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Balance with Unbounded Complexes *†

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Abstract

Given a double complex X there are spectral sequences with the E_2 terms being either H_I $(\mathrm{H}_{II}(X))$ or $\mathrm{H}_{II}(\mathrm{H}_I(X))$. But if $H_I(X) = H_{II}(X) = 0$ both spectral sequences have all their terms 0. This can happen even though there is nonzero (co)homology of interest associated with X. This is frequently the case when dealing with Tate (co)homology. So in this situation the spectral sequences may not give any information about the (co)homology of interest. In this article we give a different way of constructing homology groups of X when $\mathrm{H}_I(X)$ $=\mathrm{H}_{II}(X) = 0$. With this result we give a new and elementary proof of balance of Tate homology and cohomology.

1.Introduction

We will mainly be concerned with left R-modules over some ring R. So unless otherwise specified, the term module will mean a left R-module. By a complex (C, d) of left R-modules we mean a graded module $C = (C_n)_{n\in\mathbf{Z}}$ along with a morphism $d^C = d : C \to C$ of graded modules of degree -1 such that $d \circ d = 0$. We also use the notation $C = (C^n)$ but where d is of degree $+1$ and where we let $C_n = C^{-n}$.

Given a complex C we let $Z(C) \subset C$ be $Ker(d)$, let $B(C) = Im(d)$ and let $H(C) = Z(C)/B(C)$. If M and N are modules and $C = (C_i)$ and $D = (D^j)$ are complexes, we form complexes denoted $Hom(M, D)$ and $Hom(C, N)$ where $Hom(M, D)^j = Hom(M, D^j)$ and where $Hom(C, N)^i = Hom(C_i, N)$.

By a double complex of modules X we mean a bigraded module $(X^{(i,j)})_{(i,j)\in\mathbf{Z}\times\mathbf{Z}}$ along with morphisms d' and d'' of bidegrees $(1, 0)$ and $(0, 1)$ respectively such that $d' \circ d' = 0$,

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$d'' \circ d'' = 0$ and $d' \circ d'' + d'' \circ d' = 0$. In [2], d' and d'' are denoted d_1 and d_2 and the homology groups of X with respect to d_1 (d_2) are denoted $\mathrm{H}_I(X)$ ($\mathrm{H}_{II}(X)$).

In this paper we find it convenient to use the related notion of what Verdier ([5] , Definition 2.1.2) calls a 2-tuple complex and which we will call a bicomplex. We get a bicomplex if we take the axioms for a double complex and replace the condition $d' \circ d'' + d'' \circ d' = 0$ with the condition that $d' \circ d'' = d'' \circ d'$. We can form the additive category of bicomplexes where morphisms $f : X \to Y$ have bidegree $(0, 0)$. This category will be an abelian category which is not only equivalent to but is isomorphic to the category of double complexes.

Given the bicomplex X we let $Z'(X) \subset X$ be the bicomplex $Ker(d')$. Then we let $B'(X) = Im(d')$ and let $H'(X)$ be the quotient bicomplex $Z'(X)/B'(X)$. Note that these bicomplexes have their $d' = 0$. In a similar manner we define $Z''(X)$, $B''(X)$ and $H''(X)$.

We will tacitly assume that for a bicomplex X we have $X^{(i,j)} \cap X^{(k,l)} = \emptyset$ unless $(i, j) = (k, l)$. We then will let $x \in X$ mean that $x \in X^{(i,j)}$ for some (unique) $(i, j) \in \mathbf{Z} \times \mathbf{Z}$.

