounded-ness of $\{u_n\}$ in X , we may assume

$$|\nabla u_n|^{p(x)-2} \nabla u_n \rightharpoonup T \in (L^{p^0(\cdot)}(\Omega))^N, \quad f(x, u_n) \rightharpoonup g(x) \in L^{(p^*(\cdot))^0}(\Omega).$$

Noting $\varphi'(u_n) \rightarrow 0$ in X^* as $n \rightarrow \infty$, we also have

$$\int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla(\phi_{\varepsilon} u) dx + \int_{\Omega} |u_n|^{p(x)-2} u_n \phi_{\varepsilon} u dx = \int_{\Omega} f(x, u_n) \phi_{\varepsilon} u dx + o(1).$$

Thus,

$$(4) \quad \int_{\Omega} T \cdot \nabla(\phi_{\varepsilon} u) dx + \int_{\Omega} |u|^{p(x)} \phi_{\varepsilon} dx = \int_{\Omega} f(x, u) u \phi_{\varepsilon} dx.$$

We claim

$$(5) \quad \int_{\Omega} u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_{\varepsilon} dx \rightarrow \int_{\Omega} u T \nabla \phi_{\varepsilon} dx \text{ as } n \rightarrow \infty.$$

In fact,

$$\begin{aligned}
& \int_{\Omega} \{u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_{\varepsilon} - u T \nabla \phi_{\varepsilon}\} dx \\
&= \int_{\Omega} (u_n - u) |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_{\varepsilon} dx \\
&\quad + \int_{\Omega} u \nabla \phi_{\varepsilon} \{|\nabla u_n|^{p(x)-2} \nabla u_n - T\} dx \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

From (3), (4) and (5) it follows that

$$\begin{aligned} & \int_{\Omega} \phi_{\varepsilon} d\mu + \int_{\Omega} |u|^{p(x)} \phi_{\varepsilon} dx \\ & \leq \int_{\Omega} C |u| \phi_{\varepsilon} dx + \int_{\Omega} C_0 \phi_{\varepsilon} dv - \int_{\Omega} u T \nabla \phi_{\varepsilon} dx \\ & = \int_{\Omega} C |u| \phi_{\varepsilon} dx + \int_{\Omega} C_0 \phi_{\varepsilon} dv \\ & \quad - \left\{ \int_{\Omega} f(x, u) u \phi_{\varepsilon} dx - \int_{\Omega} |u|^{p(x)} \phi_{\varepsilon} dx - \int_{\Omega} \phi_{\varepsilon} T \cdot \nabla u dx \right\}. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we get $\mu(\{x_1\}) \leq C_0 v(\{x_1\})$ or $\mu_1 \leq C_0 v_1$.

Similarly, $\mu(\{x_j\}) \leq C_0 v(\{x_j\})$ or $\mu_j \leq C_0 v_j, j = 2, 3, \dots$.

Suppose that $\mu(\{x_j\}) > 0$ for some j . Since $\{u_n\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, there is a constant M_* such that $\int_{\Omega} |u_n|^{p^*(x)} dx \leq M_* < \infty$ for all n . If $\mu(\{x_j\}) \geq 1$, then

$v_j \geq \frac{1}{C_0} \left| \frac{v_j}{C_{p^*}^*} \right|^{\frac{p^-(x_j)}{p^{*+}(x_j)}}$. It can be rewrited as

$$v_j \geq \left| C_0 (C_{p^*}^*)^{\frac{p^-(x_j)}{p^{*+}(x_j)}} \right|^{\frac{-1}{1 - \frac{p^-(x_j)}{p^{*+}(x_j)}}}.$$

Similarly, if $\mu(\{x_j\}) < 1$ then $v_j \geq \frac{1}{C_0} \left| \frac{v_j}{C_{p^*}^*} \right|^{\frac{p^+(x_j)}{p^{*-}(x_j)}}$ and

$$v_j \geq \left| C_0 (C_{p^*}^*)^{\frac{p^+(x_j)}{p^{*-}(x_j)}} \right|^{\frac{-1}{1 - \frac{p^+(x_j)}{p^{*-}(x_j)}}}.$$

Due to the definition of M_* , we also have

$$\sum_{\mu(\{x_j\}) \geq 1} v_j + \sum_{\mu(\{x_j\}) < 1} v_j \leq M_*.$$

Noting that M_* is a constant which is only dependent on $\{u_n\}$. When C_0 (depending on M_*) is small enough, we reach a contradiction. Step 1 is completed.

Step 2. We claim that $u_n \rightarrow u$ strongly in $L^{p^*(x)}(\Omega)$ as $n \rightarrow \infty$.

Since $|u_n|^{p^*(x)} \xrightarrow{*} \nu = |u|^{p^*(x)}$, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^{p^*(x)} dx = \int_{\Omega} |u|^{p^*(x)} dx,$$

which together with $|u_n|^{p^*(x)} \rightarrow |u|^{p^*(x)}$ in measure implies that $\{|u_n|^{p^*(x)}\}$ is uniformly integrable.

Obviously,

$$|u_n - u|^{p^*(x)} \leq 2^{p^*(x)} (|u_n|^{p^*(x)} + |u|^{p^*(x)}).$$

Thus $\{|u_n - u|^{p^*(x)}\}$ is uniformly integrable. Therefore

$$\lim_{n \rightarrow \infty} \int_{\Omega} |u_n - u|^{p^*(x)} dx = \int_{\Omega} \lim_{n \rightarrow \infty} |u_n - u|^{p^*(x)} dx = 0.$$

Step 3. We claim that $u_n \rightarrow u$ strongly in X as $n \rightarrow \infty$.

Since $\varphi'(u_n) = J'(u_n) - \Psi'(u_n) \rightarrow 0$ and $u_n \rightarrow u$ strong in $L^{p^*(x)}(\Omega)$ as $n \rightarrow \infty$, we have $\Psi'(u_n) \rightarrow \Psi'(u)$ and $J'(u_n) \rightarrow \Psi'(u)$ as $n \rightarrow \infty$. As $L = J'$ is a homeomorphism, we have $u_n \rightarrow L^{-1}(\Psi'(u))$ in X as $n \rightarrow \infty$. ■

Lemma 3.6. Assume $\Theta : X \rightarrow \mathbb{R}$ is weakly-strongly continuous and $\Theta(0) = 0$, $\gamma > 0$ is a fixed number. Let

$$(6) \quad \beta_k = \beta_k(\gamma) = \sup \{ \Theta(u) \mid \|u\| \leq \gamma, u \in Z_k \},$$

then $\beta_k \rightarrow 0$ as $k \rightarrow \infty$.

Lemma 3.7. (see [23]). If $|u(\cdot)|^{q(\cdot)} \in L^{s(\cdot)/q(\cdot)}(\Omega)$, where $s(\cdot), q(\cdot) \in L_+^\infty(\Omega)$, $q(x) \leq s(x)$, then $u \in L^{s(\cdot)}(\Omega)$ and there is a number $\bar{q} \in [q^-, q^+]$ such that $\left| |u(\cdot)|^{q(\cdot)} \right|_{s(\cdot)/q(\cdot)} = (|u|_{s(\cdot)})^{\bar{q}}$.

