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# PRELIMINARY STUDY OF AN OPTIMUM VIBRATION ABSORBER FOR A MULTI-MASS SYSTEM WITH MULTIPLE EXCITATION

BY

JOHN R. BURROWS

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A

THESIS

submitted to the faculty of the

SCHOOL OF MINES AND METALLURGY OF THE UNIVERSITY OF MISSOURI

in partial fulfillment of the work required for the

Degree of

MASTER OF SCIENCE, MECHANICAL ENGINEERING

Rolla, Missouri

1961

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Approved by

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# ABSTRACT

The object of this thesis is to demonstrate, by the use of a modern digital computer, a fast, efficient method to eliminate or minimize undesirable stress conditions in a multimass vibrating system with multiple excitation. The condition desired is obtained by the addition of an optimum tuned and damped dynamic vibration absorber, for one critical speed only.

Maximum stress is the criteria used for design here and not amplitude as has been used previously by all other authors.

This solution requires the applied torque to be reevaluated as an average torque and a number of half interger harmonics. This method also demonstrates the use of a Holzer Table with complex numbers to account for damping.

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# LIST OF SYMBOLS

$K, k$ = Spring Stiffness (lb/in) linear system

$K_n$ = Torsional Spring Stiffness (in-lb/rad) torsional system, where n = any subscript

$M, m$ = Mass ($\text{lb-sec}^2/\text{in}$) linear system

$J$ = Mass Moment of Inertia ($\text{in-lb-sec}^2$) torsional system

$P_o$ = Maximum Force, $P_o \sin wt$, (lb)

$x_n$ = Linear Displacement (in), where n = any subscript

$\dot{x}_n$ = Linear Velocity (in/sec), where n = any subscript

$\ddot{x}_n$ = Linear Acceleration ($\text{in/sec}^2$), where n = any subscript

$\theta_n$ = Angular Displacement (rad), where n = any subscript

$\dot{\theta}_n$ = Angular Velocity (rad/sec), where n = any subscript

$\ddot{\theta}_n$ = Angular Acceleration ($\text{rad/sec}^2$), where n = any subscript

$w$ = Circular Frequency (rad/sec)

$w_a = \sqrt{k/m}$ = Natural Frequency of Absorber (rad/sec )

$W_a = \sqrt{K/M}$ = Natural Frequency of Main System (rad/sec)

$x_{st} = P_o/K$ = Static Deflection of Main System (in)

$u = m/M$ = Mass Ratio = absorber mass/main mass

$f = w_a/W_a$ = Frequency Ratio (natural frequencies)

$g = w/w_n$ = Forced Frequency Ratio

$c$ = Damping Factor (lb-sec/in) linear system

$c_c = 2mW_n$ = "Critical" Damping (lb-sec/in) linear system

$c$ = Damping Factor (in-lb-sec/rad) torsional system

$j = i = \sqrt{-1}$ = Imaginary Unit

$L_e$ = Equivalent Length (in)

$L_n$ = Length of shaft (in), where n = any number

$T_n$ = Applied Torque (in-lb), where n = any number

Q = Order Number of Vibration (vib/rev)

# INTRODUCTION

The problem of this thesis is a very practical one. It consists of an in-line six cylinder engine with a flywheel used to drive a generator and excitor. This system is reduced to a multimass vibration system with periodic forcing functions. An investigation of the given system will disclose that the shaft stresses at certain critical speeds will become quite large, approaching infinity. These infinitely large stresses are reduced by the addition of a tuned and damped dynamic vibration absorber, for one critical speed only.

Until the advent of modern digital computers no other practical method was available for the solution of such a problem other than approximation and trial and error, using measuring devices on the actual system. This thesis will demonstrate a numerical solution of the problem.

It is the author's belief that the method demonstrated will be of economic value to those interested in the design of such systems.

The subject was chosen because of the author's interest in the field of vibrations and dynamics and a desire for further knowledge in these fields.

The author wishes to express his sincere thanks to Dr. T.R. Faucett for his suggestion of the subject and guidance in the solution of the problem. Thanks are also due Professor R.E. Lee for arranging computer time for the solution of the problem. Thanks are also given to Dr. A.J. Miles for his encouragement in this endeavor.

# REVIEW OF LITERATURE

Frahm (1), 1909, was the first to eliminate critical vibrations at one frequency by the addition of a tuned absorber to the vibrating system.

Den Hartog (1), gives important information concerning amplitude of vibration for the simple spring mass absorber system with and without damping.

Holzer (2), 1921, was the first to use numerical methods to solve multimass vibration problems. This method, known as the Holzer Table, when first developed, did not include externally applied torques or damping coefficients.

Church (2), demonstrates the method of using applied torques in the Holzer Table.

Porter (4), has evaluated the half integer harmonic coefficients, for all four cycle gasoline engines, to be used in a Fourier Series. These coefficients were used as applied torques to the cylinders in the solution of this problem.

Lewis (3), demonstrates a convenient method of handling damping coefficients in complex form.

Nestorides (5), gives an example of the Holzer Table with damping coefficients.

## DISCUSSION

The first section of the discussion is taken directly from Den Hartog (1). This analysis of the simple spring mass system with an absorber is covered here in order for the reader to have a basic understanding of the principles involved. The symbols and figures are taken from Den Hartog (1).

Figure (1-a) represents any spring mass system where the forcing function, $P_o \sin wt$ is at or dangerously close to the natural frequency of the given system. If due to physical limitations this condition cannot be altered, a dynamic vibration absorber must be added to the system to eliminate this undesirable condition. It will be shown that if the natural frequency of the absorber $\sqrt{k/m}$ is equal to the natural frequency w, of the forcing function the main mass will not vibrate.

From Figure (1-a) the differential equations of motion for the main mass and absorber are as follows:

(1)
$$M \ddot{x}_1 + x_1 (k + K) - K x_2 = P_o \sin wt$$
$$m \ddot{x}_2 + K (x_2 - x_1) = 0$$

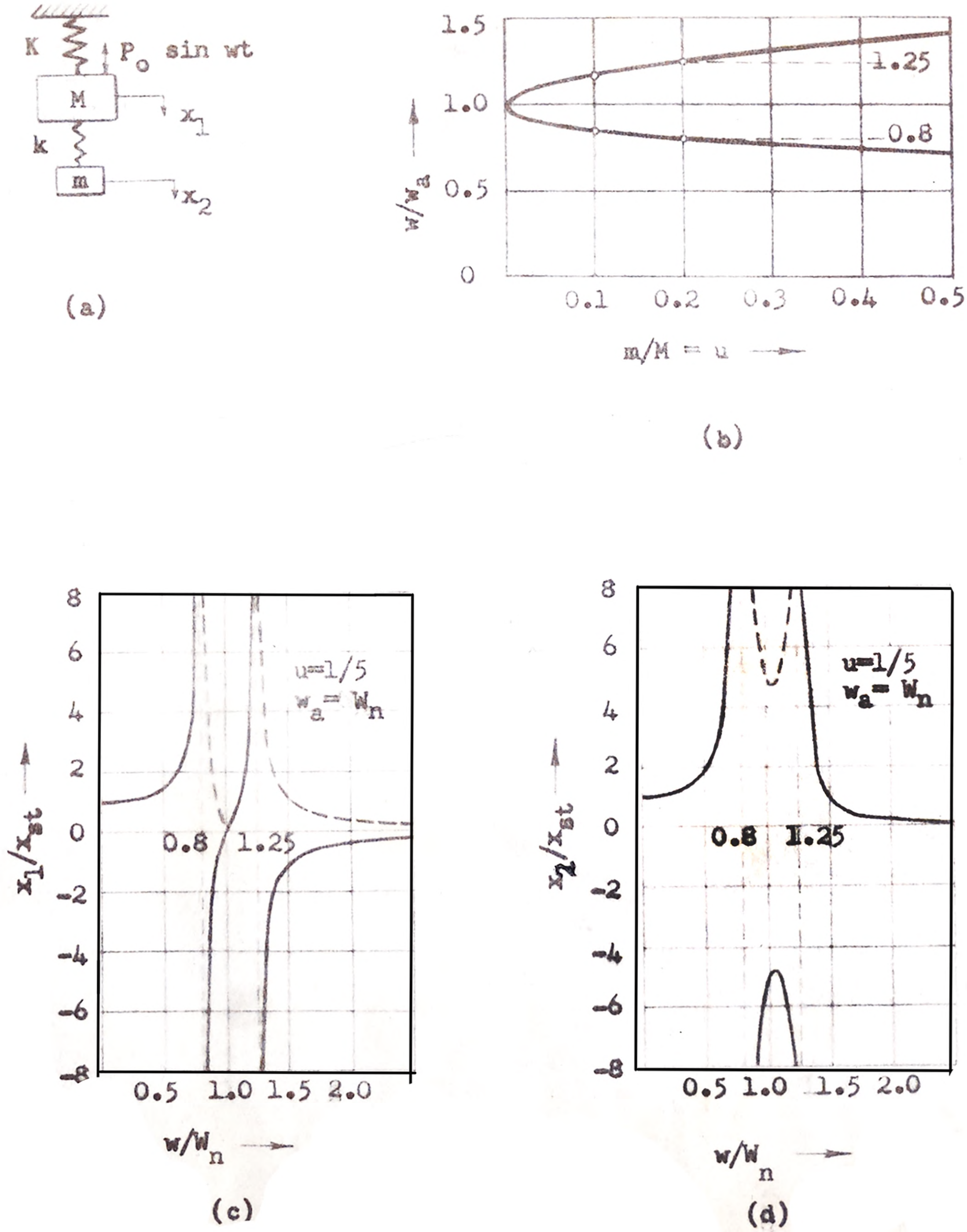
The vibrations assume the form

(2)
$$x_1 = a_1 \sin wt \qquad \dot{x}_1 = a_1 w \cos wt \qquad \ddot{x}_1 = -a_1 w^2 \sin wt$$
$$x_2 = a_2 \sin wt \qquad \dot{x}_2 = a_2 w \cos wt \qquad \ddot{x}_2 = -a_2 w^2 \sin wt$$

