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Sum Formula of Multiple Hurwitz-Zeta Values

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Abstract

Let $s_1, \dots, s_d$ be d positive integers and define the multiple t-values of depth d by

$$t(s_1, \dots, s_d) = \sum_{n_1 > \cdots > n_d \geq 1} \frac{1}{(2n_1 - 1)^{s_1} \cdots (2n_d - 1)^{s_d}},$$

which is equal to the multiple Hurwitz-zeta value $2^{-w}\zeta(s_1, \dots, s_d; -\frac{1}{2}, \dots, -\frac{1}{2})$ where $w = s_1 + \cdots + s_d$ is called the weight. For $d \leq n$, let $T(2n, d)$ be the sum of all multiple t-values with even arguments whose weight is $2n$ and whose depth is d. Recently Shen and Cai gave formulas for $T(2n, d)$ for $d \leq 5$ in terms of $t(2n)$, $t(2)t(2n-2)$ and $t(4)t(2n-4)$. In this short note we generalize Shen-Cai's results to arbitrary depth by using the theory of symmetric functions established by Hoffman.

1 Introduction

In recent years multiple zeta functions and many different variations and generalizations have been studied intensively due to their close relations to other objects in a lot of diverse branches of mathematics and physics. In particular, a large number of identities are establishes between their special values. In [4] Shen and Cai found a few very interesting equations which are similar in nature to Euler's identity of double zeta values. They gave formulas of the sum $E(2n, d)$ of multiple zeta values at even arguments of fixed depth d and weight $2n$, for $d \leq 4$. These have been generalized to arbitrary depth by Hoffman [1]. In [3] Shen and Cai turned to the following values

$$t(s_1, \dots, s_d) = \sum_{n_1 > \cdots > n_d \geq 1} \frac{1}{(2n_1 - 1)^{s_1} \cdots (2n_d - 1)^{s_d}},$$

which we call multiple t-values of depth d in this note. It is clear that this is equal to $2^{-w}\zeta(s_1, \dots, s_d; -\frac{1}{2}, \dots, -\frac{1}{2})$ where $w = s_1 + \cdots + s_d$ is called the weight. Put

$$T(2n, d) = \sum_{\substack{j_1 + \cdots + j_d = n \\ j_1, \dots, j_d \geq 1}} t(2j_1, \dots, 2j_d).$$

Using similar but more complicated ideas from [4] Shen and Cai gave a few sum formulas for $T(2n,d)$ for $d \le 5$ in [3]. For example,

$$T(2n,5) = \frac{7}{128}t(2n) - \frac{3}{64}t(2)t(2n-2) + \frac{1}{320}t(4)t(2n-4). \tag{1}$$

In this note, we shall generalize these to arbitrary depth using ideas from [1] where Hoffman applied the theory of symmetric functions to study the generating function of $E(2n,d)$. It turns out that we need both Bernoulli numbers B_j and Euler numbers E_j defined by the following generating functions respectively:

$$\frac{x}{e^x-1} = \sum_{j=0}^{\infty} B_j \frac{x^j}{j!}, \qquad \sec x = \sum_{j=0}^{\infty} (-1)^j E_{2j} \frac{x^{2j}}{(2j)!}, \tag{2}$$

and the Euler numbers $E_{2j+1} = 0$ for all $j \ge 0$.

Our main results are the following theorems.

Theorem 1.1. *For $d \le n$,*

$$T(2n,d) = \sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(-1)^j \pi^{2j}}{2^{2d-2}(2j)!d} \binom{2d-2j-2}{d-1} t(2n-2j),$$

where $t(2j) = 2^{-2j}(2^{2j}-1)\zeta(2j)$. Or, equivalently,

$$T(2n,d) = \binom{2d-2}{d-1}\frac{t(2n)}{2^{2d-2}d} - \sum_{j=1}^{\lfloor \frac{d-1}{2} \rfloor} \binom{2d-2j-2}{d-1} \frac{t(2j)t(2n-2j)}{2^{2d-3}(2^{2j}-1)B_{2j}d}.$$