Proposition 3.8. (Fountain theorem, see [52, 53]). Assume X is a Banach space, $\varphi \in C^1(X, \mathbb{R})$ is an even functional and satisfies (PS) condition, the subspace X_k, Y_k and Z_k are defined by (2). If for each $k = 1, 2, \dots$, there exist $\rho_k > \gamma_k > 0$ such that

$$(A_1) \quad \alpha_k := \inf \{ \varphi(u) \mid u \in Z_k, \|u\| = \gamma_k \} \rightarrow \infty \quad (k \rightarrow \infty);$$

$$(A_2) \quad \beta_k := \max \{ \varphi(u) \mid u \in Y_k, \|u\| = \rho_k \} \leq 0.$$

then φ has a sequence of critical values tending to $+\infty$.

Proposition 3.9. (Dual Fountain theorem, see [53]) Assume X is a Banach space, $\varphi \in C^1(X, \mathbb{R})$ is an even functional, the subspace X_k, Y_k and Z_k are defined by (2), and there is a $k_0 > 0$ such that, for each $k \geq k_0$, there exists $\rho_k > \gamma_k > 0$ such that

$$(D_1) \quad \inf \{ \varphi(u) \mid u \in Z_k, \|u\| = \rho_k \} \geq 0,$$

$$(D_2) \quad \zeta_k := \max \{ \varphi(u) \mid u \in Y_k, \|u\| = \gamma_k \} < 0,$$

$$(D_3) \quad \eta_k := \inf \{ \varphi(u) \mid u \in Z_k, \|u\| \leq \rho_k \} \rightarrow 0 \quad (k \rightarrow \infty),$$

$$(D_4) \quad \varphi \text{ satisfies } (PS)_c^* \text{ condition for every } c \in [\eta_{k_0}, 0),$$

then φ has a sequence of critical values c^k tending to 0. Moreover, $c^k \in [\eta_k, \zeta_k]$.

4. MAIN RESULTS AND PROOFS

In this section, we study the existence of infinitely many pairs of solutions via Fountain Theorem and the dual Fountain Theorem stated before, respectively.

Definition 4.1 . We say $u \in X$ is a weak solution of (P) provided

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v dx + \int_{\Omega} |u|^{p(x)-2} u \cdot v dx = \int_{\Omega} f(x, u) v dx, \forall v \in X.$$

It is easy to see that the critical points of φ correspond to the weak solutions of (P).

Remark 1. Regarding the p -Laplacian equations with the critical Sobolev growth conditions, there are many results showing that c is a critical value of φ , when the value c is less than some real number c_{∞} which depends on the best imbedding constant

$$S_p := \inf \left\{ \int_{\Omega} |\nabla u|^p dx \mid \int_{\Omega} |u|^{p^*} dx = 1 \right\} = \left(\inf_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{|\nabla u|_p}{|u|_{p^*}} \right)^p.$$

Due to the non-homogeneity in problems involving $p(x)$ -Laplacian, we could only get the best imbedding constant $C = \inf_{u \in X \setminus \{0\}} \frac{|\nabla u|_{p(\cdot)}}{|u|_{p^*(\cdot)}}$. We can see that $\left(\inf_{u \in X \setminus \{0\}} \frac{|\nabla u|_{p(\cdot)}}{|u|_{p^*(\cdot)}}$ is a function dependent on the variable x . It is difficult to get the similar results.

As an application of Theorem 2.7, we show the existence of infinitely many pairs of solutions by a perturbation argument.

Theorem 4.2. *If $f(x, t) = \mu |t|^{\alpha(x)-2} t + \lambda |t|^{q(x)-2} t, \forall (x, t) \in \bar{\Omega} \times \mathbb{R}$, satisfying $1 < q(x) \leq p^*(x), \alpha^+ < p^-$ and*

$$(7) \quad q^- > p^+,$$

and the positive parameters λ and μ satisfy one of the following conditions

(1⁰) λ is fixed, and μ is small enough;

(2⁰) $\frac{q^+}{q^+ - p^-} > \frac{q^-}{q^- - \alpha^+}$, μ is fixed and λ is small enough;

(3⁰) $\frac{q^+}{q^+ - p^-} < \frac{q^-}{q^- - \alpha^+}$, $\lambda \rightarrow 0^+$ and $\mu \rightarrow +\infty$ such that $(\frac{1}{\lambda})^{\frac{q^-}{q^- - \alpha^+} - \frac{q^+}{q^+ - p^-}} \mu^{\frac{q^-}{q^- - \alpha^+}}$ is small enough;

(4⁰) $\lambda \rightarrow 0^+$ and $\mu \rightarrow 0^+$ such that $(\frac{1}{\lambda})^{\frac{q^-}{q^- - \alpha^+} - \frac{q^+}{q^+ - p^-}} \mu^{\frac{q^+}{q^+ - \alpha^-}}$ is small enough;

(5⁰) $\lambda \rightarrow +\infty$ and $\mu \rightarrow 0^+$ such that $(\frac{1}{\lambda})^{\frac{q^+}{q^+ - \alpha^-} - \frac{q^-}{q^- - p^+}} \mu^{\frac{q^+}{q^+ - \alpha^-}}$ is small enough;

then (P) has a sequence of pairs of solutions $\{\pm u_n\}$ such that $\varphi(\pm u_n) < 0$ and $\varphi(\pm u_n) \rightarrow 0$.

Proof. Let's consider a sequence of perturbation problems as follows.

$$(P_n) \quad \begin{cases} -div(|\nabla u|^{p(x)-2} \nabla u) + |u|^{p(x)-2} u = f_n(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$f_n(x, t) = \mu |t|^{\alpha(x)-2} t + \lambda |t|^{q(x)-2-\varepsilon_n} t, \forall (x, t) \in \overline{\Omega} \times \mathbb{R},$$

where ε_n is decreasing, $\varepsilon_1 < q^- - p^+$ and $\varepsilon_n \rightarrow 0^+$. The corresponding functional of (P_n) is

$$\varphi_n(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx - \int_{\Omega} F_n(x, u) dx,$$

where $F_n(x, u) = \int_0^u f_n(x, t) dt$.

For the functionals φ_n , we will check the conditions of Proposition 3.9 item by item.

Assume $\rho_k = 1$ and k is large enough. For any $u \in Z_k$ with $\|u\|_{p(\cdot)} = \rho_k$, it follows from $q^- > p^+$, Theorem 3.2 and Lemma 3.6 that

$$\varphi_n(u) \geq C_k > 0.$$

Hence (D_1) is satisfied.

Assume $\gamma_k < \rho_k$ is small enough. For any $u \in Y_k$ with $\|u\|_{p(\cdot)} = \gamma_k$, we have

$$\varphi_n(u) \leq \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx - \int_{\Omega} \frac{\mu |u|^{\alpha(x)}}{\alpha(x)} dx \triangleq b_k < 0.$$

So (D_2) is satisfied.