2. The Main Result

Theorem 2.1. *Let X be a bicomplex such that $H'(X) = H''(X) = 0$ i.e. such that X has exact rows and columns. Then $H'(Z''(X)) = H''(Z'(X))$. In this case, if we let $H(X)$ be $H'(Z''(X)) = H''(Z'(X))$, there is a natural isomorphism of bigraded modules $H(X) \to H(X)$ of bidegree $(1, -1)$.*

Proof. It is easy to see that $H'(Z''(X)) = Z'(X) \cap Z''(X)/d'(Z''(X))$ and that $H''(Z'(X)) = Z'(X) \cap Z''(X)/d''(Z'(X))$. Hence we only need prove that $d'(Z''(X)) = d''(Z'(X))$. We argue this is so by chasing the diagram. Let $d'(x) \in d'(Z''(X))$ where $x \in Z''(X)$. Then $d''(x) = 0$. Since $H''(X) = 0$ we have $x = d''(y)$ for some $y \in X$. Since $d'(d'(y)) = 0$ we have $d'(y) \in Z'(X)$. Also $d''(d'(y)) = d'(d''(y)) = d'(x)$. So $d'(x) = d'(d''(y)) = d''(d'(y)) \in d''(Z'(X))$. So we have $d'(Z''(X)) \subset d''(Z'(X))$. A similar argument gives that $d''(Z'(X))) \subset d'(Z''(X))$ and so that $d'(Z''(X)) = d''(Z'(X))$.

So when $H'(X) = H''(X) = 0$ we let $H(X) = Z'(X) \cap Z''(X)/d'(Z''(X)) = Z'(X) \cap Z''(X)/d''(Z''(X))$. In this case we want to find an isomorphism $H(X) \to H(X)$ of bigraded modules of bidegree $(1, -1)$. Let $x + d'(Z''(X)) \in H(X) = Z'(X) \cap Z''(X)/d'(Z''(X))$. Since $x \in Z'(X) \cap Z''(X)$ we have $d''(x) = 0$. So since $H''(X) =$

0 we have $x = d''(y)$ for some $y \in X$. We claim $d'(y) \in Z'(X) \cap Z''(X)$. For $d'(d'(y)) = 0$ and $d''(d'(y)) = d'(d''(y)) = d'(x) = 0$ (since $x \in Z'(X)$). So we want to map $x + d'(Z''(X))$ to $d'(y) + d'(Z''(X))$. To see that this map is well-defined, let $\overline{x} + d'(Z''(X)) = x + d'(Z''(X))$ where $\overline{x} \in Z'(X) \cap Z''(X)$. Let $d''(\overline{y}) = \overline{x}$. Then we have $\overline{x} - x \in d'(Z''(X))$. So let $\overline{x} - x = d'(z)$ where $z \in Z''(X)$. Then since $H''(X) = 0$ we have $w \in X$ with $d''(w) = z$. Then $d''(d'(w)) = d'(d''(w)) = d'(z) = \overline{x} - x$. Since $d''(\overline{y} - y) = \overline{x} - x$ we have $d''(\overline{y} - y - d'(w)) = 0$, i.e. that $\overline{y} - y - d'(w) \in Z''(X)$. But $d'(\overline{y} - y - d'(w)) = d'(\overline{y}) - d'(y) - 0$. So $d'(\overline{y}) - d'(y) \in d'(Z''(X))$. This gives that $x + d'(Z''(X)) \mapsto d'(y) + d'(Z''(X))$ (where $d''(y) = x$) is well-defined. This map is clearly additive, natural and of bidegree $(1, -1)$. Reversing the roles of d' and d'' we get a homomorphism $H(X) \to H(X)$ of bidegree $(-1, 1)$. By construction we see that these two maps are inverses of one another, and so both are isomorphisms.

□

3. Construction of Bicomplexes

If $C = (C_i)$ and $D = (D^j)$ are complexes we construct a bicomplex denoted $Hom(C, D)$. We let $Hom(C, D)^{(i,j)} = Hom(C_i, D^j)$ and let $d' = Hom(d^C, D)$ and $d'' = Hom(C, d^D)$. Letting $X = Hom(C, D)$, the condition $H'(X) = 0$ just says that for each $j \in \mathbf{Z}$ the complex $Hom(C, D^j)$ is exact. Similarly the condition $H''(X) = 0$ just says that $Hom(C_i, D)$ is an exact complex for all $i \in \mathbf{Z}$.
If $X = Hom(C, D)$ then want to describe the bicomplexes $Z'(X)$, $Z''(X)$ and $H(X)$ under certain condition.