Substitution into the original equations gives

FIGURE 1

CURVES PERTAINING TO SIMPLE TUNED VIBRATION ABSORBER WITHOUT DAMPING



$$a_1 \, (-M \, w^2 + K + k) - k \, a_2 = P_o$$

(3)

$$-k \, a_1 + a_2 \, (-m \, w^2 + k) = 0$$

Dividing the first of equations (2) by K and the second by m gives

$$a_1 \left( 1 + \frac{k}{K} - \left[\frac{w}{W_n}\right]^2 \right) - a_2 \frac{k}{K} = x_{st}$$

(4)

$$a_1 = a_2 \left( 1 - \left[\frac{w}{w_a}\right]^2 \right)$$

Where

$x_{st} = P_o / K$ = static deflection of main mass

$w_a = \sqrt{k/m}$ = natural frequency of absorber

$W_n = \sqrt{K/M}$ = natural frequency of main system

$u = m/M$ = mass ratio = absorber mass/main mass

Solving for $a_1$ and $a_2$ in dimensionless form gives

$$\frac{a_1}{x_{st}} = \frac{1 - \left[\frac{w}{w_a}\right]^2}{\left( 1 - \left[\frac{w}{w_a}\right]^2 \right) \left( 1 + \frac{k}{K} - \left[\frac{w}{W_n}\right]^2 \right) - \frac{k}{K}}$$

(5)

$$\frac{a_2}{x_{st}} = \frac{1}{\left( 1 - \left[\frac{w}{w_a}\right]^2 \right) \left( 1 + \frac{k}{K} - \left[\frac{w}{W_n}\right]^2 \right) - \frac{k}{K}}$$

From the first of equations (4) the amplitude $a_1$ of the main mass will be zero when the numerator $1 - w^2/w_a^2$ is zero. This occurs only when the frequency of the absorber is equal to the natural frequency of the forcing function.

When $w = w_a$ the second equation reduces to

$$a_2 = - \frac{K}{k} x_{st} = - \frac{P_o}{k}$$

This means then that while the main mass has no motion the absorber mass is vibrating with a motion $-P_o/k \sin wt$. This force is then equal and opposite to the forcing function external to the system.

The above equations hold for any value of $w/W_n$. The addition of an absorber is however not required unless the original system is in resonance or near it. The resonant condition is

$$w = w_a = W_n \quad \text{or} \quad \frac{k}{m} = \frac{K}{M} \quad \text{or} \quad \frac{k}{K} = \frac{m}{M}$$

The size of the damper is defined by the ratio

$$u = \frac{m}{M}$$

For the above conditions equations (4) can be rewritten as follows

$$\frac{x_1}{x_{st}} = \frac{1 - \left[\frac{w}{w_a}\right]^2}{\left(1 - \left[\frac{w}{w_a}\right]^2\right)\left(1 + u - \left[\frac{w}{w_a}\right]^2\right) - u} \sin wt$$

(6)

$$\frac{x_2}{x_{st}} = \frac{1}{\left(1 - \left[\frac{w}{w_a}\right]^2\right)\left(1 + u - \left[\frac{w}{w_a}\right]^2\right) - u} \sin wt$$

From these two equations it will be noted that both denominators are the same, therefore both masses will have infinite amplitude for the same $w/w_a$ ratio. The denominators set equal to zero then forms the following second degree equation in $w/w_a$.

$$\left[\frac{w}{w_a}\right]^4 - \left[\frac{w}{w_a}\right]^2 (2 + u) + 1 = 0$$

Solving

$$\left[\frac{w}{w_a}\right]^2 = \left(1 + \frac{u}{2}\right) \pm \sqrt{u + \frac{u}{4}}$$

The relationship described by the above equation for the simple two mass system is shown graphically in Figure (1-b).

The effect of adding an absorber one-fifth the mass of the main system is shown in Figure (1-c) and on the absorber in Figure (1-d).

In Figures (1-c) and (1-d) the values plotted on the negative ordinate are caused by a change in the sign of the numerator or denominator. This change in sign means a 180° change in the phase angle and is of no importance here since

$$-\,x_o \sin wt = +\,x_o \sin (wt + 180°)$$

these values can therefore be plotted on the positive axis.

From the foregoing study it is apparent that the undamped dynamic vibration absorber is only effective where the frequency of the forcing function is nearly constant. In such a case $w/w_a = w/W_n = 1$ and the amplitude of the main mass will be zero. In the case of a variable speed motor such an application is useless since the one resonant speed of the original system is replaced by two resonant speeds. This condition will however be shown to be advantageous if a damping factor is added to the absorber spring.

Figure (2-a) represents the same system shown in Figure (1-a) with the addition of damping in the absorber spring.

The differential equations of motion for the main mass and absorber with damping added are as follows.

$$(7)\qquad \begin{aligned} &M \ddot{x}_1 + K x_1 + k (x_1 - x_2) + (\dot{x}_1 - \dot{x}_2) = P_o \sin wt \\ &m \ddot{x}_2 + k (x_2 - x_1) + c (\dot{x}_2 - \dot{x}_1) = 0 \end{aligned}$$

The vibrations assume the form given by equation (2) and substitution into the above equations gives $x_1$ and $x_2$ in complex form.

$$(8)\qquad \begin{aligned} &-M w^2 x_1 + K x_1 + k (x_1 - x_2) + j w c (x_1 - x_2) = P_o \\ &-m w^2 x_2 + k (x_2 - x_1) + j w c (x_2 - x_1) = 0 \end{aligned}$$

Rewriting

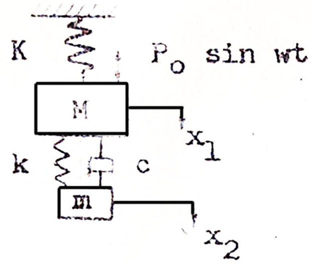
$$(9)\qquad \begin{aligned} &(-M w^2 + K + k + j w c)x_1 - (k + j w c)x_2 = P_o \\ &- (k + j w c)x_1 + (-m w^2 + k + j w c)x_2 = 0 \end{aligned}$$

Expressing $x_2$ in terms of $x_1$ from the second equation and substituting into the first gives

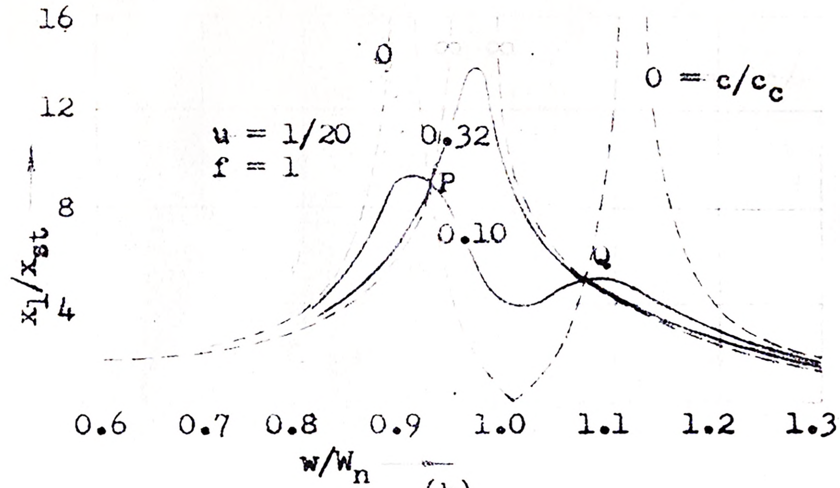
$$(10)\qquad x_1 = P_o \frac{(k - m w^2) + j w c}{\left[(-Mw^2 + k)(-mw^2 + k) - mw^2 k\right] + jwc\left[-Mw^2 + K - mw^2\right]}$$