The next three cases after (1) are

$$\begin{aligned}
T(2n,6) &= \frac{21}{512}t(2n) - \frac{7}{192}t(2)t(2n-2) + \frac{1}{256}t(4)t(2n-4),\\
T(2n,7) &= \frac{33}{1024}\zeta(2n) - \frac{15}{512}t(2)t(2n-2) + \frac{1}{256}t(4)t(2n-4) - \frac{1}{21504}t(6)t(2n-6),\\
T(2n,8) &= \frac{429}{16384}t(2n) - \frac{99}{4096}t(2)t(2n-2) + \frac{15}{4096}t(4)t(2n-4) - \frac{1}{12288}t(6)t(2n-6).
\end{aligned}$$

As we mentioned in the above the proof of Theorem 1.1 utilizes the generating function of $T(2n,d)$ defined by

$$\Phi(u,v) = 1 + \sum_{n \ge d \ge 1} T(2n,d)u^n v^d$$

for which we have the following result.

Theorem 1.2. *We have*

$$\Phi(u,v) = \cos(\pi\sqrt{(1-v)u}/2)\sec(\pi\sqrt{u}/2).$$

The next theorem involves Euler numbers and is more useful computationally when the difference between n and d is small.

Theorem 1.3. *For $d \le n$ we have*

$$T(2n,d) = \frac{(-1)^{n-d}\pi^{2n}}{4^n(2n)!}\sum_{\ell=0}^{n-d}\binom{n-\ell}{d}\binom{2n}{2\ell}E_{2\ell}. \tag{3}$$

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2 Proof of Theorem 1.2 and Theorem 1.3

We first recall some results on symmetric functions contained in [1, 2] with some slight modification. Let Sym be the subring of $\mathbb{Q}[\![x_1, x_2, \dots]\!]$ consisting of the formal power series of bounded degree that are invariant under permutations of the x_j. Define elements e_j, h_j, and p_j in Sym by the generating functions

$$\begin{aligned}
E(u) &= \sum_{j=0}^{\infty} e_j u^j = \prod_{j=1}^{\infty}(1+ux_j),\\
H(u) &= \sum_{j=0}^{\infty} h_j u^j = \prod_{j=1}^{\infty}\frac{1}{1-ux_j} = E(-u)^{-1},\\
P(u) &= \sum_{j=1}^{\infty} p_j u^{j-1} = \sum_{j=1}^{\infty}\frac{x_j}{1-ux_j} = \frac{H'(u)}{H(u)}.
\end{aligned}$$

Define a homomorphism $\mathfrak{T} : \mathrm{Sym} \to \mathbb{R}$ such that $\mathfrak{T}(x_j) = 1/(2j-1)^2$ for all $j \ge 1$. Hence for all $n \ge 1$

$$\mathfrak{T}(p_n) = t(2n) = \sum_{j\ge1}\frac{1}{(2j-1)^{2n}}.$$

First we need a simple lemma.

Lemma 2.1. *For any positive integer n let $\{2\}^n$ be the string $(2,\dots,2)$ with 2 repeated n times. Then we have*

$$t(\{2\}^n) = \frac{\pi^{2n}}{4^n(2n)!}. \tag{4}$$

Proof. It is easy to see that

$$\begin{aligned}
1+\sum_{n=1}^{\infty} t(\{2\}^n)x^n &= \prod_{j=1}^{\infty}\left(1+\frac{x}{(2j-1)^2}\right)\\
&= \prod_{j=1}^{\infty}\left(1+\frac{x}{j^2}\right) \Big/ \prod_{j=1}^{\infty}\left(1+\frac{x}{(2j)^2}\right)\\
&= \frac{\sinh(\pi\sqrt{x})}{\pi\sqrt{x}}\cdot\frac{\pi\sqrt{x}/2}{\sinh(\pi\sqrt{x}/2)}\\
&= \cosh(\pi\sqrt{x}/2)\\
&= \sum_{n=1}^{\infty}\frac{\pi^{2n}x^n}{4^n(2n)!}.
\end{aligned}$$