When $\|u\|_{p(\cdot)}$ is small enough, by direct computations, we have

$$\begin{aligned} \varphi_n(u) &= \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx \\ &\quad - \int_{\Omega} \frac{\mu |u|^{\alpha(x)}}{\alpha(x)} dx - \int_{\Omega} \frac{\lambda}{q(x) - \varepsilon_n} |u|^{q(x)-\varepsilon_n} dx \\ &\geq \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx \\ &\quad - \int_{\Omega} \frac{\mu |u|^{\alpha(x)}}{\alpha(x)} dx - \lambda \left| \frac{1}{q(x) - \varepsilon_n} \right|_{\frac{q(x)}{\varepsilon_n}} |u|_{q(\cdot)}^{q(\zeta)-\varepsilon_n} \\ &\geq \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx \\ &\quad - \int_{\Omega} \frac{\mu |u|^{\alpha(x)}}{\alpha(x)} dx - C\lambda \|u\|_{p(\cdot)}^{q(\zeta)-\varepsilon_n} \end{aligned}$$

$$\begin{aligned} &\geq \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx \\ &\quad - \int_{\Omega} \frac{\mu |u|^{\alpha(x)}}{\alpha(x)} dx - C\lambda \|u\|_{p(\cdot)}^{q(\zeta) - \varepsilon_1}. \end{aligned}$$

Noting that $q^- > p^+$, when k is large enough and $\rho_k = 1$, $\forall u \in Z_k$, $\|u\|_{p(\cdot)} \leq \rho_k$, we have

$$\varphi_n(u) \geq - \int_{\Omega} \frac{\mu |u|^{\alpha(x)}}{\alpha(x)} dx \geq -2C\mu \|u\|_{p(\cdot)} \beta_k(1) \geq -2C\mu \beta_k(1) \triangleq d_k \rightarrow 0 \text{ as } k \rightarrow \infty,$$

where $\beta_k(1)$ is defined in (6) with $\Theta(u) = \int_{\Omega} \frac{|u|^{\alpha(x)}}{\alpha(x)} dx$. Hence (D_3) is satisfied.

Similarly to the proof of Lemma 3.4, we see that every $(PS)_c^*$ sequence is bounded. Similarly to the proof of the Theorem 4.6 of [55], we see that (D_4) is satisfied.

For any $n = 1, 2, \dots$, from Proposition 3.9, we see that φ_n has a sequence of critical values $c_n^k \in [d_k, b_k]$, and for every c_n^k , φ_n has a related critical point u_n^k .

If $u_n^k \rightarrow u^k \in X$ as $n \rightarrow +\infty$, then it is easy to see that u^k is a critical point of φ , and the critical value of $\varphi(u^k) = c^k = \lim_{n \rightarrow \infty} c_n^k \in [d_k, b_k]$. Thus u^k is a nontrivial solution to (P). Since $d_k \rightarrow 0^-$ as $k \rightarrow +\infty$, we can see that φ has infinitely many solutions u^k such that $\varphi(u^k) \rightarrow 0^-$ as $k \rightarrow +\infty$.

It only remains to prove that $u_n^k \rightarrow u^k$ as $n \rightarrow \infty$ when k is large enough.

We only need to prove that $u_n^{k_0} \rightarrow \omega$ in X and the rest is completely similar.

Since $u_n^{k_0}$ is a critical point of φ_n , we have

$$(8) \quad c_n^{k_0} = J(u_n^{k_0}) - \int_{\Omega} \frac{1}{q(x) - \varepsilon_n} \lambda |u_n^{k_0}|^{q(x) - \varepsilon_n} dx - \int_{\Omega} \frac{1}{\alpha(x)} \mu |u_n^{k_0}|^{\alpha(x)} dx,$$

and

$$(9) \quad \int_{\Omega} |\nabla u_n^{k_0}|^{p(x)} dx + \int_{\Omega} |u_n^{k_0}|^{p(x)} dx = \int_{\Omega} \lambda |u_n^{k_0}|^{q(x) - \varepsilon_n} dx + \int_{\Omega} \mu |u_n^{k_0}|^{\alpha(x)} dx.$$

It follows from (8) and (9) that

$$c_n^{k_0} = \left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2) - \varepsilon_n}\right) \lambda \int_{\Omega} |u_n^{k_0}|^{q(x) - \varepsilon_n} dx + \left(\frac{1}{p(\xi_1)} - \frac{1}{\alpha(\xi_3)}\right) \mu \int_{\Omega} |u_n^{k_0}|^{\alpha(x)} dx,$$

and then

$$\begin{aligned} \left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2) - \varepsilon_n}\right) \lambda \int_{\Omega} |u_n^{k_0}|^{q(\xi_4) - \varepsilon_n} dx &= \left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2) - \varepsilon_n}\right) \lambda \int_{\Omega} |u_n^{k_0}|^{q(x) - \varepsilon_n} dx \\ &= c_n^{k_0} + \left(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)}\right) \mu \int_{\Omega} |u_n^{k_0}|^{\alpha(x)} dx \\ &\leq c_n^{k_0} + \left(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)}\right) \mu c_1 \int_{\Omega} |u_n^{k_0}|^{\alpha(\xi_5)} dx, \end{aligned}$$

where $\xi_i \in \bar{\Omega}$, $i = 1, \dots, 5$, $c_1 = 2 \sup_n |1|_{\frac{q(\cdot)-\varepsilon_n}{q(\cdot)-\varepsilon_n-\alpha(\cdot)}} < +\infty$. Thus

$$\left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2) - \varepsilon_n}\right) \lambda \left|u_n^{k_0}\right|_{q(\cdot)-\varepsilon_n}^{q(\xi_4)-\varepsilon_n} \leq \left(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)}\right) \mu c_1 \left|u_n^{k_0}\right|_{q(\cdot)-\varepsilon_n}^{\alpha(\xi_5)}$$

when n is large enough.

Therefore

$$(10) \quad \left|u_n^{k_0}\right|_{q(\cdot)-\varepsilon_n} \leq \left[\frac{\mu\left(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)}\right)c_1}{\lambda\left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n}\right)}\right]^{\frac{1}{q(\xi_4)-\alpha(\xi_5)-\varepsilon_n}},$$

which implies that

$$\begin{aligned} & \left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2) - \varepsilon_n}\right) \lambda \int_{\Omega} \left|u_n^{k_0}\right|^{q(x)-\varepsilon_n} dx = \left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2) - \varepsilon_n}\right) \lambda \left|u_n^{k_0}\right|_{q(\cdot)-\varepsilon_n}^{q(\xi_4)-\varepsilon_n} \\ & \leq \left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2) - \varepsilon_n}\right) \left[\frac{\left(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)}\right)c_1}{\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n}}\right]^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} \left(\frac{1}{\lambda}\right)^{\frac{\alpha(\xi_5)}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} \mu^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}}. \end{aligned}$$