FIGURE 2

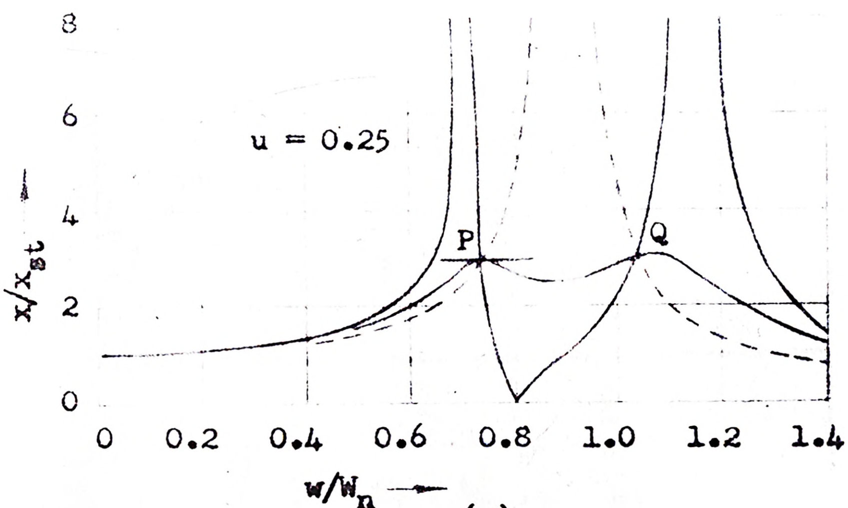
CURVES PERTAINING TO SIMPLE TUNED VIBRATION ABSORBER WITH DAMPING



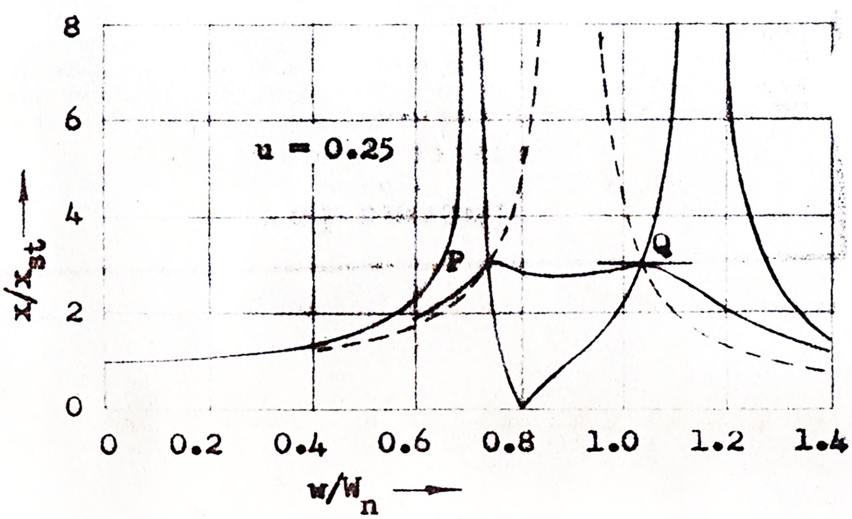

(a)




(b)




(c)




(d)

This expression is in the form

(10a) $$x_1 = P_o \cdot \frac{A + jB}{C + jD}$$

which can be reduced to the form

(11) $$x_1 = P_o \ ( A_1 + jB_1 )$$

where $A_1$ and $B_1$ are real and do not contain j. In vector representation then $x_1$ consists of one component in phase with the forcing function $P_o$ and another advanced 90°.

Equation (10) is reduced to the form of (11) as follows.

(11a) $$x_1 = P_o \frac{(A + jB)(C - jD)}{(C + jD)(C - jD)} = P_o \frac{(AC + BD) + j(BC - AD)}{C^2 + D^2}$$

The amplitude of the vector $x_1$ may be expressed as

(12) $$x_1 = P_o \sqrt{A_1{}^2 + B_1{}^2}$$

Therefore

(12a) $$\frac{x_1}{P_o} = \sqrt{\left[\frac{AC + BD}{C^2 + D^2}\right]^2 + \left[\frac{BC - AD}{C^2 + D^2}\right]^2}$$

$$\frac{x_1}{P_o} = \frac{A^2 + B^2}{C^2 + D^2}$$

Applying this result to equation (10) gives

(13) $$\frac{x_1{}^2}{P_o{}^2} = \frac{(k - mw^2) + w^2c^2}{\left[(-Mw^2+k)(+mw^2+k) - mw^2k\right]^2 + w^2c^2\left[-Mw^2+K - mw^2\right]^2}$$

Equation (13) may be rewritten in dimensionless form using the following symbols.

$u = m/M$ = mass ratio = absorber mass/main mass

$w_a = \sqrt{k/m}$ = natural frequency of absorber

$w_n = \sqrt{K/M}$ = natural frequency of main system

(14) $f = w_a/W_n$ = frequency ratio (natural frequencies)

$g = w/W_n$ = forced frequency ratio

$x_{st} = P_o/K$ = static deflection of system

$c_c = 2\ m\ W_n$ = "critical" damping

(15) $$\frac{x_1}{x_{st}} = \frac{(2 \frac{c}{c_c} g^2)^2 + (g^2 - f^2)}{(2\frac{c}{c_c} g)^2(g^2-1+ug^2)^2 + [uf^2g^2 - (g^2-1)(g^2-f^2)]^2}$$

This algebraic operation reduces equation (13) from a function of seven variables to a function of four variables as expressed above.

A plot of the amplitude ratio $x_1/x_{st}$ as a function of the frequency ratio g is shown in Figure (2-b) where $f = 1$ and $u = 1/20$. It is of interest to note from Figure (2-b) that as c varies from zero to infinity the system changes from one with a simple absorber to one with a single degree of freedom with a mass ratio of 21/20 M. Two other curves for $c/c_c = 0.10$ and 0.32 are also shown in Figure (2-b).

It will be noted from Figure (2-b) that all four curves intersect at the points P and Q. It can be proven that all curves pass through these two points and are independent of damping at this point. By changing the relative "tuning" $f = w_a/W_n$, the points P and Q can be shifted up and down for the curve c = 0. The proof of such statements is long and tedious and shall therefore be omitted. It will be considered sufficient only to state that the most favorable curve is the one where the amplitude of P equals the amplitude of Q and passes with a horizontal tangent through either of the points as shown in Figures (2-c) and (2-d). From Figure (2-c) it can be seen that the curve horizontal at P is not horizontal at Q and Figure (2-d) shows the opposite condition. Either condition is close to the optimum damping.

The study of the simple spring mass absorber system involves all the basic principles. An expansion of these principles is given in the remainder of the discussion.

The system shown in Figure (3) represents a problem encountered in the design of internal combustion engine systems.

From the given data the system was dynamically balanced and reduced to an equivalent system as shown in Figure (4).

In order to dynamically balance the engine it was necessary to add 2.92 # at a radius of 3.25 " to cranks one and six, thereby increasing the polar mass moment of inertia from 0.232715 to 1.032715 in-lb-sec$^2$.