This finishes the proof of the lemma. $\square$

Now let $N_{n,d}$ be the sum of all the monomial symmetric functions corresponding to partitions of n having length d. Then clearly

$$\mathfrak{T}(N_{n,d}) = T(2n,d).$$

As in [1] we may define

$$\mathcal{F}(u,v) = 1+\sum_{n\geq d\geq 1} N_{n,d}u^n v^d,$$

then $\mathfrak{T}$ sends $\mathcal{F}(u,v)$ to the generating function

$$\Phi(u,v) = 1+\sum_{n\geq d\geq 1} T(2n,d)u^n v^d.$$

By Lemma 2.1 we have

$$\mathfrak{T}(e_n) = t(\{2\}^n) = \frac{\pi^{2n}}{4^n(2n)!}. \tag{5}$$

Hence

$$\mathfrak{T}(E(u)) = \cosh(\pi\sqrt{u}/2),$$

and

$$\mathfrak{T}(H(u)) = \mathfrak{T}(E(-u)^{-1}) = 1/\cosh(\pi\sqrt{-u}/2) = \sec(\pi\sqrt{u}/2).$$

Thus by [1, Lemma 1] $\mathcal{F}(u,v) = E((v-1)u)H(u)$ and we get

$$\begin{aligned}
\Phi(u,v) = \mathfrak{T}(E((v-1)u)H(u)) &= \cosh(\pi\sqrt{(v-1)u}/2)\sec(\pi\sqrt{u}/2)\\
&= \cos(\pi\sqrt{(1-v)u}/2)\sec(\pi\sqrt{u}/2).
\end{aligned}$$

This proves Theorem 1.2.

Setting $v = 1$ in Theorem 1.2 we obtain

$$\Phi(u, 1) = \sec(\pi\sqrt{u}/2).$$

This yields immediately the following identity by (2)

$$\mathfrak{T}(h_n) = \sum_{d=1}^{n} T(2n, d) = \frac{(-1)^n E_{2n}\pi^{2n}}{4^n(2n)!}. \tag{6}$$

Now by [1, Lemma 2] we have

$$N_{n,d} = \sum_{\ell=0}^{n-d} \binom{n-\ell}{d} (-1)^{n-d-\ell} h_\ell e_{n-\ell}.$$

Applying the homomorphism $\mathfrak{T}$ and using equation (4) and (6) we get Theorem 1.3 immediately.

3 Proof of Theorems 1.1 and a combinatorial identity

We now rewrite the generating function $\Phi(4u, v)$ as follows using Theorem 1.2:

$$\Phi(4u, v) = \sum_{d\geq 0} v^d \tilde{G}_d(u) = \sec(\pi\sqrt{u})\cos(\pi\sqrt{(1-v)u}) = \sec(\pi\sqrt{u})\sum_{j=0}^{\infty}\frac{\pi^{2j}}{(2j)!}(v-1)^j u^j.$$

Let D be the differential operator with respect to u. Then

$$\begin{aligned}
\tilde{G}_d(u) =& (-1)^d \sec(\pi\sqrt{u})\sum_{j\geq d}\frac{(-1)^j\pi^{2j}u^j}{(2j)!}\binom{j}{d} \\
=& \sec(\pi\sqrt{u})\cdot\frac{(-u)^d}{d!}\cdot D^d\sum_{j\geq d}\frac{(-1)^j\pi^{2j}u^j}{(2j)!} \\
=& \sec(\pi\sqrt{u})\cdot\frac{(-u)^d}{d!}\cdot D^d\cos(\pi\sqrt{u}) \\
=& -\frac{\pi^2}{2}\sec(\pi\sqrt{u})\cdot\frac{(-u)^d}{d!}\cdot D^{d-1}\frac{\sin(\pi\sqrt{u})}{\pi\sqrt{u}} \\
=& \frac{\pi^2 u}{2d}\frac{\tan(\pi\sqrt{u})}{\pi\sqrt{u}}G_{d-1}(u)
\end{aligned}$$