Thus,

$$\begin{aligned} & c_n^{k_0} + \left(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)}\right) \mu \int_{\Omega} \left|u_n^{k_0}\right|^{\alpha(x)} dx \\ & = \left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2) - \varepsilon_n}\right) \lambda \int_{\Omega} \left|u_n^{k_0}\right|^{q(x)-\varepsilon_n} dx \\ & \leq \left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2) - \varepsilon_n}\right) \left[\frac{\left(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)}\right)c_1}{\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n}}\right]^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} \left(\frac{1}{\lambda}\right)^{\frac{\alpha(\xi_5)}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} \mu^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}}. \end{aligned}$$

Then we have

$$\begin{aligned} J_n(u_n^{k_0}) & = c_n^{k_0} + \int_{\Omega} \frac{1}{q(x) - \varepsilon_n} \lambda \left|u_n^{k_0}\right|^{q(x)-\varepsilon_n} dx + \int_{\Omega} \frac{\mu}{\alpha(x)} \left|u_n^{k_0}\right|^{\alpha(x)} dx \\ & \leq C \left\{ 2 \left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)}\right) \left[\frac{\left(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)}\right)c_1}{\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n}}\right]^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} \right. \\ & \quad \left. \left(\frac{1}{\lambda}\right)^{\frac{\alpha(\xi_5)}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} \mu^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} + |d_{k_0}| \right\}. \end{aligned}$$

It implies that $\{u_n^{k_0}\}$ is bounded.

From Theorem 2.7, we have

$$u_n^{k_0} \rightharpoonup \omega \text{ in } W_0^{1,p(\cdot)}(\Omega), \quad \left|\nabla u_n^{k_0}\right|^{p(x)} \overset{*}{\rightharpoonup} \mu \text{ in } \mathbb{M}(\bar{\Omega}), \quad \left|u_n^{k_0}\right|^{q(x)-\varepsilon_n} \overset{*}{\rightharpoonup} v^\# \text{ in } \mathbb{M}(\bar{\Omega}).$$

Set

$$C_q^* = \sup\{|\omega_\varepsilon|_{q(\cdot)}^{q^+} + 1 \mid \omega_\varepsilon \in W_0^{1,p(\cdot)}(\Omega), |\nabla \omega_\varepsilon|_{p(\cdot)} \leq 1\}$$

and $0 < C_q^* < +\infty$. The limit measures are of the form

$$\begin{aligned} \mu &= |\nabla \omega|^{p(x)} + \sum_{j \in J} \mu_j \delta_{x_j} + \tilde{\mu}, \mu(\bar{\Omega}) \leq 1, \\ v^\# &= |\omega|^{p^*(x)} + \sum_{j \in J} v_j^\# \delta_{x_j}, v^\#(\bar{\Omega}) \leq C_q^* + |\Omega|, \end{aligned}$$

where $x_j \in \bar{\Omega}$, J is a countable set, $\tilde{\mu} \in M(\bar{\Omega})$ is a nonatomic measure.

Similarly to the proof of Lemma 3.5, we have

$$\mu_j \leq \lambda v_j^\# \leq \lambda v_j \leq \lambda C_q^* \max\{\mu_j^{\frac{q^+}{p^-}}, \mu_j^{\frac{q^-}{p^+}}\},$$

and then

$$\begin{aligned} v_j^\# &\geq \left| \lambda (C_q^*)^{\frac{p^-}{q^+}} \right|^{\frac{-1}{1-\frac{p^-}{q^+}}}, \text{ when } \mu(\{x_j\}) \geq 1, \\ v_j^\# &\geq \left| \lambda (C_q^*)^{\frac{p^+}{q^-}} \right|^{\frac{-1}{1-\frac{p^+}{q^-}}}, \text{ when } 0 < \mu(\{x_j\}) < 1, \end{aligned}$$

which implies that there exist only finite $v_j^\# \neq 0$. Without loss of generality, we may assume that $v_j^\# > 0$ for exact $j = 1, \dots, k$, and then

$$\sum_{j \in J} v_j^\# = \sum_{\mu(\{x_j\}) \geq 1} v_j^\# + \sum_{0 < \mu(\{x_j\}) < 1} v_j^\# \geq \sum_{\mu(\{x_j\}) \geq 1} a \left(\frac{1}{\lambda}\right)^{\frac{q^+}{q^+ - p^-}} + \sum_{0 < \mu(\{x_j\}) < 1} a \left(\frac{1}{\lambda}\right)^{\frac{q^-}{q^- - p^+}}.$$

Therefore

$$(11) \quad \sum_{j \in J} v_j^\# \geq \begin{cases} b \left(\frac{1}{\lambda}\right)^{\frac{q^+}{q^+ - p^-}}, \lambda < 1 \\ b \left(\frac{1}{\lambda}\right)^{\frac{q^-}{q^- - p^+}}, \lambda \geq 1 \end{cases},$$

where b is a constant which is independent on (n, λ, μ) .

On the other hand,

$$\int_{\Omega} |u_n^{k_0}|^{q(x) - \varepsilon_n} dx = |u_n^{k_0}|_{q(\cdot) - \varepsilon_n}^{q(\xi_4) - \varepsilon_n} \leq \left[\frac{(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)}) C_1}{\lambda (\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2) - \varepsilon_n})} \right]^{\frac{q(\xi_4) - \varepsilon_n}{q(\xi_4) - \varepsilon_n - \alpha(\xi_5)}} \mu^{\frac{q(\xi_4) - \varepsilon_n}{q(\xi_4) - \varepsilon_n - \alpha(\xi_5)}},$$

which implies that

$$(12) \quad \sum_{j=1}^k v_j^\# \leq \lim_{n \rightarrow \infty} \int_{\Omega} |u_n^{k_0}|^{q(x) - \varepsilon_n} dx \leq c_\# \left(\frac{1}{\lambda}\right)^{\frac{q(\xi_4)}{q(\xi_4) - \alpha(\xi_5)}} \mu^{\frac{q(\xi_4)}{q(\xi_4) - \alpha(\xi_5)}},$$

where $c_{\#}$ is a constant independent on (λ, μ, n) .

From (11) and (12), we have

$$\left. \begin{aligned} & b\left(\frac{1}{\lambda}\right)^{\frac{q^+}{q^+-p^-}}, \lambda < 1 \\ & b\left(\frac{1}{\lambda}\right)^{\frac{q^-}{q^-+p^+}}, \lambda \geq 1 \end{aligned} \right\} \leq \sum_{j=1}^k v_j^{\#} \leq \begin{cases} c_{\#}\left(\frac{1}{\lambda}\right)^{\frac{q^-}{q^-+\alpha^+}} \mu^{\frac{q^+}{q^+-\alpha^-}}, \lambda < 1, \mu < 1, \\ c_{\#}\left(\frac{1}{\lambda}\right)^{\frac{q^-}{q^-+\alpha^+}} \mu^{\frac{q^-}{q^-+\alpha^+}}, \lambda < 1, \mu \geq 1, \\ c_{\#}\left(\frac{1}{\lambda}\right)^{\frac{q^+}{q^+-\alpha^-}} \mu^{\frac{q^+}{q^+-\alpha^-}}, \lambda \geq 1, \mu < 1, \\ c_{\#}\left(\frac{1}{\lambda}\right)^{\frac{q^+}{q^+-\alpha^-}} \mu^{\frac{q^-}{q^-+\alpha^+}}, \lambda \geq 1, \mu \geq 1, \end{cases} .$$

Under one of the conditions of (1⁰)-(5⁰), we get a contradiction.