The equivalent length of crankshaft, between crankthrows,

FIGURE 3 - GIVEN SYSTEM

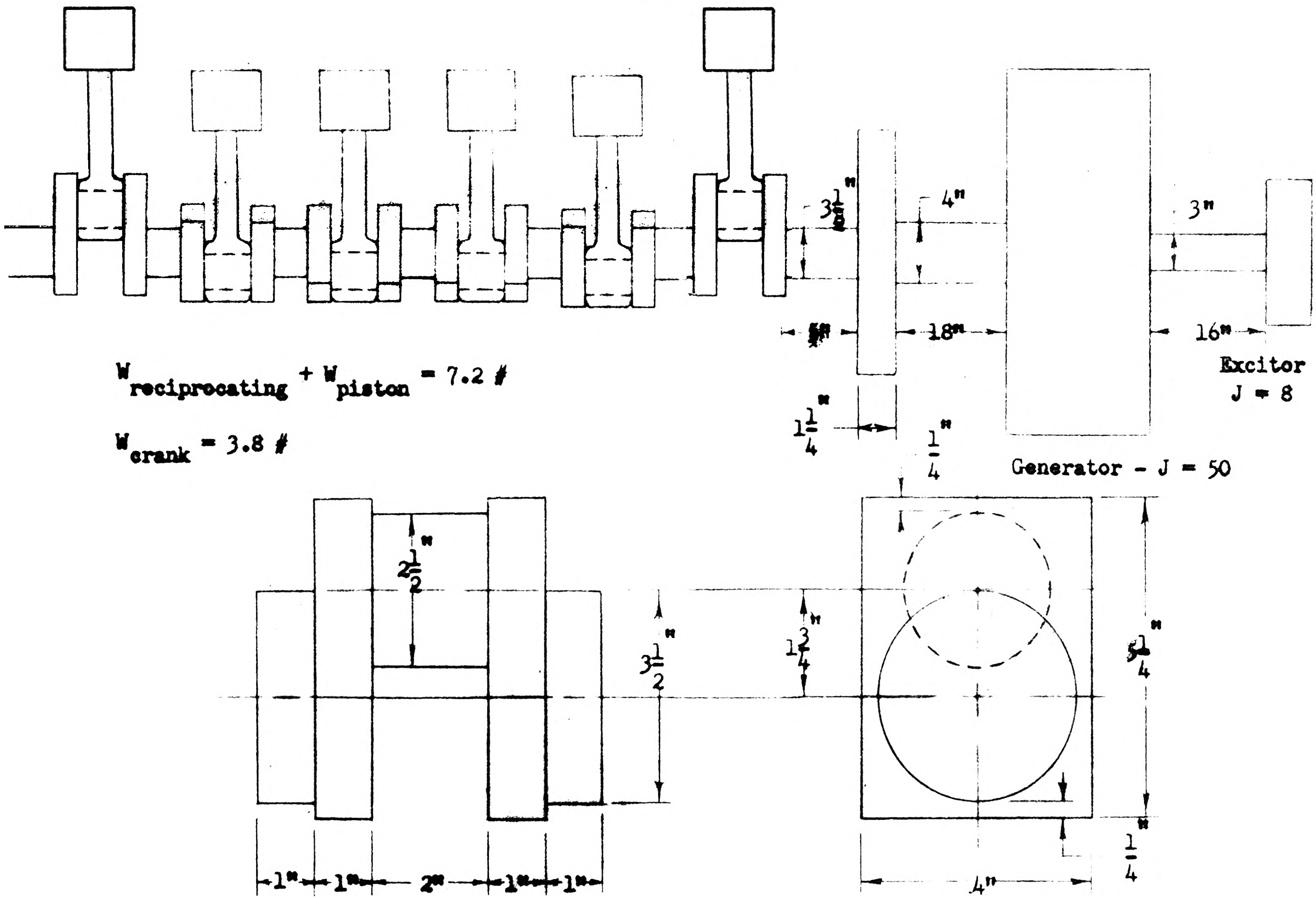
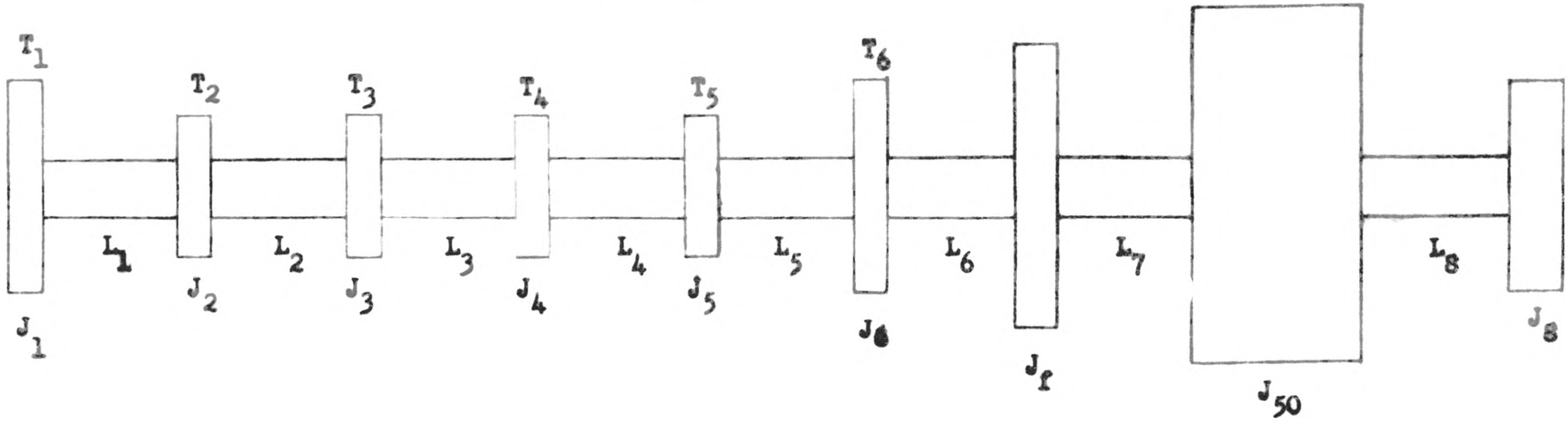


**FIGURE 4 - EQUIVALENT SYSTEM**
**For 3 1/2" Diameter Shaft**




| | | |
|---|---|---|
| $J_1$ = 1.032715 in-lb-sec$^2$ | | |
| $J_2$ = .232715 in-lb-sec$^2$ | $L_1$ = 10.1 in | $K_{1-2}$ = 17.5 • $10^6$ in-lb/rad |
| $J_3$ = .232715 in-lb-sec$^2$ | $L_2$ = 10.1 in | $K_{2-3}$ = 17.5 • $10^6$ in-lb/rad |
| $J_4$ = .232715 in-lb-sec$^2$ | $L_3$ = 10.1 in | $K_{3-4}$ = 17.5 • $10^6$ in-lb/rad |
| $J_5$ = .232715 in-lb-sec$^2$ | $L_4$ = 10.1 in | $K_{4-5}$ = 17.5 • $10^6$ in-lb/rad |
| $J_6$ = 1.032715 in-lb-sec$^2$ | $L_5$ = 10.1 in | $K_{5-6}$ = 17.5 • $10^6$ in-lb/rad |
| $J_f$ = 3.46 in-lb-sec$^2$ | $L_6$ = 9.05 in | $K_{6-f}$ = 19.5 • $10^6$ in-lb/rad |
| $J_{50}$ = 50.00 in-lb-sec$^2$ | $L_7$ = 10.55 in | $K_{f-50}$ = 16.75 • $10^6$ in-lb/rad |
| $J_8$ = 8.00 in-lb-sec$^2$ | $L_8$ = 29.6 in | $K_{50-8}$ = 5.97 • $10^6$ in-lb/rad |

was determined from Carters equation (2) for solid journals and crankpins.

$$(16)\quad L_e = d_e^4 \left[ \frac{e + 0.8a}{D_j^4} + \frac{0.75b}{D_c^4} + \frac{1.5r}{a\ c^3} \right]$$

From this equivalent length the shaft torsional stiffness could be determined

$$(17)\quad K = \frac{G \pi d_e^4}{32 L_e}$$

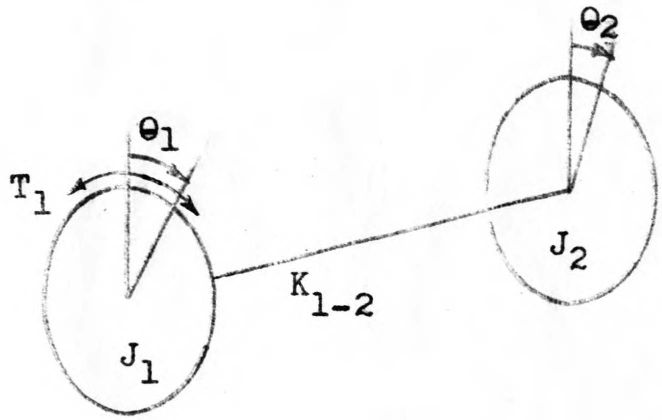
The shaft stiffness between cylinder number six and the flywheel was determined by multiplying the aforementioned stiffness by two and treating the additional length of 3 1/2" diameter as one shaft acting on another in series. The remaining shafts were changed to 3 1/2" diameter for convenience only.

The calculations mentioned above are long and tedious and serve no purpose other than to show the origin of the equivalent system used in this example.

The given system may now be analyzed by a method developed by Holzer (2) in 1921. This method, as later devised, may be used for free or forced vibrations with or without damping. This numerical method of analyzing the system was developed from the differential equations of each mass and is well suited for solution on the digital computer.

A study of the first two vibrating masses will lead to the general equations used in the Holzer Table.

Mass $J_1$ $\theta_1 > \theta_2$




(18) $T_1 - J_1 \ddot{\theta}_{1v} = K_{1-2} (\theta_1 - \theta_2)$

since

$\theta_{1v} = \theta_1 \cos wt$ $T_{1v} = T_1 \cos wt$

$\dot{\theta}_{1v} = -\theta_1 w \sin wt$

$\ddot{\theta}_{1v} = -\theta_1 w^2 \cos wt$

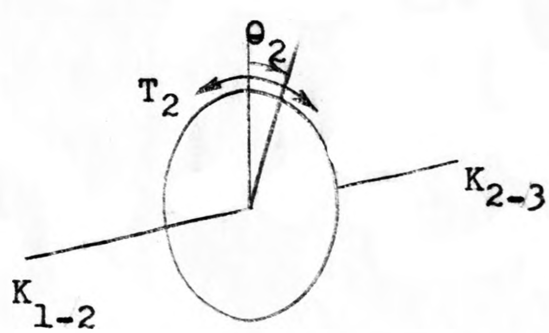
Substituting in (18) gives

(18a) $T_1 \cos wt - J_1 ( - \theta_1 w^2) \cos wt = K_{1-2} ( \theta_1 - \theta_2 ) \cos wt$

And solving for $\theta_2$

(19) $$\theta_2 = \theta_1 - \frac{T_1 + J_1 \theta_1 w^2}{K_{1-2}}$$

Mass $J_2$ $\theta_1 > \theta_2 > \theta_3$



(20) $T_{2v} - J_2 \ddot{\theta}_{2v} = -K_{1-2}(\theta_{1v} - \theta_{2v}) + K_{2-3}(\theta_{2v} - \theta_{3v})$

Since

$\theta_{2v} = \theta_2 \cos wt$ $\quad \theta_{3v} = \theta_3 \cos wt$ $\quad T_{2v} = T_2 \cos wt$

$\dot{\theta}_{2v} = -\theta_2 w \sin wt$ $\quad \dot{\theta}_{3v} = -\theta_3 w \sin wt$

$\ddot{\theta}_{2v} = -\theta_2 w^2 \cos wt$ $\quad \ddot{\theta}_{3v} = -\theta_3 w^2 \cos wt$

(20a) $T_2 + J_2 \theta_2 w^2 = -(T_1 + J_1 \theta_1 w^2) + K_{2-3} (\theta_2 - \theta_3)$

And solving for-$\theta_3$ gives

(21) $$\theta_3 = \theta_2 - \frac{(T_1+T_2) + w^2 (J_1 \theta_1 + J_2 \theta_2)}{K_{2-3}}$$

From the above analysis the following general equations can be derived.