by [1, (12)] (the definition of G_k is defined on page 9). By [1, Lemma 3] we have

$$\tilde{G}_d(u) = -\frac{\pi^2 u}{2d}\sum_{j=0}^{\lfloor\frac{d-2}{2}\rfloor}\frac{(-4\pi^2 u)^j}{2^{2d-3}(2j+1)!}\binom{2d-2j-3}{d-1} \tag{7}$$

$$+\frac{\pi\sqrt{u}}{2d}\tan(\pi\sqrt{u})\sum_{j=0}^{\lfloor\frac{d-1}{2}\rfloor}\frac{(-4\pi^2 u)^j}{2^{2d-2}(2j)!}\binom{2d-2j-2}{d-1} \tag{8}$$

$$=\frac{\pi\sqrt{u}}{2d}\tan(\pi\sqrt{u})\sum_{j=0}^{\lfloor\frac{d-1}{2}\rfloor}\frac{(-4\pi^2 u)^j}{2^{2d-2}(2j)!}\binom{2d-2j-2}{d-1} + \text{terms of degree} < d.$$

It is well-dnown that

$$\tan x = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}2^{2m}(2^{2m}-1)B_{2m}x^{2m-1}}{(2m)!}.$$

Hence

$$\frac{\pi\sqrt{u}}{2}\tan(\pi\sqrt{u}) = \sum_{m=1}^{\infty}4^m t(2m)u^m.$$

Therefore $T(2n,d)$ is the coefficient of u^n in

$$\tilde{G}_d(u/4) = \frac{1}{d}\sum_{m=2}^{\infty}t(2m)u^m\sum_{j=0}^{\lfloor\frac{d-1}{2}\rfloor}\frac{(-\pi^2 u)^j}{2^{2d-2}(2j)!}\binom{2d-2j-2}{d-1}.$$

This implies Theorem 1.1 immediately. Notice that by comparing Theorem 1.1 and Theorem 1.3 we get the following identity of between Bernoulli numbers and Euler numbers.

Theorem 3.1. *For all $d \le n$*

$$\sum_{j=0}^{\lfloor\frac{d-1}{2}\rfloor}\frac{(2^{2n-2j}-1)B_{2n-2j}}{2^{2d-1}d}\binom{2d-2j-2}{d-1}\binom{2n}{2j} = \frac{(-1)^{n-d}\pi^{2n}}{4^n(2n)!}\sum_{\ell=0}^{n-d}\binom{n-\ell}{d}\binom{2n}{2\ell}E_{2\ell}.$$

Further we have

$$\sum_{j=0}^{\lfloor\frac{d-1}{2}\rfloor}\frac{(2^{2n-2j}-1)B_{2n-2j}}{2^{2d-1}d}\binom{2d-2j-2}{d-1}\binom{2n}{2j}$$
$$=\begin{cases} 0, & \text{if } n < d < 2n; \\ \dfrac{n}{2^{2d-1}d}\dbinom{2d-2n-1}{d-1}, & \text{if } d \ge 2n. \end{cases}$$

Proof. We only need to show the second identity. Notice that when $d > n$ the coefficient of $u^n v^d$ is 0 in $\Phi(u, v)$. Thus the coefficient of u^n in $\tilde{G}_d(u/4)$ is zero. By (7) and (8) we have

$$\begin{aligned}
&\sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(2^{2n-2j}-1)B_{2n-2j}}{2^{2d-1}d} \binom{2d-2j-2}{d-1} \binom{2n}{2j} \\
=&\frac{(-1)^n (2n)!}{(2\pi)^{2n}} \times \text{Coeff. of } u^n \text{ of (7) (i.e. } j = n-1) \\
=&\begin{cases} 0, & n < d < 2n; \\ \dfrac{n}{2^{2d-1}d} \dbinom{2d-2n-1}{d-1}, & d \geq 2n, \end{cases}
\end{aligned}$$

as desired. □

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