Thus $v_j^{\#} = \mu_j = 0$ for $j = 1, 2, \dots$. Therefore

$$\lim_{n \rightarrow \infty} \int_{\Omega} |u_n^{k_0}|^{q(x)-\varepsilon_n} dx = \int_{\Omega} |\omega|^{q(x)} dx.$$

Obviously, $|u_n^{k_0}|^{q(x)-\varepsilon_n} \rightarrow |\omega|^{q(x)}$ a.e. on Ω . Thus $\{|u_n|^{q(x)-\varepsilon_n}\}$ is uniformly integrable. It is easy to see that

$$|u_n^{k_0}|^{q(x)-\frac{q(x)}{q(x)-1}\varepsilon_n} \leq 1 + |u_n^{k_0}|^{q(x)-\varepsilon_n} .$$

Then $\{|u_n^{k_0}|^{q(x)-\frac{q(x)}{q(x)-1}\varepsilon_n}\}$ is uniformly integrable. Obviously, $|u_n^{k_0}|^{q(x)-\varepsilon_n-2} u_n^{k_0} \rightarrow |\omega|^{q(x)-2} \omega$ a.e. on Ω , then

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left| |u_n^{k_0}|^{q(x)-\varepsilon_n-2} u_n^{k_0} - |\omega|^{q(x)-2} \omega \right|^{\frac{q(x)}{q(x)-1}} dx = 0.$$

It means that $\Psi'_n(u_n^{k_0}) \rightarrow \Psi'(\omega)$ in X^* , where $\Psi_n(u_n^{k_0}) = \int_{\Omega} F_n(x, u_n^{k_0}) dx$. Thus $u_n^{k_0} \rightarrow L^{-1}(\Psi'(\omega))$. ■

Remark 2. In [10], Cao and Yan dealt with the existence of infinitely many solutions of the following Laplacian equation Dirichlet problems involving critical nonlinearities and Hardy potential

$$\text{(I)} \quad \begin{cases} -\Delta u - \frac{\mu}{|x|^2} u = |u|^{2^*-2} u + au \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases}$$

by considering the following perturbed problem which is of subcritical growth,

$$\text{(I}_n\text{)} \quad \begin{cases} -\Delta u - \frac{\mu}{|x|^2} u = |u|^{2^*-2-\varepsilon_n} u + au \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases}$$

By applying a local Pohozaev identity, Cao and Yan proved that the solution $\{u_n\}$ of (I_n) converges to a solution u of (I) . Because of the nonhomogeneity of $p(x)$ -Laplacian, the Pohozaev identity can't be obtained by the usual methods. So the method in [10] is very hard to be directly used in dealing with the $p(x)$ -Laplacian problems.

Assume that

$$(13) \quad f(x, t) = g(x, t) + h(x) |t|^{p^*(x)-2} t, \forall (x, t) \in \overline{\Omega} \times \mathbb{R}.$$

Denote

$$\Phi(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx - \int_{\Omega} G(x, u) dx.$$

where $G(x, t) = \int_0^t g(x, s) ds$.

Theorem 4.3. *Assume that (13) is satisfied, and (P) satisfies the following conditions (F_1) - (F_5) :*

(F_1) Ω is a radially symmetric domain with respect to the origin, $1 < p < N$ and the function p is radial, i.e.

$$p(x) = p(|x|) \text{ for any } x \in \overline{\Omega},$$

(F_2) g and h are radial with respect to the space variable x , i.e.

$$g(x, t) = g(|x|, t) \text{ and } h(x) = h(|x|) \text{ for any } (x, t) \in \overline{\Omega} \times \mathbb{R},$$

(F_3) $f(x, -t) = -f(x, t)$ for any $(x, t) \in \overline{\Omega} \times \mathbb{R}$,

(F_4) $|g(x, t)| \leq C(1 + |t|^{\theta(x)-1})$ for any $(x, t) \in \overline{\Omega} \times \mathbb{R}$, where the function $\theta(\cdot) \in C^1(\overline{\Omega})$ satisfies $p(x) < \theta(x) < p^*(x)$,

(F_5) $h(0) = 0$, $h \in C(\overline{\Omega})$ and $[h(\cdot)]^{\frac{-\theta(\cdot)}{p^*(\cdot)-\theta(\cdot)}} \in L^1(\Omega)$.

Then (P) has a sequence of radial solutions $\{\pm u_m\}$ such that $\varphi(\pm u_m) \rightarrow +\infty$.

Proof. First, we will prove that φ satisfies the (PS) condition.

Let $\{u_n\}$ be a (PS) sequence. We claim that $\{u_n\}$ is bounded. Similarly to the proof of Lemma 3.4, we have

$$\begin{aligned} & c + \|u_n\|_{p(\cdot)} \\ & \geq \varphi(u_n) - (\varphi'(u_n), \frac{1}{\theta(x)} u_n) \\ & = \int_{\Omega} \frac{1}{p(x)} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) dx \\ & \quad - \int_{\Omega} F(x, u_n) dx - \int_{\Omega} \frac{1}{\theta(x)} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) dx \\ & \quad + \int_{\Omega} \frac{1}{\theta(x)} u_n f(x, u_n) dx + \int_{\Omega} \frac{1}{\theta^2(x)} u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \theta(x) dx \end{aligned}$$

$$\begin{aligned}
 &\geq \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{\theta(x)}\right) (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) dx + \int_{\Omega} \left(\frac{1}{\theta(x)} - \frac{1}{p^*(x)}\right) h(x) |u_n|^{p^*(x)} dx \\
 &\quad - \int_{\Omega} \frac{|\nabla \theta(x)|}{\theta^2(x)} |u_n| |\nabla u_n|^{p(x)-1} dx - 2C_1 \int_{\Omega} |u_n|^{\theta(x)} dx - C_2 \\
 &\geq l_1 \int_{\Omega} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) dx + \int_{\Omega} \left(\frac{1}{\theta(x)} - \frac{1}{p^*(x)}\right) h(x) |u_n|^{p^*(x)} dx \\
 &\quad - \int_{\Omega} \frac{|\nabla \theta(x)|}{\theta^2(x)} |u_n| |\nabla u_n|^{p(x)-1} dx - 2C_1 \int_{\Omega} |u_n|^{\theta(x)} dx - C_2,
 \end{aligned}$$

where $l_1 = \inf_{x \in \Omega} \left(\frac{1}{p(x)} - \frac{1}{\theta(x)}\right)$.