The inertia torque developed by mass n

(22) Torque inertia $= J_n \theta_n w^2$

The total torque acting on shaft n

(23) $$\text{Torque} = w^2 \sum_1^n J_n \theta_n + \sum_1^n T_n$$

The amplitude $\theta_n$ of disk n

(24) $$\theta_n = \theta_{n-1} - \frac{w^2 \sum_1^n J_n \theta_n + \sum_1^n T_n}{K_{n-1}}$$

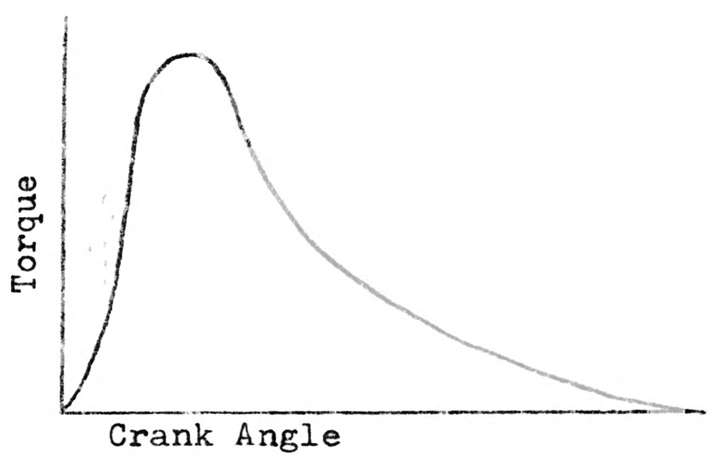
A method illustrating this analysis is given in Table I. The amplitude of the first mass is assumed to be 1.0000 A radian. This value is placed in column 4. Column 5 is obtained by multiplying column 3 by column 4. Column 5 is then added to column 6 to give $w^2 \sum_1^n J_n \theta_n + \sum_1^n T_n$. Dividing.

TABLE I

| (1) Mass No | (2) J | (3) $Jw^2$ | (4) A | (5) $JA\ w^2$ | (6) $JuAw^2 +\ Tn$ | (7) k | (8) $\frac{JAw^2 +\ Tu}{k}$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | .0243 | 1,000A | .0243A | .0243A + 1 | 1 | .0243A + 1 |
| 2 | 1 | .0243 | .9757A - 1 | .0237A - .0243 | .0480A + 1.9757 | 1 | .0480A + 1.9757 |
| 3 | 1 | .0243 | .9277A - 2.9757 | .0225A - .0724 | .0705A + 2.9033 | 1 | .0705A + 2.9033 |
| 4 | 1 | .0243 | .8572A - 5.8790 | .0208A - .1430 | .0913A + 3.7603 | 1 | .0913A + 3.7603 |
| 5 | 1 | .0243 | .7659A - 9.6393 | .0186A - .2340 | .1099A + 4.5263 | 1 | .1099A + 4.5263 |
| 6 | 1 | .0243 | .6560A - 14.165 | .0159A - .344 | .1258A + 5.182 | 1 | .1258A + 5.182 |
| 7 | 1 | .0243 | .5302A - 19.347 | .0129A - .471 | .1387A + 5.711 | 1 | .1387A + 5.711 |
| 8 | 1 | .0243 | .3915A - 25.058 | .0095A - .610 | .1482A - 6.101 | .5 | .2964A - 12.202 |
| 9 | 71 | 1.725 | .0951A - 37.260 | .1640A - 64.20 | .3122A - 70.30 | 1.34 | .2334 - 52.50 |
| 10 | 19.5 | .474 | -.1379A + 15.25 | -.0653A + 7.23 | .2469A - 63.07 | | |

column 6 by column 7 gives column 8. Subtracting the value of column 8 from line 1 column 4 gives the amplitude of mass 2 and the procedure is repeated until line 10, column 6 where the torque remaining on the system is determined. In this easy to follow example the remaining torque is set equal to zero and the equation solved for A. The torque acting on any shaft may now be determined.

In the foregoing discussion it was mentioned only that the exciting torque was a periodic function. The approximate shape of the torque versus crank angle is shown below.




Fourier (8) has shown that a periodic curve of this type may be expressed as a constant and a series of sine or cosine terms of various amplitudes whose frequencies are integral multiples of periodic motion as shown below.

(25) $$f(x) = \frac{a_o}{2} + \sum_{1}^{\infty} (a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi x}{L})$$

Since the sum of any two corresponding terms may be written as

$$(26) \qquad a_n \cos \frac{n\pi}{L} t + b \quad \sin \frac{n\pi}{L} t = c_n \sin \left(\frac{n\pi}{L} t + \phi_n\right)$$

Expanding

$$(26a) \qquad c_n \sin \left(\frac{n\pi}{L} t + \phi\right) = c_n \sin \frac{n\pi}{L} t \cos \phi_n + c_n \sin \phi_n \cos \frac{n\pi}{L} t$$

Then

$$a_n = c_n \sin \phi_n, \quad b_n = c_n \cos \phi_n, \quad c_n = \sqrt{a_n^2 + b_n^2}$$

$$(26b) \qquad \phi_n = \arctan \frac{a_n}{b_n}$$

And equation (26) may be rewritten as

$$(27) \quad f(x) = \frac{a_o}{2} + \sum_1^{\infty} c_n \sin \left(\frac{n\pi}{L} t + \phi_n\right)$$

The problem now becomes one of determining the value of the various coefficients of equation (26) and substituting them in equation (27). This may be done by dividing the abscissa of the given period, in this case $720^{\circ}$, into z equal intervals and measuring the ordinate $y_k$ at the end of each interval. The values of $y_k$ are entered in a table, and the coefficients are calculated by the following formulas.

$$(28) \qquad \frac{a_o}{2} = \frac{1}{z} \sum_{k=1}^{k=z} y_k$$

$$(29) \qquad a_n = \frac{2}{z} \sum_{k=1}^{k=z} y_k \cos \left(720 \frac{nk}{z}\right)$$

$$(30) \qquad b_n = \frac{2}{z} \sum_{k=1}^{k=z} y_k \sin \left(720 \frac{nk}{z}\right)$$

Where k is the number of the interval having the displacement $y_k$ (2).

The values of these coefficients, for four cycle gasoline engines, have been determined by F.P. Porter (4) for any mean effective pressure and are shown in Figure (5). Half integral coefficients higher than 12 are not shown on Figure (5) because they are of insufficient amplitude to have a marked effect on the system.

From the information given in Figure (5) it is now possible to change the disturbing torque into harmonic components whose frequencies are half integral multiples of the operating frequency. Half integral multiples occur because the power stroke on a four cycle internal combustion engine occurs on every other revolution of the crankshaft.

Since any of the harmonic frequencies are capable of exciting the system into resonance, all possible conditions must be investigated. The investigation of the given system can be thoroughly understood by the information given in Figure (6).

The effect of all six cylinders must now be obtained by studying the phase diagrams for the crank arrangement used. The harmonic torque phase diagram shown in Figure (6-c,d,e,f, & g) have a vectorial sum of zero. In Figure (6-h) the harmonic torque vectors add or act in phase and are therefore dangerous. Such a condition is known as a major order vibration. All other orders are called minor orders and are neglected in this investigation since their effect is small compared to the major order vibrations.