Proposition 3.1. *If C is an exact complex and D is any complex, then $Z'(Hom(C, D)) \cong Hom(Z(C), D)$ where the isomorphism is an isomorphism of bicomplexes.*

Proof. We have $Z'(Hom(C, D))^{(i,j)}$ is by definition the kernel of the map $Hom(C_i, D^j) \to Hom(C_{i+1}, D^j))$. But this kernel is $Hom(C_i/B_i(C), D^j)$. Since C is exact $C_i/B_i(C) \cong Z_i(C)$. So as graded modules we have $Z'(Hom(C, D)) \cong Hom(Z(C), D)$. Clearly these are isomorphisms of bicomplexes. □

We prove the next result in a similar manner.

Proposition 3.2. *If D is an exact complex then $Z''(Hom(C, D)) \cong Hom(C, Z(D))$ where the isomorphism is an isomorphism of bicomplexes.*

Theorem 3.3. *If C and D are both exact complexes and if $Hom(C, D)$ has ex-*

act rows and columns then for each $(i,j) \in \mathbf{Z} \times \mathbf{Z}$ *we have* $H(Hom(C,D))^{(i,j)} \cong H^j(Hom(Z_i(C),D)) \cong H^i(Hom(C,Z^j(D)))$.

Proof. To get the two isomorphisms we use the two descriptions of $H(Hom(C,D))$. We first use that $H(Hom(C,D)) = H''(Z'Hom(C,D))$. Since $Z'(Hom(C,D)) \cong Hom(Z(C),D)$ as bicomplexes we have that $H(Hom(C,D)) \cong H''(Hom(Z(C),D)$. Since $H''(Hom(Z(C),D))^{(i,j)} = H^j Hom(Z_i(C),D))$, we get the first isomorphism. Using the other description of $H(Hom(C,D))$ we get the second isomorphism. □

Corollary 3.4. *For any* $n \in \mathbf{Z}$ *we have* $H^n(Hom(Z_0(C),D)) \cong H^n(Hom(C,Z^0(D)))$.

Proof. Using the natural isomorphism of Theorem 2.1 we have that $H(Hom(C,D))^{(n,0)} \cong H(Hom(C,D))^{(0,n)}$. Using the two isomorphisms of Theorem 3.3 we have $H(Hom(C,D))^{(0,n)} \cong H^n(Hom(Z_0(C),D))$ and $H(Hom(C,D))^{(n,0)} \cong H^n(Hom(C,Z^0(D)))$. □

If P is an exact sequence of projective modules and E an exact sequence of injective modules, then $Hom(P,E)$ has exact rows and columns. So this bicomplex satisfies the hypotheses of Theorem 3.3. If M and N are a Gorenstein projective and injective module respectively and if P and E are a complete projective and injective resolution of M and of N respectively, then Corollary 3.4 says $H(Hom(M,E)) \cong H(Hom(P,N))$. This is what is meant by Tate balance of cohomology. These (common) cohomology groups are denoted $\widehat{Ext}^n(M,N)$ (see [1], section 4 for definitions and notation).

Remarks 3.5. The balance over a Gorenstein ring was first proved by Iacob ([4], Example 1, pg.2024) and then by Asadollahi and Salarian [1] when the ring is local, Gorenstein and the first module finitely generated. Christensen and Jorgensen in [3] used the inventive idea of a pinched complex to give a different proof of the general balance result.

If we use the tensor product instead of the homomorphism functor, we get results analogous to the above. The proof of these results will be easy and obvious modifications of those for the Hom functor. So we will just state the results. Note that if C and D are complexes of left and right R-modules respectively, then we can form a bicomplex which we will denote $C \otimes D$.

Theorem 3.6. *If* C *is an exact complex of right* R*-modules and* D *is an exact* *complex of left R-modules and if the bicomplex $C \otimes D$ has exact rows and columns then $H^n(Z_0(C) \otimes D) \cong H^n(C \otimes Z_0(D))$ for any $n \in \mathbf{Z}$.*

To get examples where this result can be applied we only need assume that C and D are both exact complexes of flat modules. Then clearly $C \otimes D$ will have exact rows and columns. With this result we get balance of Tate homology.

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