Since $[h(\cdot)]^{\frac{-\theta(\cdot)}{p^*(\cdot)-\theta(\cdot)}} \in L^1(\Omega)$ and $\theta \in C^1(\overline{\Omega})$, similarly to the proof of Lemma 3.4, we have

$$\begin{aligned}
 &\frac{|\nabla \theta(x)|}{\theta^2(x)} |u_n| |\nabla u_n|^{p(x)-1} \\
 &\leq C_3 \frac{1}{p(x)} \frac{1}{\varepsilon_1^{p(x)}} \left\{ \frac{p^*(x) - p(x)}{p^*(x)} \varepsilon_1^{\frac{-p^*(x)p(x)}{p^*(x)-p(x)}} [h(x)]^{\frac{-p(x)}{p^*(x)-p(x)}} + \frac{p(x)}{p^*(x)} \varepsilon_1^{p^*(x)} h(x) |u_n|^{p^*(x)} \right\} \\
 &\quad + C_3 \frac{p(x) - 1}{p(x)} \varepsilon_1^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)},
 \end{aligned}$$

and

$$\begin{aligned}
 |u_n|^{\theta(x)} &\leq \frac{p^*(x) - \theta(x)}{p^*(x)} \left\{ \frac{1}{\varepsilon_1} [h(x)]^{\frac{-\theta(x)}{p^*(x)}} \right\}^{\frac{p^*(x)}{p^*(x)-\theta(x)}} + \frac{\theta(x)}{p^*(x)} \left\{ \varepsilon_1 [h(x)]^{\frac{\theta(x)}{p^*(x)}} |u_n|^{\theta(x)} \right\}^{\frac{p^*(x)}{\theta(x)}} \\
 &= \frac{p^*(x) - \theta(x)}{p^*(x)} \left(\frac{1}{\varepsilon_1}\right)^{\frac{p^*(x)}{p^*(x)-\theta(x)}} [h(x)]^{\frac{-\theta(x)}{p^*(x)-\theta(x)}} + \frac{\theta(x)}{p^*(x)} (\varepsilon_1)^{\frac{p^*(x)}{\theta(x)}} h(x) |u_n|^{p^*(x)}.
 \end{aligned}$$

Noting that $\frac{\theta(x)}{p^*(x)-\theta(x)} > \frac{p(x)}{p^*(x)-p(x)}$, when positive constants ε_1 is small enough, we have

$$c + \|u_n\|_{p(\cdot)} \geq \varphi(u_n) - (\varphi'(u_n), \frac{1}{\theta(x)} u_n) \geq \frac{2l_1}{3} \int_{\Omega} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) dx - C_4.$$

Thus $\{u_n\}$ is bounded. Consequently,

$$u_n \rightharpoonup u \text{ in } W_0^{1,p(\cdot)}(\Omega), \quad |\nabla u_n|^{p(x)} \overset{*}{\rightharpoonup} \mu \text{ in } \mathbb{M}(\overline{\Omega}), \quad |u_n|^{p^*(x)} \overset{*}{\rightharpoonup} \nu \text{ in } \mathbb{M}(\overline{\Omega}).$$

Set

$$C_{p^*}^* = \sup \{ |\omega_{\varepsilon}|_{p^*(\cdot)}^{p^*+} + 1 \mid \omega_{\varepsilon} \in W_0^{1,p(\cdot)}(\Omega), |\nabla \omega_{\varepsilon}|_{p(\cdot)} \leq 1 \}$$

and $0 < C_{p^*}^* < +\infty$. According to Proposition 2.6, the limit measures are of the form

$$\begin{aligned} \mu &= |\nabla u|^{p(x)} + \sum_{j \in J} \mu_j \delta_{x_j} + \tilde{\mu}, \mu(\bar{\Omega}) \leq 1, \\ v &= |u|^{p^*(x)} + \sum_{j \in J} \nu_j \delta_{x_j}, v(\bar{\Omega}) \leq C^*, \end{aligned}$$

where $x_j \in \bar{\Omega}$, and J is a countable set, and $\tilde{\mu} \in M(\bar{\Omega})$ is a nonatomic positive measure.

Similarly to the proof of Lemma 3.5, we may assume that $\mu_j > 0$ for $j = 1, \dots, k$, and the rest are zero.

If $x_k \neq 0$, since (P) is radial, it is easy to see that for any $x \in \bar{\Omega}$ such that $|x| = |x_k|$ and $x \notin \{x_1, \dots, x_k\}$, we have $\mu(x) = \mu(x_k)$. It is a contradiction. Thus $\mu(x) = 0$ for any $x \neq 0$.

For every $\varepsilon > 0$, we set $\phi_\varepsilon(x) = \phi(\frac{x}{\varepsilon})$, $x \in \bar{\Omega}$, where $\phi \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $B_1\{0\}$ and $\phi \equiv 0$ in $\mathbb{R}^N \setminus B_2\{0\}$ and $|\nabla \phi| \leq 2$. Since $\varphi'(u_n) \rightarrow 0$ in X^* as $n \rightarrow \infty$ and $\{u_n\}$ is bounded, we have $(\varphi'(u_n), \phi_\varepsilon u_n) \rightarrow 0$, and then

$$\begin{aligned} (\varphi'(u_n), \phi_\varepsilon u_n) &= \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla(\phi_\varepsilon u_n) dx \\ &\quad + \int_{\Omega} |u_n|^{p(x)-2} u_n \phi_\varepsilon u_n dx - \int_{\Omega} f(x, u_n) \phi_\varepsilon u_n dx \\ &= \int_{\Omega} |\nabla u_n|^{p(x)} \phi_\varepsilon dx + \int_{\Omega} |u_n|^{p(x)} \phi_\varepsilon dx \\ &\quad + \int_{\Omega} u_n |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \phi_\varepsilon dx - \int_{\Omega} f(x, u_n) \phi_\varepsilon u_n dx \rightarrow 0. \end{aligned}$$

Without loss of generality, we may assume that $u_n \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$. From (5), we have

$$(14) \quad \int_{\Omega} \phi_\varepsilon d\mu + \int_{\Omega} |u|^{p(x)} \phi_\varepsilon dx + \int_{\Omega} uT \cdot \nabla \phi_\varepsilon dx - \int_{\Omega} g(x, u) \phi_\varepsilon u dx - \int_{\Omega} h(x) \phi_\varepsilon dv = 0.$$

It is easy to see that

$$\begin{aligned} \left| \int_{\Omega} uT \cdot \nabla \phi_\varepsilon dx \right| &\leq |T|_{\frac{p(\cdot)}{p(\cdot)-1}} \|u \nabla \phi_\varepsilon\|_{p(\cdot)}, \\ \int_{\Omega} \left| |\nabla \phi_\varepsilon|^{p(x)} \right|^{\left(\frac{p^*(x)}{p(x)}\right)^0} dx &= \int_{B(0,2\varepsilon)} |\nabla \phi_\varepsilon|^N dx \leq \left(\frac{2}{\varepsilon}\right)^N \text{meas} B(0, 2\varepsilon) = \frac{4^N}{N} \varpi_N, \end{aligned}$$

where ϖ_N is the Hausdorff measure of the unit ball of \mathbb{R}^N , and

$$\int_{\Omega} |u \nabla \phi_\varepsilon|^{p(x)} dx = \int_{B(0,2\varepsilon)} |u \nabla \phi_\varepsilon|^{p(x)} dx \leq 2 \left\| |\nabla \phi_\varepsilon|^{p(x)} \right\|_{\left(\frac{p^*(\cdot)}{p(\cdot)}\right)^0} \left\| |u|^{p(x)} \right\|_{\frac{p^*(\cdot)}{p(\cdot)}, B(0,2\varepsilon)}.$$