In order to show the effect of the major order vibrations,

FIGURE 5

RESULTANT HARMONIC COEFFICIENTS FOR TORQUE/AR CURVES
4 CYCLE GASOLINE ENGINE

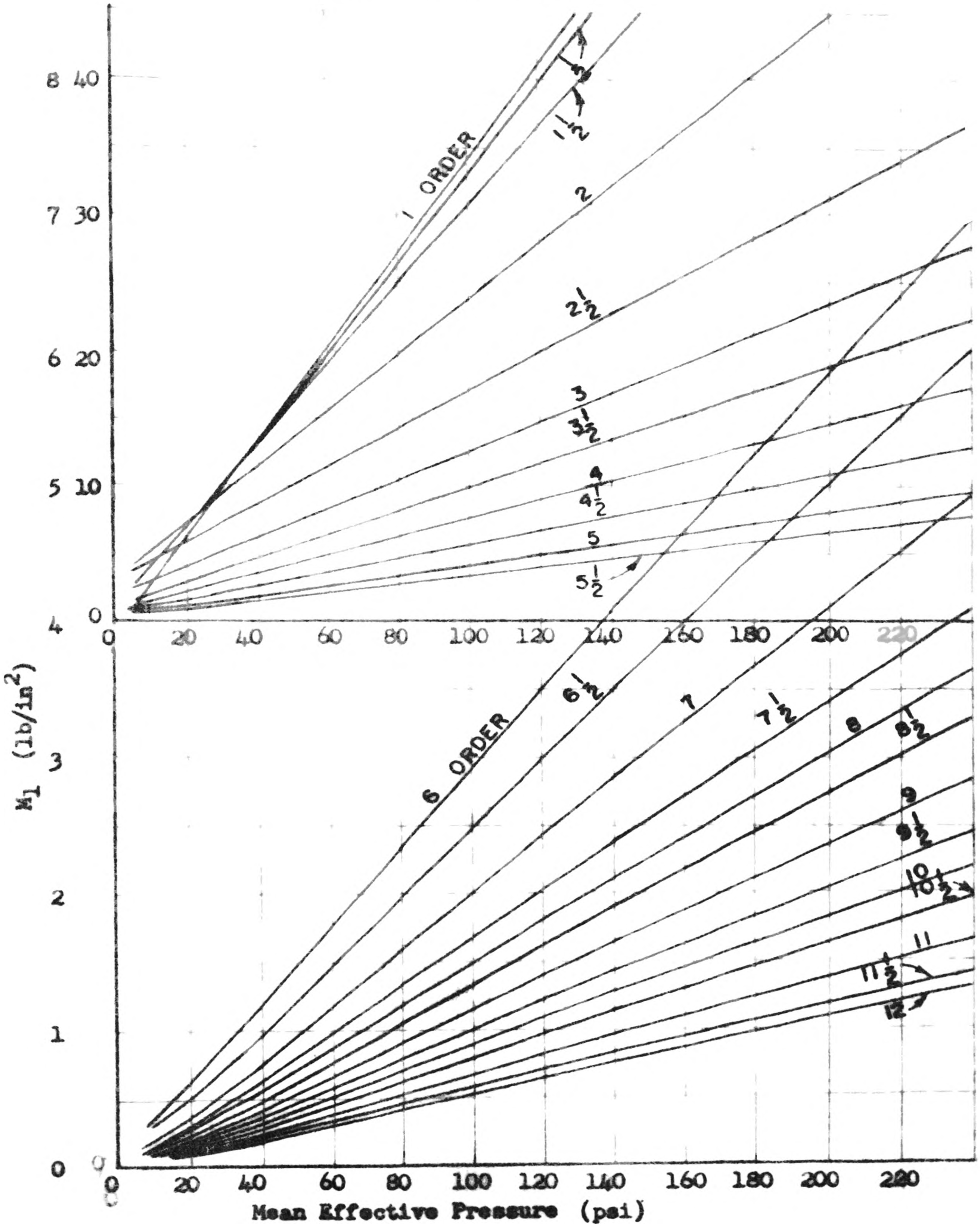
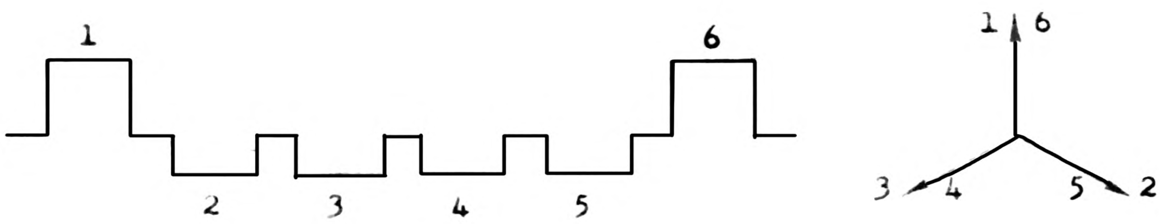


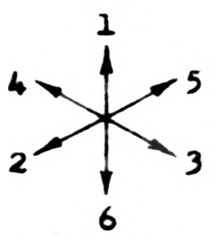
**FIGURE 6**
**HARMONIC TORQUE PHASE DIAGRAMS**




Firing Order 1 - 5 - 3 - 6 - 2 - 4
(a)

| Cylinder | Original System | Order Number 1/2 | 1 | 1 1/2 | 2 | 2 1/2 | 3 |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 120 | 60 | 120 | 180 | 240 | 300 | 360 |
| 3 | 240 | 120 | 240 | 360 | 480 | 600 | 720 |
| 6 | 360 | 180 | 360 | 540 | 720 | 900 | 1080 |
| 2 | 480 | 240 | 480 | 720 | 960 | 1200 | 1440 |
| 4 | 600 | 300 | 600 | 900 | 1200 | 1500 | 1800 |

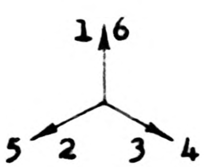
(b)



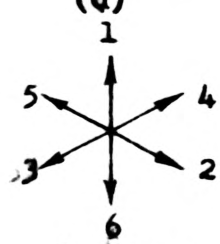

**Orders**
**1/2, 3 1/2, 6 1/2, etc.**
**(c)**


**Orders**
**1, 4, 7, etc.**
**(d)**






**Orders**
**2, 5, 8, etc.**
**(f)**




**Orders**
**2 1/2, 5 1/2, 8 1/2, etc.**
**(g)**

**Orders**
**1 1/2, 4 1/2, 7 1/2, etc.**
**(e)**

**Orders**
**3, 6, 9, etc.**
**(h)**

the natural frequencies of the system, without disturbing torques must first be determined. This was accomplished on a digital computer using a Holzer Table, with the disturbing torque reduced to zero. The results of the calculations are shown on Figure (7), with a natural frequency at 146 and 200 cycles per second.

The following equation gives the relationship between the exciting frequency and engine rpm.

$$f_{exciting} = N \frac{rev}{min} \cdot Q \frac{vib}{rev} \cdot \frac{1\ min}{60\ sec}$$

(31) $$N = \frac{60\ f_e}{Q}\ rpm$$

The first major order number is three. At the first natural frequency the engine speed would be

$$N = \frac{60}{3}(146) = 2{,}920\ rpm$$

Since this is outside the normal operating speed of the engine, maximum 2,200 rpm, it may be disregarded.

The next major order number is six. This critical major order vibration falls within the operating range and 2,000 rpm for the second mode of vibration. The first critical speed 1,460 rpm will be investigated.

The value of T for the sixth harmonic is taken from Figure (5) and is

$$4.07 \frac{\#}{in} \cdot \frac{\pi 4^2}{4}\ in^2 \cdot 1.75\ in = 89.5\ in\text{-}lbs$$

At this point there must be a deviation from the criterion governing the simple spring mass absorber system.

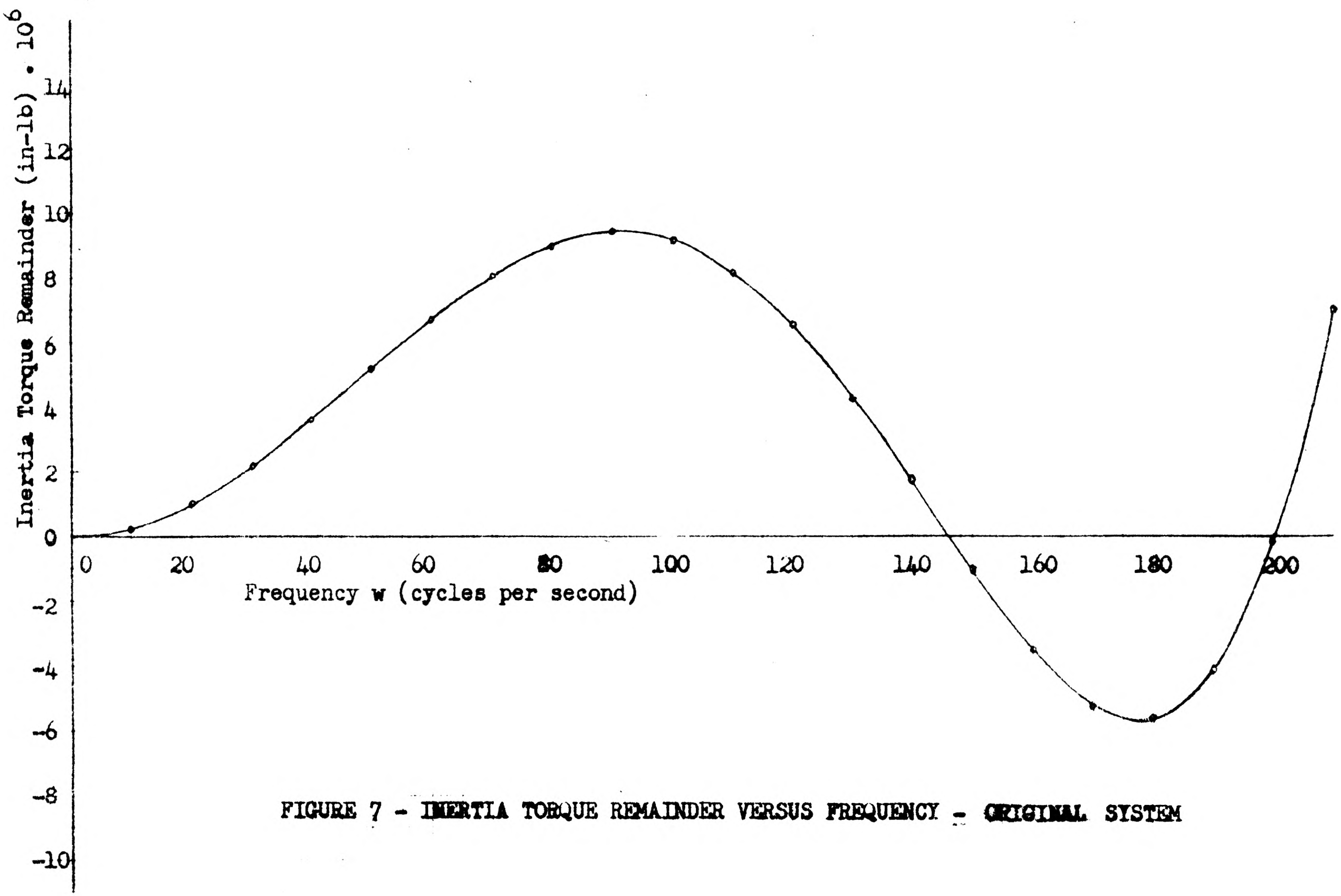



FIGURE 7 - INERTIA TORQUE REMAINDER VERSUS FREQUENCY - ORIGINAL SYSTEM

In this problem displacement is not the controlling factor but rather maximum stress in any part of the system. This represents a considerable departure from the method used for the simple absorber. This stress can be effectively controlled by the addition of a properly designed absorber and the results will be similar to those shown in Figure (2-c). The displacement however will be of importance in the choice of a front or rear absorber. In this system the maximum displacement was at the rear of the system for the first mode of vibration, thereby indicating the use of a rear absorber.