Then, we have

$$(15) \quad \left| \int_{\Omega} uT \cdot \nabla \phi_{\varepsilon} dx \right| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

It follows from (14) and (15) that

$$(16) \quad \mu(0) - h(0)v(0) = 0.$$

Combining (F₂), (F₅) and (16), we have $\mu(0) = 0$. Thus $\mu(x) = 0$ for any $x \in \overline{\Omega}$. Proposition 2.6 guaranties that $v(x) = 0$ for any $x \in \overline{\Omega}$. Thus $v_j = \mu_j = 0$ for $j = 1, 2, \dots$. Thus

$$\lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^{p^*(x)} dx = \int_{\Omega} |u|^{p^*(x)} dx.$$

Noting that we also have $u_n \rightarrow u$ a.e. in Ω , we conclude that $\{|u_n|^{p^*(\cdot)}\}$ is uniformly integrable. Thus $\{|u_n - u|^{p^*(\cdot)}\}$ is uniformly integrable. According to the Vitaly Theorem, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} |u_n - u|^{p^*(x)} dx = \int_{\Omega} \lim_{n \rightarrow \infty} |u_n - u|^{p^*(x)} dx = 0.$$

It means that $\Psi'(u_n) \rightarrow \Psi'(u)$ in X^* . Then $u_n \rightarrow u_1^* := L^{-1}(\Psi'(u))$. Therefore, $u_n \rightarrow u$ in $W_0^{1,p(\cdot)}(\Omega)$. So φ satisfies the (PS) condition.

Next, we will prove that $\Psi(\cdot)$ is weak-strong continuous.

If $u_n \rightharpoonup u$, we only need to prove

$$(17) \quad \lim_{n \rightarrow \infty} \int_{\Omega} h(x) |u_n|^{p^*(x)} dx = \int_{\Omega} h(x) |u|^{p^*(x)} dx.$$

Obviously, $\{u_n\}$ is bounded in X . From the above proof, we see that $v\{x\} = 0$ when $x \neq 0$. Therefore, for any $\delta > 0$, we have

$$\int_{\Omega \setminus \overline{B(0,\delta)}} |h(x) |u_n|^{p^*(x)} - h(x) |u|^{p^*(x)}| dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

For any $\varepsilon > 0$, it follows from the boundedness of $\{u_n\}$ that

$$\begin{aligned} & \int_{\Omega} |h(x) |u_n|^{p^*(x)} - h(x) |u|^{p^*(x)}| dx \\ &= \int_{\Omega \setminus \overline{B(0,\delta)}} |h(x) |u_n|^{p^*(x)} - h(x) |u|^{p^*(x)}| dx + \int_{\overline{B(0,\delta)}} |h(x) |u_n|^{p^*(x)} - h(x) |u|^{p^*(x)}| dx \\ &\leq \max_{x \in \overline{B(0,\delta)}} |h(x)| \int_{\Omega} (|u_n|^{p^*(x)} + |u|^{p^*(x)}) dx + \frac{\varepsilon}{2} \leq \varepsilon, \end{aligned}$$

when δ is small enough and n is large enough. Thus (17) is valid.

Similarly to the proof of Theorem 4.3 in [55], we see that the conditions (A_1) and (A_2) of Proposition 3.8 (Fountain Theorem) are satisfied. So φ has a sequence of critical points $\{\pm u_m\}$ in $W_{0,r}^{1,p(\cdot)}(\Omega)$ such that $\varphi(\pm u_m) \rightarrow +\infty$, where

$$W_{0,r}^{1,p(\cdot)}(\Omega) = \left\{ u \in W_0^{1,p(\cdot)}(\Omega) \mid u \text{ is radial} \right\},$$

and it is easy to see that $\{\pm u_m\}$ are radial solutions of (P). ■

REFERENCES

1. B. Abdellaoui, V. Felli and I. Peral, Some remarks on systems of elliptic equations doubly critical in the whole \mathbb{R}^N , *Calc. Var.*, **34** (2009), 97-137.
2. E. Acerbi and G. Mingione, Regularity results for a class of functionals with nonstandard growth, *Arch. Ration. Mech. Anal.*, **156** (2001), 121-140.
3. E. Acerbi and G. Mingione, Gradient estimates for the $p(x)$ -Laplacian system, *J. Reine Angew. Math.*, **584** (2005), 117-148.
4. A. Almeida and P. Hästö, Besov spaces with variable smoothness and integrability, *J. Funct. Anal.*, **258(5)** (2010), 1628-1655.
5. C. O. Alves, Existence of solution for a degenerate $p(x)$ -Laplacian equation in \mathbb{R}^N , *J. Math. Anal. Appl.*, **345** (2008), 731-742.
6. S. Antontsev and S. Shmarev, Elliptic equations and systems with nonstandard growth conditions: Existence, uniqueness and localization properties of solutions, *Nonlinear Analysis*, **65(4)** (2006), 728-761.
7. T. Bartsch, S. J. Peng and Z. T. Zhang, Existence and non-existence of solutions to elliptic equations related to the Caffarelli-Kohn-Nirenberg inequalities, *Calc. Var.*, **30** (2007), 113-136.
8. H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.*, **36** (1983), 437-477.
9. H. Brézis and M. Willem, On some nonlinear equations with critical exponents, *J. Functional Analysis*, **255(9)** (2008), 2286-2298.
10. D. M. Cao and S. S. Yan, Infinitely many solutions for an elliptic problem involving critical Sobolev growth and Hardy potential, *Calc. Var.*, **38** (2010), 471-501.
11. J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, *Calc. Var.*, **3** (1995), 493-512.
12. K. C. Chang, *Critical point theory and applications*, Shanghai Scientific and Technology Press, Shanghai, 1986.
13. Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Applied Mathematics*, **66(4)** (2006), 1383-1406.