The size of the absorber, as in the simple system with no damping, controls the amount of deviation from the critical speed as shown in Figure (1-b). For this system an absorber with approximately 5% of the total polar mass moment of inertia was chosen. The results of this addition to the system for c equal to infinity (absorber locked) is shown in Figure (9). This lowers the natural frequency of the system to 1,262 rpm.

A value of K, with c = 0, must now be chosen so the ordinates of points P and Q as shown in Figure (9) are equal. This was accomplished by imposing a torsional stiffness between the absorber mass and the last mass of the main system in the program and using an absorber frequency equal to the natural frequency of the locked system as the first approximation.

$$f = \frac{1}{2} \quad \frac{K}{J}$$

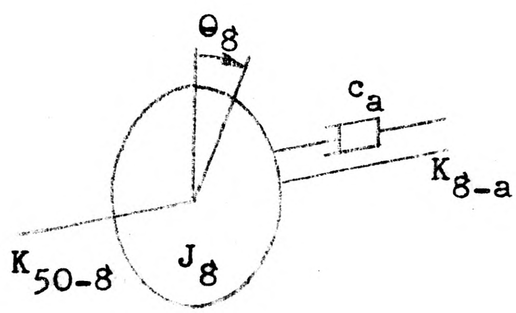
$$K = (126)^2 \cdot 4\pi^2 \cdot (3.25) = 2{,}043{,}000 \text{ in-lb/rad}$$

This value of K was ultimately adjusted to an optimum value of 1,650,000 in-lb/rad. The results of which are shown in Figure (9).

It now becomes necessary to determine a value of c such that the stress amplitude curve in Figure (9) will pass with a horizontal tangent through point P in Figure (9).

The differential equations of motion of each mass in the system are obtained in the same manner shown earlier in the discussion on page 17. The following equation illustrates the method of handling the damping coefficients in the Holzer Table.

Mass $J_8$ $\theta_{50}$ $\theta_8$ $\theta_a$




$$(32) \qquad J_8 \ddot{\theta}_{8v} = K_{50-8} (\theta_{50v} - \theta_{8v}) - K_{8-a} (\theta_{8v} - \theta_{av}) - c_a (\dot{\theta}_{8v} - \dot{\theta}_{av})$$

Since

$$\Theta_{8v} = \Theta_8 \cos wt \qquad \Theta_{av} = \Theta_a \cos wt$$

$$\dot{\Theta}_{8v} = - \Theta_8 \, w \sin wt \qquad \dot{\Theta}_{av} = - \Theta_a \, w \sin wt$$

$$\ddot{\Theta}_{8v} = - \Theta_8 \, w^2 \cos wt$$

Substituting in (32) gives

$$\text{(32a)} \quad J_8 \Theta_8 \, w^2 \cos wt = -K_{50-8} \, (\Theta_{50} - \Theta_8) \cos wt + K_{8-a} \, (\Theta_8 - \Theta_a) \cos wt - c_a w \, (\dot{\Theta}_8 - \dot{\Theta}_a) \sin wt$$

From the equation for $J_{50}$ (not shown)

$$\text{(32b)} \quad J_8 \, \Theta_8 \, w^2 \cos wt + (\Sigma T_{1-6} + w^2 \Sigma J_{1-50} \, \Theta_{1-50}) \cos wt = (\Theta_8 - \Theta_a)(K_{8-a} \cos wt - c_a \, w \sin wt)$$

From the identity (sin wt = - i cos wt) equation (32b) may be rewritten.

$$\text{(33)} \quad \Theta_a = \Theta_8 - \frac{(\Sigma T_{1-6} + w^2 \Sigma J_{1-8} \, \Theta_{1-8}) \cos wt}{(K_{8-a} + i \, c_a \, w) \cos wt}$$

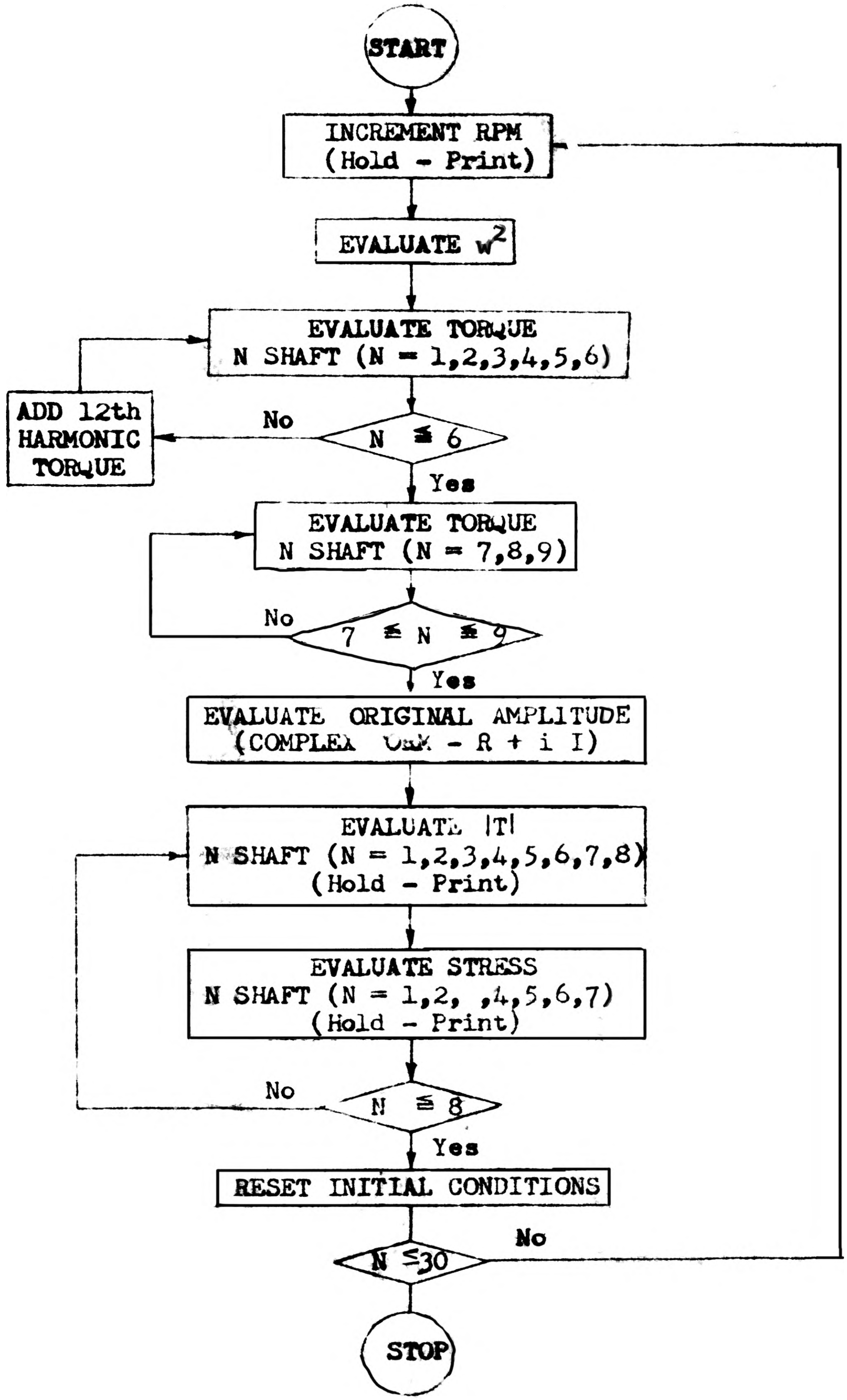
And rationalizing it is now possible to replace a spring stiffness by a complex coupling coefficient in the Holzer Table as shown below.

$$\text{(34)} \quad \Theta_a = \Theta_8 - \left( \frac{K_{8-a} - i \, c_a w}{K_{8-a}^2 + c_a^2 w^2} \right) (\Sigma T_{1-6} + w^2 \Sigma J_{1-8} \Theta_{1-8})$$

The program for the solution of the problem was rewritten in the general form to include all values of K and c. The flow chart of this program is shown in Figure (8) and an example at 1,400 rpm, c = 1,462 and K = 1,650,000 is shown in Table II.

The optimum value of c shown in Figure (9) was determined by selecting two frequencies, one slightly above and another slightly below point P Figure (9) and equating the stress at these two frequencies. This causes the stress amplitude curve to pass through point P with approximately zero slope.

FIGURE 8
FLOW CHART



# TABLE II

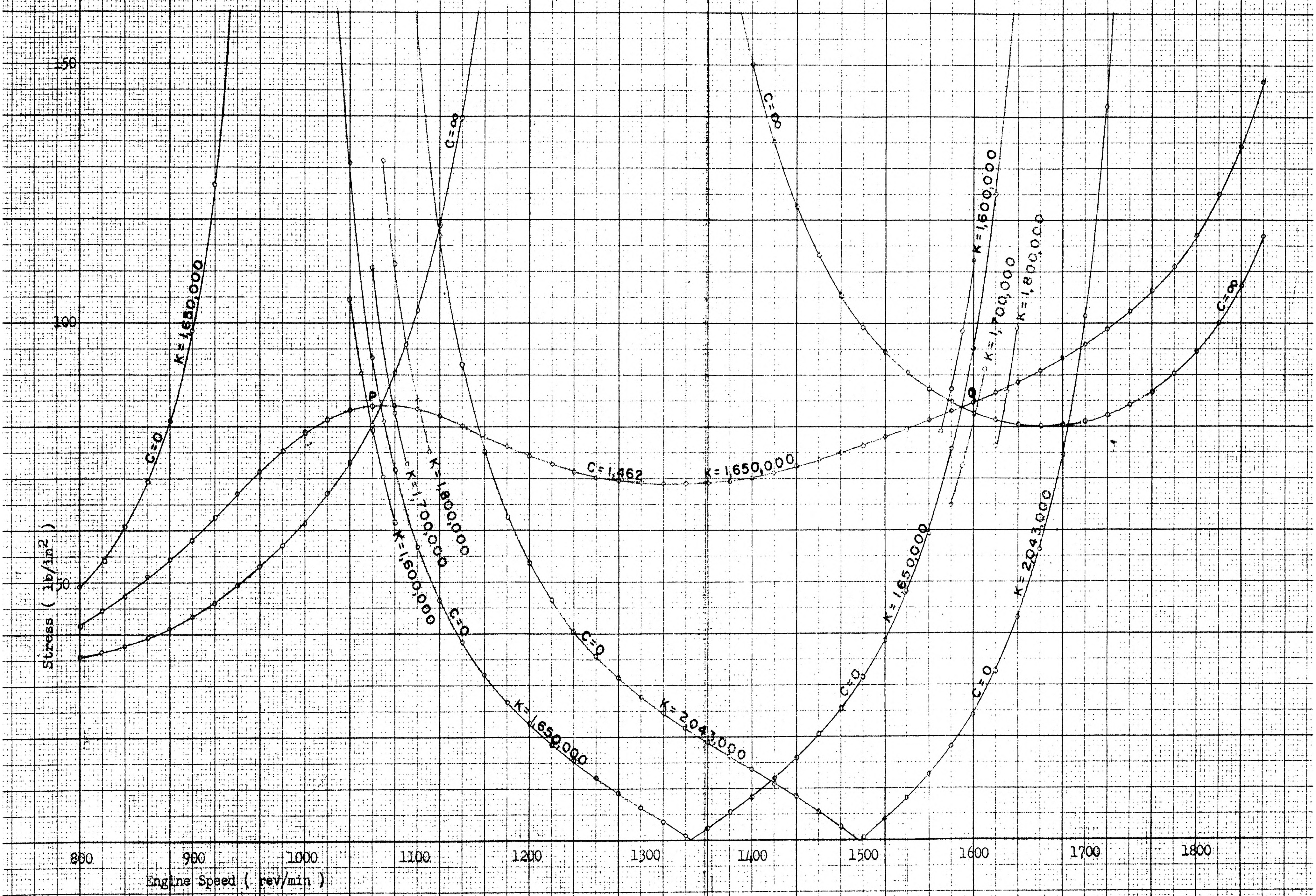
140 cps $w^2$=773,777.14  K=1,650,000  c=1,462

| (1) Item | (2) J | (3) $Jw^2$ | (4) $\theta$ | (5) $Jw^2\theta$ | (6) $\Sigma Jw^2\theta$ | (7) K | (8) $\Delta\theta$ |
|---|---|---|---|---|---|---|---|
| 1 | 1.032715 | 799,091.25 | 1.0000000 A<br>+0.000000 | 779,091.25 A<br>+0.000000 | 779,091.25 A<br>+89.503982 | $17.5\cdot10^6$ | .045662358 A<br>$+5.1145134\cdot10^6$ |
| 2 | .232715 | 180,069.54 | .95433764 A<br>$-5.1145134\cdot10^6$ | 171,847.14 A<br>-.92096808 | 970,938.40 A<br>+178.08699 | $17.5\cdot10^6$ | .055482195 A<br>$+10.176400\cdot10^6$ |
| 3 | .232715 | 180,069.54 | .89885544 A<br>$-15.290913\cdot10^6$ | 161,856.49 A<br>-2.7534278 | 1,132,794.9 A<br>+264.83754 | $17.5\cdot10^6$ | .064731136 A<br>$+15.133575\cdot10^6$ |
| 4 | .232715 | 180,069.54 | .83412430 A<br>$-30.424488\cdot10^6$ | 150,200.38 A<br>-5.4785235 | 1,282,995.2 A<br>+348.86300 | $17.5\cdot10^6$ | .073314015 A<br>$+19.935029\cdot10^6$ |
| 5 | .232715 | 180,069.54 | .76081029 A<br>$-50.359517\cdot10^6$ | 136,998.76 A<br>-9.0682153 | 1,419,994.0 A<br>+429.29877 | $17.5\cdot10^6$ | .081142516 A<br>$+24.531359\cdot10^6$ |
| 6 | 1.032715 | 799,091.26 | .67966777 A<br>$-74.890876\cdot10^6$ | 543,116.57 A<br>-59.844642 | 1,963,110.6 A<br>+458.95811 | $19.5\cdot10^6$ | .10067234 A<br>$+23.536314\cdot10^6$ |
| 7 | 3.46 | 2,677,268.8 | .57899543 A<br>$-98.427190\cdot10^6$ | 1,550,126.4 A<br>-263.51605 | 3,513,236.9 A<br>+195.44206 | $16.75\cdot10^6$ | .20974549 A<br>$+11.668183\cdot10^6$ |
| 8 | 50.00 | 38,688,854 | .36924993 A<br>$-110.09537\cdot10^6$ | 14,285,857 A<br>-4,259.4639 | 17,799,094 A<br>-4,064.0219 | $5.97\cdot10^6$ | 2.9814229 A<br>$-680.74071\cdot10^6$ |
| 9 | 8.00 | 6,190,216.8 | -2.6121729 A<br>$+570.64533\cdot10^6$ | -16,169,917 A<br>+3,532.4183 | 1,629,177. A<br>-531.60356 | Multiply by<br>$.37702177\cdot10^{-6}$<br>$+i1.29385819\cdot10^{-6}$ | .61423519 A<br>+i.47874701 A |
| 10 | 3.25 | 2,514,775.6 | [illegible] A<br>-1.47874701 A | -8,113,692.5 A<br>-11,203,941.3 A | -6,484,515.5 A<br>-11,203,941.3 A | + | $-200.42611\cdot10^{-6}$<br>$-1156.21605\cdot10^{-6}$ |
| | | | [illegible]<br>+[illegible]$\cdot10^{-6}$ | [illegible],939.[illegible]7<br>+1392.84832 | [illegible]<br>+1392.8432 = 0 | | |

Solving for A

$A = 220.69114\cdot10^{-6} + i\,19.608117\cdot10^{-6}$

FIGURE 9 - STRESS VERSUS ENGINE SPEED - FOR DESIGN OF OPTIMUM TUNED AND DAMPED DYNAMIC VIBRATION ABSORBER - FIRST MODE OF VIBRATION



# CONCLUSIONS

The results of this investigation are very good. The addition of the tuned and damped absorber to the system has reduced the stress at 1,262 rpm, absorber locked, from infinity to a value less than 100 psi.

The optimum tuned and damped stress curve shown in Figure (9) does not compare favorably to the ideal case shown in Figure (2-c). The cause of this lack of comparison is due to the close proximity of the second mode of vibration, at 2,000 rpm. The curve shown in Figure (9) is however the best possible stress condition for the first mode of vibration using a rear single absorber. It will be noted that the afforementioned curve approaches the c = 0 curve at increased rpm indicating the optimum condition.

The infinite stress for the second mode of vibration at 2,000 rpm must be eliminated in the same manner. For this case however a front absorber is required since the maximum deflection occurs at the front of the system.

The foregoing discussion should be considered a preliminary study of the method of analysis since there are several areas left to be investigated. A general program for many masses with a front, and front and rear absorber would be of great interest. Also the design of an actual absorber with the required spring constants and damping co-efficient would be very informative.

The author suggests that further worthwhile study could be done in this area.

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# VITA

John Raymond Burrows was born in St. Louis, Missouri, August 30, 1935. He attended Southwest High School in St. Louis and graduated in June 1953. In September 1953 the author was enrolled in the Engineering Curriculum at Harris Teachers Junior College and graduated in September 1955 with an Associate of Arts Degree in Engineering. In September 1955 the author was enrolled at the Missouri School of Mines and Metallurgy in the Mechanical Engineering Department and in January became a Co-Op student with McDonnell Aircraft Corporation. After graduation in June 1958 the author accepted employment with Standard Oil of California in La Habra, California. In September 1960 the author entered the Missouri School of Mines and Metallurgy as a teaching assistant to do graduate work.

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