14. A. Coscia and G. Mingione, Hölder continuity of the gradient of $p(x)$ -harmonic mappings, *Comptes Rendus de l'Académie des Sciences - Ser: I - Mathematics* **328** (1999), 363-368.
15. Y. B. Deng and J. X. Wang, Nonexistence of radial node solutions for elliptic problems with critical Sobolev exponents, *Nonlinear Analysis*, **71** (2009), 172-178.
16. L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics 2017, Springer-Verlag, Berlin, 2011.
17. L. Diening, P. Hästö and A. Nekvinda, Open problems in variable exponent Lebesgue and Sobolev spaces, in: *FSDONA04 Proceedings*, P. Drábek, J. Rákosník (eds.), Milovy, Czech Republic, 2004, pp. 38-58.
18. Y. H. Ding and F. H. Lin, Solutions of perturbed Schrödinger equations with critical nonlinearity, *Calc. Var.*, **30** (2007), 231-249.
19. X. L. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424-446.
20. X. L. Fan, Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form, *J. Differential Equations*, **235** (2007), 397-417.
21. X. L. Fan and Q. H. Zhang, Existence of solutions for $p(x)$ -Laplacian Dirichlet problem, *Nonlinear Anal.*, **52** (2003), 1843-1852.
22. X. L. Fan, Q. H. Zhang and D. Zhao, Eigenvalues of $p(x)$ -Laplacian Dirichlet problem, *J. Math. Anal. Appl.*, **302** (2005), 306-317.
23. X. L. Fan, Solutions for $p(x)$ -Laplacian Dirichlet problems with singular coefficients, *J. Math. Anal. Appl.*, **312** (2005), 464-477.
24. Y. Q. Fu, The principle of concentration compactness in $L^{p(x)}$ spaces and its application, *Nonlinear Analysis*, **71(5-6)** (2009), 1876-1892.
25. Y. Q. Fu and X. Zhang, Multiple solutions for a class of $p(x)$ -Laplacian equations in \mathbb{R}^N involving the critical exponent, *Proc. R. Soc. Edinb. A*, **466** (2010), 1667-1686.
26. J. P. Garcia Azorero and I. Peral Alonso, Hardy inequalities and some critical elliptic and parabolic problems, *J. Differential Equations*, **144** (1998), 441-476.
27. N. Ghoussoub and C. Yuan, Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents, *Trans. Amer. Math. Soc.*, **12** (2000), 5703-5743.
28. Q. Q. Guo and P. C. Niu, Nodal and positive solutions for singular semilinear elliptic equations with critical exponents in symmetric domains, *J. Differential Equations*, **245(12)** (2008), 3974-3985.
29. Z. C. Guo, Q. Liu, J. B. Sun and B. Y. Wu, Reaction-diffusion systems with $p(x)$ -growth for image denoising, *Nonlinear Anal-Real.*, **12** (2011), 2904-2918.
30. Z. C. Guo, J. B. Sun, D. Z. Zhang and B. Y. Wu, Adaptive perona-malik model based on the variable exponent for image denoising, *IEEE T. Image Process*, **21** (2012), 958-967.
31. A. El Hamidi, Existence results to elliptic systems with nonstandard growth conditions, *J. Math. Anal. Appl.*, **300** (2004), 30-42.

32. P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen, The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values, *Potential Anal.*, **25** (2006), 205-222.
33. P. Harjulehto, P. Hästö and V. Latvala, Harnack's inequality for $p(\cdot)$ -harmonic functions with unbounded exponent p , *J. Math. Anal. Appl.*, **352(1)** (2009), 345-359.
34. P. Harjulehto, P. Hästö, V. Latvala and O. Toivanen, Critical variable exponent functionals in image restoration, *Appl. Math. Lett.*, Accepted.
35. D. Kang, On the elliptic problems with critical weighted Sobolev Hardy exponents, *Nonlinear Anal.*, **66(5)** (2007), 1037-1050.
36. T. Kopalani, Interpolation theorems for variable exponent Lebesgue spaces, *J. Funct. Anal.*, **257** (2009), 3541-3551.
37. O. Kováčik and J. Rákosník, On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, *Czechoslovak Math. J.*, **41** (1991), 592-618.
38. F. Li, Z. B. Li and L. Pi, Variable exponent functionals in image restoration, *Appl. Math. Comput.*, **216** (2010), 870-882.
39. S. Z. Lian, W. J. Gao, C. L. Cao and H. J. Yuan, Study of the solutions to a model porous medium equation with variable exponent of nonlinearity, *J. Math. Anal. Appl.*, **342(1)** (2008), 27-38.
40. P. L. Lions, The concentration-compactness principle in the calculus of variations; the limit case, *Part 1,2. Revista Mat. Iberoamericana*, **1(2,3)** (1985), 45-121, 145-201.
41. P. L. Lions, The concentration-compactness principle in the calculus of variations, locally compact case, *Part 1,2. Ann. Inst. H. Poincaré*, **1(1 & 4)** (1984), 109-145, 223-283.
42. P. Marcellini, Regularity and existence of solutions of elliptic equations with (p,q) -growth conditions, *J. Differ Equations*, **90** (1991), 1-30.
43. P. J. McKenna and W. Reichel, A priori bounds for semilinear equations and a new class of critical exponents for Lipschitz domains, *J. Functional Analysis*, **244(1)** (2007), 220-246.
44. M. Mihăilescu, Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$ -Laplace operator, *Nonlinear Analysis*, **67** (2007), 1419-1425.
45. M. Mihăilescu, G. Moroşanu and D. Stancu-Dumitru, Equations involving a variable exponent Grushin-type operator, *Nonlinearity*, **24** (2011), 2663-2680.
46. M. Mihăilescu, V. Rădulescu and D. Repovš, On a non-homogeneous eigenvalue problem involving a potential: An Orlicz-Sobolev space setting, *J. Math. Pures Appl.*, **93(2)** (2010), 132-148.
47. A. Moameni, Existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in \mathbb{R}^N , *J. Differential Equations*, **229(2)** (2006), 570-587.
48. E. Nakai and Y. Sawano, Hardy spaces with variable exponents and generalized Campanato spaces, *J. Funct. Anal.*, **262** (2012), 3665-3748.

49. M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Math 1784, Springer-Verlag, Berlin, 2000.
50. S. G. Samko, Densness of $C_0^\infty(\mathbb{R}^N)$ in the generalized Sobolev spaces $W^{m,p(x)}(\mathbb{R}^N)$, *Dokl. Ross. Akad. Nauk*, **369(4)** (1999), 451-454.
51. S. Samko, On a progress in the theory of Lebesgue spaces with variable exponent: Maximal and singular operators, *Integral Transforms Spec. Funct.*, **16** (2005), 461-482.
52. M. Struwe, "Variational Methods", *Applications to nonlinear partial differential equations and Hamiltonian systems*, Springer-Verlag, Berlin, 1990.
53. M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.
54. Q. H. Zhang, Existence and asymptotic behavior of positive solutions for variable exponent elliptic systems, *Nonlinear Analysis*, **70(1)** (2009), 305-316.
55. Q. H. Zhang, Existence of solutions for $p(x)$ -Laplacian equations with singular coefficients in \mathbb{R}^N , *J. Math. Anal. Appl.*, **348** (2008), 38-50.
56. Q. H. Zhang, Existence of positive solutions to a class of $p(x)$ -Laplacian equations with singular nonlinearities, *Appl. Math. Lett.*, Accepted.
57. X. X. Zhang and X. P. Liu, The local boundedness and Harnack inequality of $p(x)$ -Laplace equation, *J. Math. Anal. Appl.*, **332(1)** (2007), 209-218.
58. V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR. Izv.*, **29** (1987), 33-36.

Yuan Liang and Xianbin Wu
Junior College
Zhejiang Wanli University
Ningbo, Zhejiang 315100
P. R. China.
E-mail: wxb3210@zwu.edu.cn

Qihu Zhang
Department of Mathematics and Information Science
Zhengzhou University of Light Industry
Zhengzhou, Henan 450002
P. R. China.
E-mail: zhangqihu@yahoo.com
zhangqh1999@yahoo.com.cn

Chunshan Zhao
Department of Mathematical Sciences
Georgia Southern University
Statesboro, GA 30460
U.S.A.